Newton's Method and Localization

Workshop on Analytical Aspects of Mathematical Physics

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Overview

Diagonalizing the Hamiltonian is a goal in quantum theory.

I would like to discuss a method for doing it in the context of two problems:

localization in the Anderson model
and
many-body localization for a quantum Ising model with disorder.

This will be a good opportunity to develop and discuss a mixture of perturbative and nonperturbative arguments in a multiscale setting.

Joint work with Tom Spencer.

Anderson Model

The Hamiltonian on Z^d is $H = H_0 + J$ or:

$$H_{ij} = \begin{cases} v_i, & i = j; \\ J_0, & |i - j| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

where v_i is a random potential.

Localization: Eigenfunctions concentrated at centers with exponential tails. They are occasionally spread out a bit when resonances occur.

Many-body Hamiltonian

Start with a simple 2 by 2 Hamiltonian on each site:

$$H_0 = \bigotimes_{i \in \Lambda} \begin{pmatrix} h_i & J \\ J & -h_i \end{pmatrix} = \bigotimes_{i \in \Lambda} (h_i \sigma_i^z + J \sigma_i^x)$$

where h_i is a random field. Then let

$$H = H_0 + \sum_{\langle i,j \rangle} J_0 \sigma_i^z \sigma_j^z$$

This is a quantum Ising model with a random magnetic field.

Many-body localization: Eigenfunctions are approximately product states, with exponentially decaying entanglement. Occasional longer-range entanglement can happen when resonances occur.

Many-body localization

We would like to diagonalize this Hamiltonian and understand the nature of the eigenfunctions for small J and/or small J_0. What does localization mean in this context?

- In the Anderson model, localization means that the eigenfunctions closely resemble the $J = 0$ eigenfunctions, which are δ-functions at the sites in Λ.

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- In the many-body context, it should mean that the spin variables should (with rare exceptions) resemble those of the $J = 0$ states. That is, a prescribed set of 1's and -1's in the lattice. Let's call the state-labeling spin configuration $\sigma_i^{\mathrm{label}}$.

Many-body localization

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- The many-body wave function should be "concentrated" on configurations "close to" $\sigma_i^{\mathrm{label}}$. This means in particular that each wave function is approximately a product state, with very little entanglement.

Background on many-body localization: A special kind of quantum phase transition

Basko, Aleiner & Altschuler: Annals of Physics 321, 1126 (2006)
Pal & Huse: Phys. Rev. B 82, 174411 (2010), arxiv:1010:1992

Weak Randomness:	Strong randomness:
ergodic	localized
long-range entanglement	short range entanglement
thermalized	non-thermalized

Thermalized means an eigenstate at energy E would look locally like a statistical ensemble of states in thermal equilibrium.
Localized means one bare state σ^{label} (plus rare localized variations of it) predominates.

Desired Result

We would like to diagonalize H with a set of eigenfunctions given by graphical expansions with explicit bounds, including bounds on probabilities of rare events. Then, because of the smallness or rarity of deviations from σ^{label}, one should be able to show that

$$\mathrm{Av}_h \ \mathrm{Av}_{\sigma^{\text{label}}} \ |\langle \sigma_0^z \rangle_{\psi_{\sigma^{\text{label}}}}| \text{ is close to } 1,$$

which shows that for most h's and for most states the state follows the label. In the thermalized case, this would be a mixture of many (unperturbed) states and hence presumably 0.

This is analogous to the situation for the mixed-state classical Ising model at low temperature:

$$\langle \cdot \rangle_{\text{mixed}} = \frac{1}{2}\langle \cdot \rangle_+ + \frac{1}{2}\langle \cdot \rangle_-$$

with

$$\langle \sigma_0 \rangle_{\text{mixed}} = 0, \qquad |\langle \sigma_0 \rangle_\pm| \text{ close to } 1.$$

A very simple example: One spin

$$\begin{pmatrix} h & J \\ J & -h \end{pmatrix} = h\sigma^z + J\sigma^x$$

Here h is random, and J is small.

Perturbation theory for eigenvalues and eigenfunctions converges, provided $|h|$ is larger than some cutoff ε.

So with high probability, the eigenfunctions are close to $\binom{1}{0}$ & $\binom{0}{1}$.

We seek analogous control over the eigenfunctions of the many-body Hamiltonian H.

Resonances

For a warm-up, consider the Anderson Model.

Perturbation theory works if there are gaps between eigenvalues. This is a problem because the eigenvalues become dense as $\Lambda \to Z^d$. Luckily, J couples nearest neighbors only, so we only need to worry about nearest neighbor resonances (for the moment):

$$\langle i, j \rangle \text{ resonant if } |\Delta E| < \varepsilon$$

for some transition involving the spins at i, j. (Energies are the diagonal matrix elements of H_0.)

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- For the spin chain, $J_0 \sigma_i^z \sigma_j^z$ transitions the state only on $< i, j >$. The bond is resonant if some ΔE is small.

Perturbation Theory

Let us stay away from resonant bonds and put

$$A_{ij} = \frac{J_{ij}}{E_i - E_j}.$$

First order perturbation theory:

$$\begin{aligned} \psi_i^{(1)} &= \psi_i + \sum_j \frac{J_{ij}}{E_i - E_j}\, \psi_j \\ &= \sum_j (I + A)_{ij} \psi_j. \end{aligned}$$

Here $\psi_i(j) = \delta_{ij}$ are the unperturbed eigenvectors.

Effective Hamiltonian

Instead, use $\Omega = e^{-A}$ for the basis change (preserves norm).

$$\psi_i^{(1)} = \sum_j \Omega_{ij}^{\mathrm{tr}} \psi_j.$$

Renormalized Hamiltonian:

$$H^{(1)} = \Omega^{\mathrm{tr}} H \Omega$$

Related Ideas: Similarity Renormalization

Brockett-Wegner Flow
Brockett: Lin. Alg. Appl. 1991
Wegner: Ann. Phys. 1994

Similarity Renormalization
Glazek & Wilson: Phys. Rev. D48, 5863 (1993)

Block diagonalization methods for quasiperiodic Schroedinger operators
Eliasson: Acta Math. 1997

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&= H_0 + J - J + [A, J] + \frac{[A, -J + [A, J]]}{2!} + \dots \\
&= H_0 + \sum_{n=1}^{\infty} \frac{n}{(n+1)!} (\mathrm{ad}\, A)^n J \\
&= H_0 + J^{(1)}.
\end{aligned}$$

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- The diagonal part of $J^{(1)}$ is absorbed into H_0 as energy corrections.
- The columns of $\Omega = e^{-A}$ are the eigenstates to leading order.
- Expand the exponential to get a graphical expansion for the eigenfunctions: Graphs are connected (Anderson) or disconnected (many-body)–forms polymer expansions.

KAM scheme (Newton's method)

We would like to continue the procedure, perturbing in the off-diagonal part of $H^{(1)}$, which is now $O(J_0^2)$.

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- Exception: terms associated with resonant blocks.

Diagonalization within blocks

Solution is to diagonalize $H_0 + J^{\mathrm{int}}$ within (small) resonant blocks, using a matrix O.

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- No nontrivial bounds available for O: wavefunctions may involve a high degree of entanglement within a block.
- $H^{(1')} = O^{\mathrm{tr}} H^{(1)} O$ now has no $O(J_0)$ terms at all (except for some really large blocks, which have to wait until the couplings are small enough).

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- Higher order diagrams have many energy denominators which are individually OK but one has to show they are OK in combination.

Controlling the Basis Change Generator A

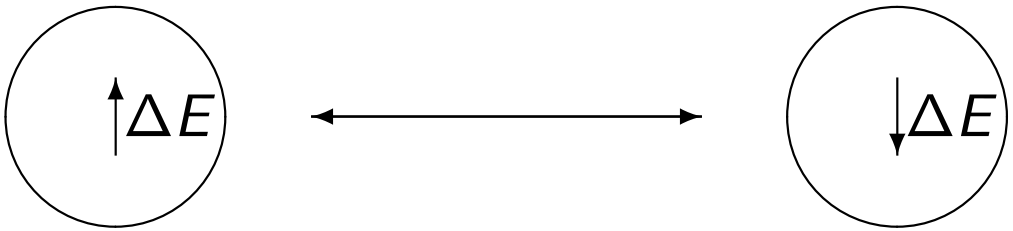
The basis change matrix is $\Omega = e^{-A}$ with $A = \sum_g A(g)$, where g is a connected multiscale graph.

If g has no loops, then $\mathrm{Av}_h |A(g)| \leq (J_0 \log \varepsilon)^{|g|}$ away from resonant regions from previous scales (from $\int_{|h| \geq \varepsilon} \frac{1}{h} d\lambda(h) \leq \log \varepsilon$).

Average behavior much better than worse case, so define new resonances by the condition $|A(g)| \geq (J_0/\varepsilon)^{|g|}$. Then by the Markov inequality,

$$P\left(|A(g)| \geq (J_0/\varepsilon)^{|g|}\right) \leq \frac{(J_0 \log \varepsilon)^{|g|}}{(J_0/\varepsilon)^{|g|}} = (\varepsilon \log \varepsilon)^{|g|}$$

Block-Block Resonances



How to control the probability of resonance between blocks? This is a density of states issue.

No Wegner bound available.

Do energy differences within a block vary with the randomness?

Interaction with block's neighbors creates enough width in the distribution of ΔE to control the probability of resonance.

Loops

If g has loops, then there are duplicated or non-independent denominators, so we have integrals like $\int_{|h|\geq\varepsilon} \frac{1}{h^2} d\lambda(h) \leq \varepsilon^{-1}$ which lead to negative powers of ε. This weakens the bound on the probability of the resonance to tree decay:

$$P\left(|A(g)| \geq (J_0/\varepsilon)^{|g|}\right) \leq (\varepsilon \log \varepsilon)^{|T(g)|}.$$

But this is sufficient to control the sum over pairs of (potentially) resonant states in $X = T(g)$ (which are no more than $4^{|X|}$ in number).