Solution 12

1. Displacement operators and coherent states

(i) For $Z = \alpha a^* - \bar{\alpha} a$, $Y = \beta a^* - \bar{\beta} a$ and $[Z, Y] = -\alpha\bar{\beta}[a^*, a] - \bar{\alpha}\beta[a, a^*] = \alpha\bar{\beta} - \bar{\alpha}\beta = -2\mathrm{i}\,\mathrm{Im}(\bar{\alpha}\beta)$ we have $[[Z, Y], Z] = [[Z, Y], Y] = 0$ and hence $\mathrm{e}^{Z+Y} = \mathrm{e}^Z \mathrm{e}^Y \mathrm{e}^{-[Z,Y]/2}$. Thus

$$V(\alpha + \beta) = V(\alpha)V(\beta)\mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\bar{\alpha}\beta)}\,, \tag{8}$$

which, together with $(\alpha \leftrightarrow \beta)$, yields

$$V(\alpha)V(\beta) = \mathrm{e}^{-2\mathrm{i}\,\mathrm{Im}(\bar{\alpha}\beta)}V(\beta)V(\alpha)\,. \tag{9}$$

By a similar consideration with $Z = -\mathrm{i}sP$, $Y = \mathrm{i}tX$, or by (9) with $U_P(s) = V(s/\sqrt{2})$, $U_X(t) = V(\mathrm{i}t/\sqrt{2})$, we have

$$U_P(s)U_X(t) = \mathrm{e}^{-\mathrm{i}st}U_X(t)U_P(s)\,.$$

Hence

$$\begin{aligned} [V(\alpha), V(\beta)] = 0 \quad &\Leftrightarrow \quad \mathrm{Im}(\bar{\alpha}\beta) = \pi n\,, \\ [U_P(s), U_X(t)] = 0 \quad &\Leftrightarrow \quad st = 2\pi n\,, \end{aligned}$$

$n \in \mathbb{Z}$.

(ii) Using $W_\alpha = V(\alpha)\psi_0$ and $V(\alpha) = \mathrm{e}^{\alpha a^*}\mathrm{e}^{-\bar{\alpha}a}\mathrm{e}^{-|\alpha|^2/2}$ we get

$$\langle \psi_0, V(\alpha)\psi_0\rangle = \mathrm{e}^{-|\alpha|^2/2}\langle \psi_0, (\mathrm{e}^{\bar{\alpha}a})^*(\mathrm{e}^{-\bar{\alpha}a})\psi_0\rangle = \mathrm{e}^{-|\alpha|^2/2}\,.$$

The last equality above follows from

$$\mathrm{e}^{-\bar{\alpha}a}\psi_0 = \sum_{k=0}^{\infty}\frac{(-\bar{\alpha})^k}{k!}a^k\psi_0 = \psi_0 + \sum_{k=1}^{\infty}\frac{(-\bar{\alpha})^k}{k!}a^k\psi_0 = \psi_0 = \mathrm{e}^{\alpha a}\psi_0\,,$$

which in turn follows from $a\psi_0 = 0$. The identity $V(\alpha)^* = V(\alpha)^{-1} = V(-\alpha)$ together with (8) then implies

$$\begin{aligned} \langle W_{\alpha_1}, W_{\alpha_2}\rangle &= \langle \psi_0, V(\alpha_1)^*V(\alpha_2)\psi_0\rangle = \langle \psi_0, V(-\alpha_1)V(\alpha_2)\psi_0\rangle \\ &= \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\overline{\alpha_1}\alpha_2)}\langle \psi_0, V(\alpha_2 - \alpha_1)\psi_0\rangle = \mathrm{e}^{\mathrm{i}\,\mathrm{Im}(\overline{\alpha_1}\alpha_2)}\mathrm{e}^{-|\alpha_2-\alpha_1|^2/2}\,. \end{aligned}$$

(iii) Since $V(\alpha)\psi_0 = \mathrm{e}^{-|\alpha|^2/2}\mathrm{e}^{\alpha a^*}\psi_0$, the function in the hint is

$$f(\alpha) = \mathrm{e}^{|\alpha|^2/2}\langle \psi, W_\alpha\rangle = \mathrm{e}^{|\alpha|^2/2}\langle \psi, V(\alpha)\psi_0\rangle = \langle \psi, \mathrm{e}^{\alpha a^*}\psi_0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle \psi, \psi_n\rangle\,, \tag{10}$$

where the eigenstates $\psi_n = (a^*)^n\psi_0/\sqrt{n!}$ form an orthonormal basis. Hence $\langle \psi, W_\alpha\rangle = 0$ $(\alpha \in \mathbb{C})$ implies that the coefficient for every power of α in (10) vanishes, i.e. $\langle \psi, \psi_n\rangle = 0$ $(n \in \mathbb{N})$, and therefore $\psi \equiv 0$.

(iv) By the hint, we have to show

$$\frac{1}{\pi}\int_{\mathbb{C}} \langle W_\beta, W_\alpha\rangle\langle W_\alpha, W_\gamma\rangle \,\mathrm{d}(\mathrm{Re}\alpha)\,\mathrm{d}(\mathrm{Im}\alpha) = \langle W_\beta, W_\gamma\rangle\,.$$

By (1) we have $\langle W_\beta, W_\alpha\rangle\langle W_\alpha, W_\gamma\rangle = \mathrm{e}^F$, where

$$\begin{aligned} F &= -\frac{1}{2}\left(|\beta-\alpha|^2 + |\alpha-\gamma|^2\right) + \mathrm{i}\,\mathrm{Im}(\bar\beta\alpha + \bar\alpha\gamma) \\ &= -\frac{1}{2}\left(|\beta-\gamma|^2 + 2\,|\alpha|^2 - 2\,\mathrm{Re}(\bar\beta\alpha + \bar\alpha\gamma - \bar\beta\gamma)\right) + \mathrm{i}\ \mathrm{Im}(\bar\beta\alpha + \bar\alpha\gamma) \\ &= \left(-\frac{1}{2}\,|\beta-\gamma|^2 + \mathrm{i}\,\mathrm{Im}\,\bar\beta\gamma\right) - |\alpha|^2 + \bar\beta\alpha + \bar\alpha\gamma - \bar\beta\gamma\,. \end{aligned}$$

We are thus left to show that

$$\frac{\mathrm{e}^{-\bar\beta\gamma}}{\pi}\int_{\mathbb{C}} \mathrm{e}^{-|\alpha|^2+\bar\beta\alpha+\bar\alpha\gamma}\,\mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha) = 1\,. \tag{11}$$

The integral can be written as the product of the two factors

$$\begin{aligned} \int \mathrm{e}^{-(\mathrm{Re}\,\alpha)^2+(\bar\beta+\gamma)\,\mathrm{Re}\,\alpha}\,\mathrm{d}(\mathrm{Re}\,\alpha) &= \sqrt{\pi}\mathrm{e}^{(\bar\beta+\gamma)^2/4}\,, \\ \int \mathrm{e}^{-(\mathrm{Im}\,\alpha)^2+\mathrm{i}(\bar\beta-\gamma)\,\mathrm{Im}\,\alpha}\,\mathrm{d}(\mathrm{Im}\,\alpha) &= \sqrt{\pi}\mathrm{e}^{-(\bar\beta-\gamma)^2/4}\,, \end{aligned}$$

where $(\bar\beta+\gamma)^2 - (\bar\beta-\gamma)^2 = 4\bar\beta\gamma$. This proves (11) and therefore (2). The second version (3) follows by the substitution of variables given in the exercise.

2. Particle in the plane with transverse magnetic field

(ia) Classically, we have the Lorentz force $F_L = (e/c)v \times B$ which is orthogonal to v: $F_L \cdot v = 0$. Thus the kinetic energy is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv^2) = mv\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = F_L\cdot v = 0\,,$$

and therefore also $|v|$: $\mathrm{d}v^2/\mathrm{d}t = \mathrm{d}|v|^2/\mathrm{d}t = 0$. Hence the particle is moving on a circle of some radius R, which already implies that the vectors on the two sides of eq. (5) point in the same direction. Moreover, we know now that F_L equals the corresponding centripetal force F_Z, i.e.

$$\frac{e}{c}|v|B = |F_L| = |F_Z| = m\frac{|v|^2}{|R|}\,,$$

where the first equation follows from v being perpendicular to B and $eB > 0$. This proves (5), as well as $|v| = \omega|R|$ with $\omega = eB/mc$. In the situation of the hint we have $x = R$, and therefore

$$p = mv + \frac{e}{c}A = -\frac{e}{2c}B\times R$$

has constant absolute value. The quantization $n\hbar$ applies to

$$|p||R| = \frac{e}{2c}B|R|^2 = \frac{m}{2}\omega|R|^2\,,$$

and yields the radii $|R_n|^2 = (2c/eB)n\hbar$. Comparison with $E = m|v|^2/2 = m(\omega|R|)^2/2$ in turn yields the energies $E_n = n\hbar\omega$. The degeneracy of the level E_n can be computed heuristically: for a fixed center, the state takes a circular ring between his radius $|R_n|$ and the one of its neighbour level, $|R_{n+1}|$ (or $|R_{n-1}|$). The area amounts to $F = \pi(|R_{n+1}|^2 - |R_n|^2) = hc/eB$, $(h = 2\pi\hbar)$. Without any restrictions to the centers of the circles, those can be put in a way such that the circular rings of states corresponding to the same level do not overlap. Considering only the area (and not the geometry), this yields a density $\rho = 1/F$ of the centers, which is the degeneracy per unit area, and thus (7) follows.

(ib) Let $A = (-Bx_2, 0)$. Then

$$H = \frac{1}{2m}\big((p_1 + \frac{eB}{c}x_2)^2 + p_2^2\big)\,,$$

which does not contain x_1. Hence H commutes with p_1 and therefore leaves the eigenspaces of p_1 invariant. These eigenspaces consist of the functions $\varphi(x_2)\mathrm{e}^{\mathrm{i}kx_1}$ (i.e. each $k \in \mathbb{R}$ corresponds to an eigenspace), where $\varphi(x_2) \in L^2(\mathbb{R})$ is constant w.r.t. x_1. Moreover, since $L^2(\mathbb{R}^2) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and since the functions $\mathrm{e}^{\mathrm{i}kx_1}$ span $L^2(\mathbb{R})$, the eigenspaces of p_1 cover the whole space. This justifies the ansatz $\psi(x_1, x_2) = \mathrm{e}^{\mathrm{i}kx_1}\varphi(x_2)$ for the eigenfunctions. Plugging it into the time-independent Schrödinger equation $H\psi = E\psi$ yields

$$\big(\frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2(x_2 + \frac{\hbar kc}{eB})^2\big)\varphi = E\varphi\ .$$

Up to a translation in 2-direction, this is the eigenvalue equation of the 1-dimensional harmonic oscillator. The energies and the eigenfunctions (normed w.r.t. ξ, but not w.r.t. x_1) are

$$E_n = \hbar\omega\big(n + \frac{1}{2}\big)\ , \qquad \psi_n(x_1, x_2) = \mathrm{e}^{\mathrm{i}kx_1}\frac{\pi^{-1/4}}{\sqrt{2^n n!}}H_n(\xi)\mathrm{e}^{-\xi^2/2}\ , \qquad (n \in \mathbb{N})$$

with

$$\xi = \sqrt{\omega m\hbar^{-1}}\big(x_2 + \frac{\hbar kc}{eB}\big)\ .$$

The degeneracy is infinite, since E_n is independent of $k \in \mathbb{R}$. By considering a big but finite domain $(x_1, x_2) \in [0, L_1] \times [0, L_2]$, for example with periodic boundary conditions in 1-direction, one gets finite degeneracy. Then k is quantized, $kL_1 = 2\pi m$ $(m \in \mathbb{Z})$, and further restricted by the fact that the centers of the oscillators,

$$x_2^{(0)} = -\frac{\hbar kc}{eB} = -\frac{hc}{eBL_1}m\,,$$

are roughly located in $[0, L_2]$, and therefore the ψ_n as well. Hence we have $(eB/hc)L_1(L_2 + O(1))$ possibilities for m, each of them providing a state of energy E_n. This confirms (7).

Let now $A = B(-x_2, x_1)/2$. Then

$$\begin{aligned} H &= \frac{1}{2m}\big((p_1 + \frac{eB}{2c}x_2)^2 + (p_2 - \frac{eB}{2c}x_1)^2\big) \\ &= \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\tilde{\omega}^2}{2}(x_1^2 + x_2^2) - \tilde{\omega}(x_1p_2 - x_2p_1)\,, \end{aligned}$$

where $\tilde{\omega} = eB/2mc$. Here $L = x_1p_2 - x_2p_1$ is the (canonical) angular momentum, i.e. $(0,0,L)$ in 3 dimensions. Let $a_\pm^*$ be as in (6). Using $[x_i, x_j] = [p_i, p_j] = 0, [x_i, p_j] = \imath\hbar\delta_{ij}, (i,j = 1,2)$, we get $[a_+, a_+^*] = [a_-, a_-^*] = 1$, $[a_+^{\#_+}, a_-^{\#_-}] = 0$, ($\#_\pm =$ nothing, $*$). These are the commutation relations of two independent harmonic oscillators. We have

$$2\hbar\tilde{\omega}N_\pm = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{m\tilde{\omega}^2}{2}(x_1^2 + x_2^2) \pm \tilde{\omega}(x_1p_2 - x_2p_1) - \hbar\tilde{\omega} \ .$$

Hence

$$H = \hbar\tilde{\omega}(2N_- + 1) \ , \qquad L = \hbar(N_+ - N_-) \ .$$

The energy eigenvalues are $E_{n_-} = 2\hbar\tilde{\omega}(n_- + \frac{1}{2})$, which coincides with (4) since $\omega = 2\tilde{\omega}$. The eigenfunctions are $(n_+!n_-!)^{-1/2}(a_+^*)^{n_+}(a_-^*)^{n_-}\psi_0$, $(n_+, n_- \in \mathbb{N})$, where $a_\pm\psi_0 = 0$. Here, the eigenvalues are infinitely degenerate since they are independent of n_+.

(iia) The position x, the center r and the radius R satisfy $x = r + R$. By (5) we have $\Pi = -\beta R^\perp$, where $\beta = (e/c)B$ and $a^\perp = (-a_2, a_1)$ is the vector $a = (a_1, a_2)$ turned by 90°. Hence $R = \beta^{-1}\Pi^\perp$ and

$$r = x - \beta^{-1}\Pi^\perp \, ;$$

i.e.

$$r_1 = x_1 + \beta^{-1}\Pi_2 \ , \qquad r_2 = x_2 - \beta^{-1}\Pi_1$$

in components.

(iib)

$$\mathrm{i}[\Pi_1, \Pi_2] = -\frac{e}{c}\left(\mathrm{i}[p_1, A_2] + \mathrm{i}[A_1, p_2]\right) = -\frac{e\hbar}{c}\left(\partial_1 A_2 - \partial_2 A_1\right) = -\beta\hbar \, ,$$
$$\mathrm{i}[r_1, r_2] = \beta^{-1}\left(-\mathrm{i}[x_1, \Pi_1] + \mathrm{i}[\Pi_2, x_2] - \mathrm{i}\beta^{-1}[\Pi_2, \Pi_1]\right) = \beta^{-1}\hbar \, .$$

The commutators $[\Pi_i, r_j]$ vanish, e.g.

$$\mathrm{i}[\Pi_1, r_1] = \mathrm{i}[\Pi_1, x_1] + \beta^{-1}\mathrm{i}[\Pi_1, \Pi_2] = \hbar - \hbar = 0 \, ,$$
$$\mathrm{i}[\Pi_1, r_2] = \mathrm{i}[\Pi_1, x_2] - \beta^{-1}\mathrm{i}[\Pi_1, \Pi_1] = 0 \, .$$

(iic) We have $H = (2m)^{-1}(\Pi_1^2 + \Pi_2^2)$ and therefore $[H, r_i] = 0$ by the preceding exercise. The expression for H and $\mathrm{i}[\Pi_2, \Pi_1] = \beta\hbar$ correspond to the harmonic oscillator

$$H = \frac{\omega}{2}(P^2 + X^2) \, , \qquad \mathrm{i}[P, X] = \hbar \, ,$$

and arise from the latter by the replacements $\omega \rightsquigarrow m^{-1}$, $p \rightsquigarrow \Pi_2$, $x \rightsquigarrow \Pi_1$, $\hbar \rightsquigarrow \beta\hbar$; and consequently the eigenvalues by

$$\hbar\omega\big(n + \frac{1}{2}\big) \qquad \rightsquigarrow \qquad \frac{\beta\hbar}{m}\big(n + \frac{1}{2}\big) \, ,$$

$(n \in \mathbb{N})$. They are infinitely degenerate because of the additional degree of freedom r_1 (or r_2).