Problem Set 10

1. Time-dependent Hamilton operators

Consider a time-dependent Hamiltonian $H(t)$ and the corresponding Schrödinger equation

$$\mathrm{i}\partial_t\psi(t) = H(t)\psi(t)\,, \tag{1}$$

where we have set $\hbar = 1$. Define the operator $U(t, s)$ through

$$\mathrm{i}\partial_t U(t,s) = H(t)U(t,s)\,, \quad U(s,s) = \mathbb{1}\,. \tag{2}$$

In this problem we do not concern ourselves with issues of domains of unbounded operators and the existence and uniqueness of solutions of the above ODE (see Reed and Simon Vol.2, X.12 for details on these questions).

(i) Write the time evolution in the Schrödinger and the Heisenberg picture.

(ii) Show that $U(t, s)$ is unitary, and that $U(t, s)U(s, r) = U(t, r)$.

(iii) The classical Hamiltonian of a particle in a time-dependent electromagnetic field is

$$H(x,p) = \frac{1}{2m}(p - \frac{e}{c}A(t,x))^2 + e\varphi(t,x), \quad (x, p \in \mathbb{R}^3)\,.$$

Show that Hamilton's equations of motion reproduce Newton's equations of motion with the Lorentz force.

(iv) Let $H(t)$ be the Hamiltonian corresponding to $H(x, p)$ in (iii). Find the transformation behaviour of the wave function under the gauge transformation $\tilde{A}(t, x) = A(t, x) + \nabla\chi(t, x)$, $\tilde{\varphi}(t, x) = \varphi(t, x) - \frac{1}{c}\partial_t\chi(t, x)$.

2. Self-adjointness of Schrödinger operators with Coulomb potential

Let

$$H = -\hbar^2\Delta + w(x) \tag{3}$$

be a formal Hamilton operator acting on $L^2(\mathbb{R}^3)$. The goal of this exercise is to show that H is self-adjoint if w is the Coulomb potential $w(x) = d|x|^{-1}$.

(i) Prove *Hardy's inequality*

$$\langle\psi\,, |x|^{-2}\psi\rangle \leqslant \frac{4}{(n-2)^2}\langle\psi\,, -\Delta\psi\rangle$$

for $n \geqslant 3$ and all $\psi \in D(-\Delta)$.

Hint: For $\alpha \in \mathbb{R}$ define $A_i := \mathrm{i}\partial_i + \mathrm{i}\alpha x_i|x|^{-2}$. Compute the right-hand side of

$$0 \leqslant \sum_{i=1}^{n}\langle\psi\,, A_i^*A_i\psi\rangle\,,$$

and optimize in α.

(ii) Use Hardy's inequality and the Kato-Rellich theorem to prove that H defined in (3) with the Coulomb potential $w(x) = d|x|^{-1}$ is self-adjoint on the domain of self-adjointness of $-\hbar^2\Delta$.

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