Problem Set 7

1. Field of a uniformly moving charge

A point charge e is moving on an inertial trajectory $\underline{x} = \underline{v}t$, $\underline{v} = (v, 0, 0)$. Compute the fields $\underline{E}(\underline{x}, t)$, $\underline{B}(\underline{x}, t)$. For a fixed distance $|\underline{x}|$ from the charge, what is the direction in which $\underline{E}(\underline{x}, t = 0)$ is the strongest and the weakest respectively?

Hint: First compute the fields in the rest frame of the particle.

2. Dual field tensor

Define the dual field tensor

$$\mathcal{F}_{\rho\sigma} = \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \tag{1}$$

and the dual current

$$\mathcal{J}_{\nu\rho\sigma} = j^{\mu} \varepsilon_{\mu\nu\rho\sigma} \,,$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor with $\varepsilon_{0123} = +1$ (parity of the permutation $(0123) \mapsto (\mu\nu\rho\sigma)$).

Remark: Here duality is not meant in terms of the dual space, but of the Hodge duality (inessential for this exercise). It follows that $\mathcal{F}$ and $\mathcal{J}$ are tensors.

(i) Express the tensor components $\mathcal{F}_{\mu\nu}$ in terms of $\underline{E}$ and $\underline{B}$. The duality turns out to be an electric-magnetic one.

(ii) Show: Maxwell's equations read

$$\mathcal{F}^{\mu\nu}{}_{,\mu} = 0 \,,$$
$$\mathcal{F}_{\rho\sigma,\mu} + \mathcal{F}_{\mu\rho,\sigma} + \mathcal{F}_{\sigma\mu,\rho} = -\frac{1}{c} \mathcal{J}_{\rho\sigma\mu} \,.$$

Hint: Show

$$\mathcal{F}_{\rho\sigma,\mu} + \mathcal{F}_{\mu\rho,\sigma} + \mathcal{F}_{\sigma\mu,\rho} = F^{\alpha\nu}{}_{,\alpha} \varepsilon_{\mu\nu\rho\sigma} \,. \tag{2}$$

In the notation of $\underline{E}$ and $\underline{B}$, the left hand sides of the homogeneous and inhomogeneous Maxwell's equations emerge from each other under $(\underline{E}, \underline{B}) \rightsquigarrow (-\underline{B}, \underline{E})$. The equations above express this symmetry in relativistic notation.

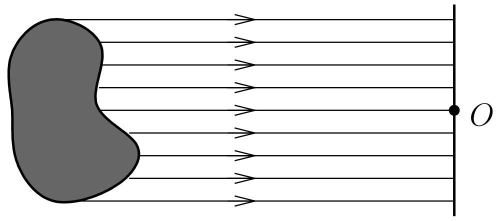
3. Seeing is not measuring

Descriptions of special relativity in popular science (e.g. Gamow 1940) claim that moving objects appear to be deformed; namely due to Lorentz contraction in the direction of motion, and the absence of it in the transversal direction. This is not quite true (Penrose, Terrell 1959; s. also Kraus and Borchers[1] 2005).

[1] www3.interscience.wiley.com/cgi-bin/fulltext/109926451/PDFSTART

To measure a length is not the same as to see a length. A measured length is defined to be the coordinate difference of two points of an object at the same time (w.r.t. some reference frame) (s. exercise 6.1(ii)). The length that we see however, is given by light signals which we receive at the same time. The image thus combines earlier distant events with later closer ones. Though a moving object looks shorter than an identical object at rest at the same position in the image, it does not look deformed but rotated. I.e. like a co-moving observer could see the object, if he had the right angle of view (compare aberration, exercise 6.3(iii)). The latter would interpret the shortening simply as a perspective effect. The claim below allows us to understand this. There are the following restrictions: (i) the above only holds for small scales: the significant transformation is the one of the direction of sight, which is Möbius (s. exercise 6.2). Therefore the transformation preserves angles, but moving objects covering a large part of the field of vision appear to be deformed. (ii) Their colour and brightness is changed.

Consider an observer O who shines, at time $t = 0$, parallel light from an object onto a photographic plate perpendicular to it. (A camera which is focused on a distant object gives the same picture up to scaling.)

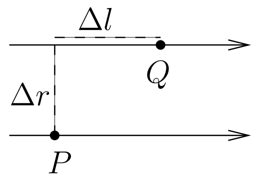


A second observer O', with common origin $(t, \underline{x}) = (0, 0) = (t', \underline{x}')$ but moving w.r.t. O, does the same at time $t' = 0$. (Remark: due to aberration, the plates of O and O' are not parallel in general.) Claim: both get the same picture. In particular, there is no length contraction.

For the derivation: the picture is the result of capturing simultaneously "light particles" which fly side by side (or at least it can be thought of like that). The claim thus follows from the following observations, which have to be justified:

Parallel inertial trajectories are mapped onto parallel inertial trajectories under Lorentz transformations. The property of light particles to fly side by side ($\Delta l = 0$ in the figure), is Lorentz-invariant. If this property is given, then their spatial distance Δr is Lorentz-invariant.

Hint: Two light particles P, Q may fly shifted. Consider two events $\mathcal{P}$, $\mathcal{Q}$ concerning the particles P, Q respectively (e.g. to strike the plate). How does $\langle \mathcal{P} - \mathcal{Q}, \mathcal{P} - \mathcal{Q} \rangle$ depend on their time difference Δt?



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