Descriptive and Prescriptive Reasoning in Integrated Assessment Climate Policy Models

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Writing nearly 20 years ago, Alan Manne (1995) highlighted the crucial role of the real annual discount rate for cost benefit analysis in climate policy. Manne summarizes as follows:

> With a real annual discount rate of 5% or more - and there are no significant climate impacts for half a century - these impacts have a present value that is virtually negligible. Within a cost-benefit framework, it then becomes exceedingly difficult to justify any near-term actions other than no-regrets policies. In the greenhouse debate, it is important to draw a clear distinction between prescriptive and descriptive reasoning. A philosopher or an economist may counsel a low or a zero rate of time preference, but this advice does not provide a good description of the collective outcome of individual choices. In particular, it implies an unrealistically rapid increase in the rate of savings and investment.

Climate policy analysts who counsel a low rate of time preference face a fundamental challenge: economic growth models with low or zero discount rates are inconsistent with market interest rates. In the Ramsey optimal growth framework, changes in the utility time preference produce implausible changes in aggregate investment without a significant change in the marginal productivity of capital. Using technical language, Manne declares that integrated assessment models based on very low discount rates “fail to pass the laugh test.”

Our paper picks up where Manne left off. We contemplate a framework for climate policy design which treats the economy as a second best situation, adding ideas from Frank Ramsey’s 1927 paper, “A Contribution to the Theory of Taxation”. To clarify our persepctive we begin from optimal taxation in which we design and implement taxes to maximize social welfare. Commonly, the optimal taxation involves minimizing the distortions caused by taxes which are levied to pay for public goods. In the optimal tax framework with a limited set of tax instruments, distortion and inefficiency may be unavoidable.

Some more words about the literature are needed here. Writing in the same volume where Manne’s paper appeared, Schelling [1995] casts the climate problem in terms of integenerational equity. Whether economics is able to address the deeper ethical issues is unclear. Rutherford [2013] echos Schelling’s observations regarding the role of kinship in attitudes toward the welfare of generations in the distant future. Hampicke [2011] likewise cautions about the limitations of the discounted utilitarianism as a framework for the assessment of climate policy. Indeed, Pindyck [2013] reminds us of a wide range of shortcomings in the set of integrated assessment models applied to climate policy questions.

The key points we make in this paper are as follows:

i. It is possible to distingush prescriptive and descriptive elements in integrated assessment models and thereby provide a plausible, logically consistent assessment of optimal policies under low or zero discount rates.

ii. The introduction of separate private and social discount rates permits calibration to an exogenous baseline growth path. This provides finer control over the assumptions underlying a given simulation and clarifies the distinction between exogenous and endogenous variables.

iii. Bilevel programming imposes only limited implementational overhead. Whether the underlying model is specified as an equilibrium system or as an optimization problem facing a representative agent, modern mathematical programming tools only require that the modeller specify the social welfare maximand.

iv. Computational experiments demonstrate that large scale model instances, even models which are are non-convex in terms of the policy instruments, can be processed using standard extended mathematical programming (EMP) solvers.

1 Dynamic General Equilibrium Analysis

Dynamic general equilibrium models are widely employed analytical frameworks for applied policy analysis. These models are attractive largely because they can describe *transitional impacts* of policy interventions. For example, carbon abatement measures planned

for implementation several years hence will alter the profitability of investments in coal-fired generating capacity today, and the analyst requires models which can assess the implications of these policies for current investment and employment. Faced with relatively few parsimonious alternatives, the economists traditionally rely on an assumption of intertemporal optimizing behavior as a starting point for performing such calculations.

The academic domain inhabited by dynamic economic equilibrium models can be uncomfortable due to the heterogeneity of interested parties. Policy models may be evaluated either on the basis of theoretical clarity or on the basis of empirical consistency. Simpler models are typically more compelling when they are relatively transparent, but simpler formulations can be difficult to reconcile with the demands of the policy analysis community.

This typically implies that a model should be able to provide congruence with a variety of alternative competing forecasts of GDP growth, energy demand and market interest rates. Unfortunately, most parsimonious theoretical models typically treat these times series as model outputs rather than model inputs. It remains the delicate task of the modeller to come up with an analytic apparatus which at is both concise and flexible.

Our paper highlights the issues which arise in reconciling theory and practice for dynamic general equilibrium analysis. Three illustrative models are considered. The first is a canonical model of intertemporal choice which illustrates the improvements in model flexibility which can be achieved with a modest increase in model complexity. The second model is a closed economy Ramsey growth model with two infinitely lived representative agents. These agents differ with respect to population growth, labor productivity growth and size. Calibration strategies which seem to work effectively in the single agent setting are less easily applied with multiple consumers, particularly when hetergeneous growth rates induce implausible levels of lending or borrowing. The final model is a simple two region energy-economy model which is calibrated to independent forecasts of regional GDP, regional energy demand and international energy prices. The model is parameterized with rought values for OECD and non-OECD regions over a 100 year horizon. The model provides a concrete example of how to work backwards from baseline equilibrium outcomes to infer consistent technology and preference parameters.

Thus, this paper addresses fundamental methodological challenges which arise in the calibration of Ramsey growth models to exogenous assumptions about a baseline growth path. The sample applications presented here are are primarily motivated by applications for energy and climate policy analysis, but these techniques could be readily adapted in other fields of economics.

When we build a *calibrated* equilibrium model, the model is formulated so that it *replicates* observed economic activities. In the static setting, this is fairly straightforward, because the benchmark database is typically based on a social accounting dataset. Subsequent policy analysis is then based on *counter-factual* simulations. These simulations explore how the simulated economy respond to changes in exogenous parameters, e.g. tax rates, technology, terms of trade or primary factor endowments.

We can formalize this description by first designating the price (p) and quantity (q) variables which define an equilibrium by a vector z which is a concatenation of q and p. The equilibrium calculation depends on the base year equilibrium values, $\bar{z}$, the base year and counterfactual policy parameters, $\bar{\tau}$ and τ, and the elasticities, σ. The counterfactual equilibrium conditions can be expressed as:

$$\mathscr{E}(z;\tau,\sigma;\bar{z},\bar{\tau}) \perp z \geq 0$$

in which "$\perp$" designates complementary slackness between the equilibrium conditions and the equilibrium variables (e.g., the equilibrium could involve zero prices or idle production activities).

The equilibrium conditions define equilibrium variables as an implicit function of the remaining parameters,$z(\tau;\bar{\tau},\bar{z},\sigma)$. Most model applications involve assessing how changes in τ produce changes in z^*, contingent on assumptions and empirical evidence regarding elasticities, σ

Benchmark replication provides a necessary (but not sufficient) verification of a model specification. When policy parameters assume reference year values, the model should replicate the reference equilibrium. In terms of the model portrayed as an implicit function, benchmark replication consists of verifying the following identity:

$$z(\bar{\tau};\bar{\tau},\bar{z},\sigma) = \bar{z}$$

Elasticites, σ, are typically *free parameters*, and thus the benchmark replication identity should hold for any assumed elasticity values. The benchmark equilibrium, however, ties down one parameter apiece, so the model is able to accommodate as many "calibrated parameters" as there are variables in the benchmark dataset.

The extension of these methods to *dynamic* settings introduces a number of methodological challenges. We typically then work from model inputs consisting of a set of base year transactions, and part of the model input consists of assumptions regarding future development of the economy. The key input assumption concerns *potential GDP*, typically embodied in a time series of labor supply measured in efficiency units. In addition, a baseline simulation may include assumptions about the future evolution of energy demand, accounting for autonomous energy enhancing improvements (AEEI).

When a dynamic model is based on a *steady-state* growth path, the construction of a baseline growth path is more or less equivalent to working with a static model. In the absence of changes in demographics or technology, the baseline equilibrium growth path involves proportional scaling of base year demands, all equilibrium values can be inferred directly from the reference year, growth rate and interest rate.

When a dynamic model is instead based on a *non-stationary* baseline, model calibration can be considerably more difficult. In such models, consumer choices over consumption are driven by intertemporal choice, and the parameters of intertemporal preferences can be only indirectly calibrated, on the basis of base year interest rates and assumptions regarding future growth. Additionally, particular aspects of the baseline growth path may be exogenous inputs. In climate policy analysis, the time path of energy demands through the model horizon is often taken as exogenous. In a neoclassical framework, technical change is exogenous, yet it remains a subtle maneuver to adjust technology parameters so as to maintain consistency between the computed baseline growth path and the exogenous energy demand profile.

If the benchmark dataset is based on a single year, $t = 0$, the model projects values from this year into one or more years in the future, based on an exogenous set of assumptions regarding future growth, denoted here by a vector ϕ. The implicit function which projects from the base year into the future, consistent with reference policy instruments:

$$z_t = z_t(\tau; \bar{\tau}, \bar{z}_0, \phi, \sigma)$$

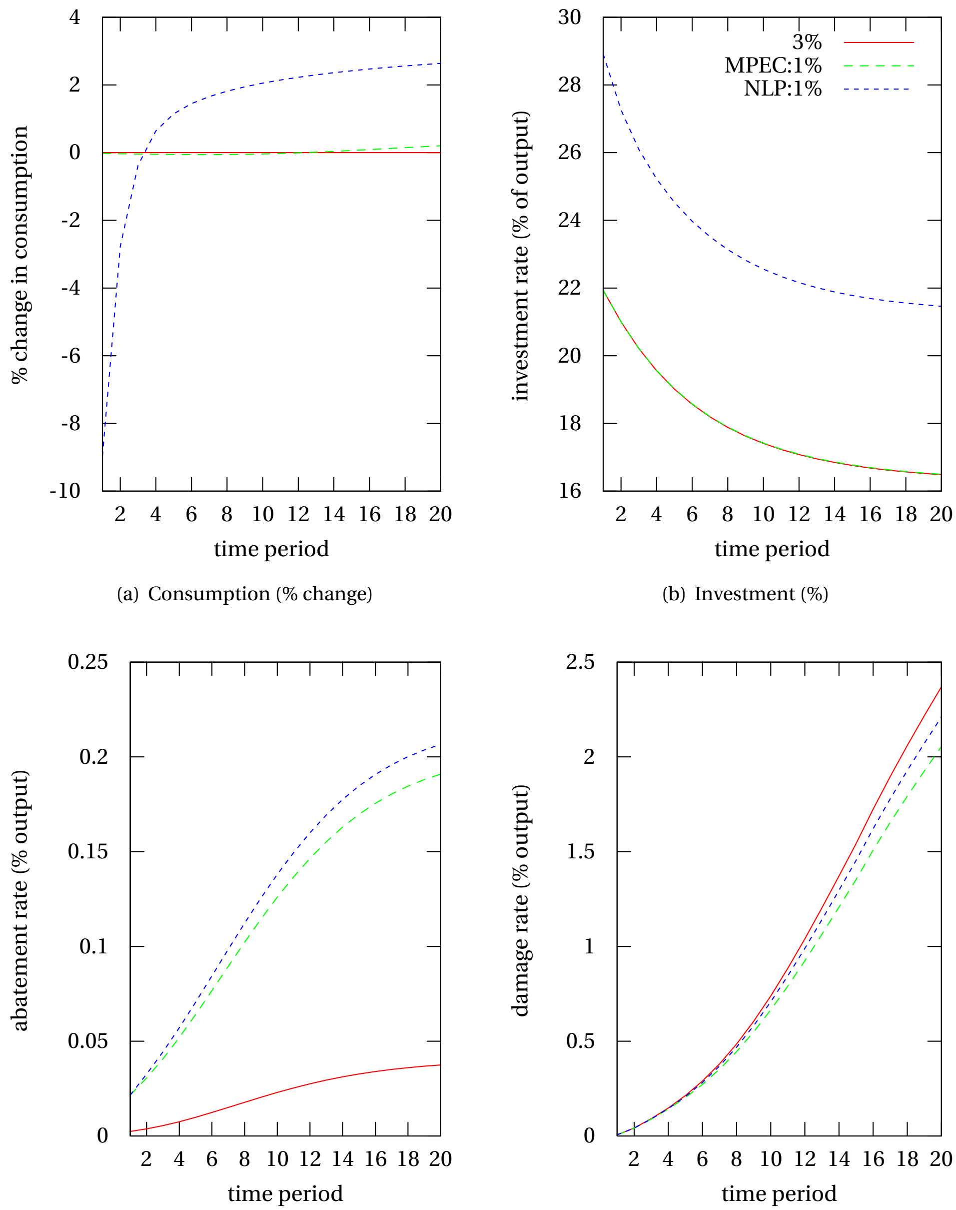
The baseline growth assumptions include a diverse collection of factors. Demographic growth assumptions are invariably part of these inputs, but there may be a number of other assumptions, such as those regarding productivity growth and autonomous changes in energy intensity.

If the model runs over T time periods, and there are n equilibrium variables per period, we then have $n \times (T - 1)$ more variables than there are parameters. Calibration of model parameters therefore requires an implicit or explicit assumptions about the baseline growth path.

Thus, there are two distinct issues involved in setting up the reference equilibrium. First, there is the issue of baseline growth rates and interest rates, equilibrium values which are directly related to assumptions about discounting, factor productivity and population growth. Second, one may be concerned about how technologies and preferences change over time and affect intra-temporal allocations. In climate policy applications, for example, an important *input* to the modeling process involves changes over time in the baseline demand for energy. In trade policy analyses, we need to account for how heterogeneous GDP growth affects import demand, export supply and inter-regional terms of trade. All of these considerations are part of the “art” of formulating a dynamic equilibrium model.

2 The DICE Model

Nordhaus [2008] uses a Ramsey growth model, DICE, to study long-term climate policy issues.



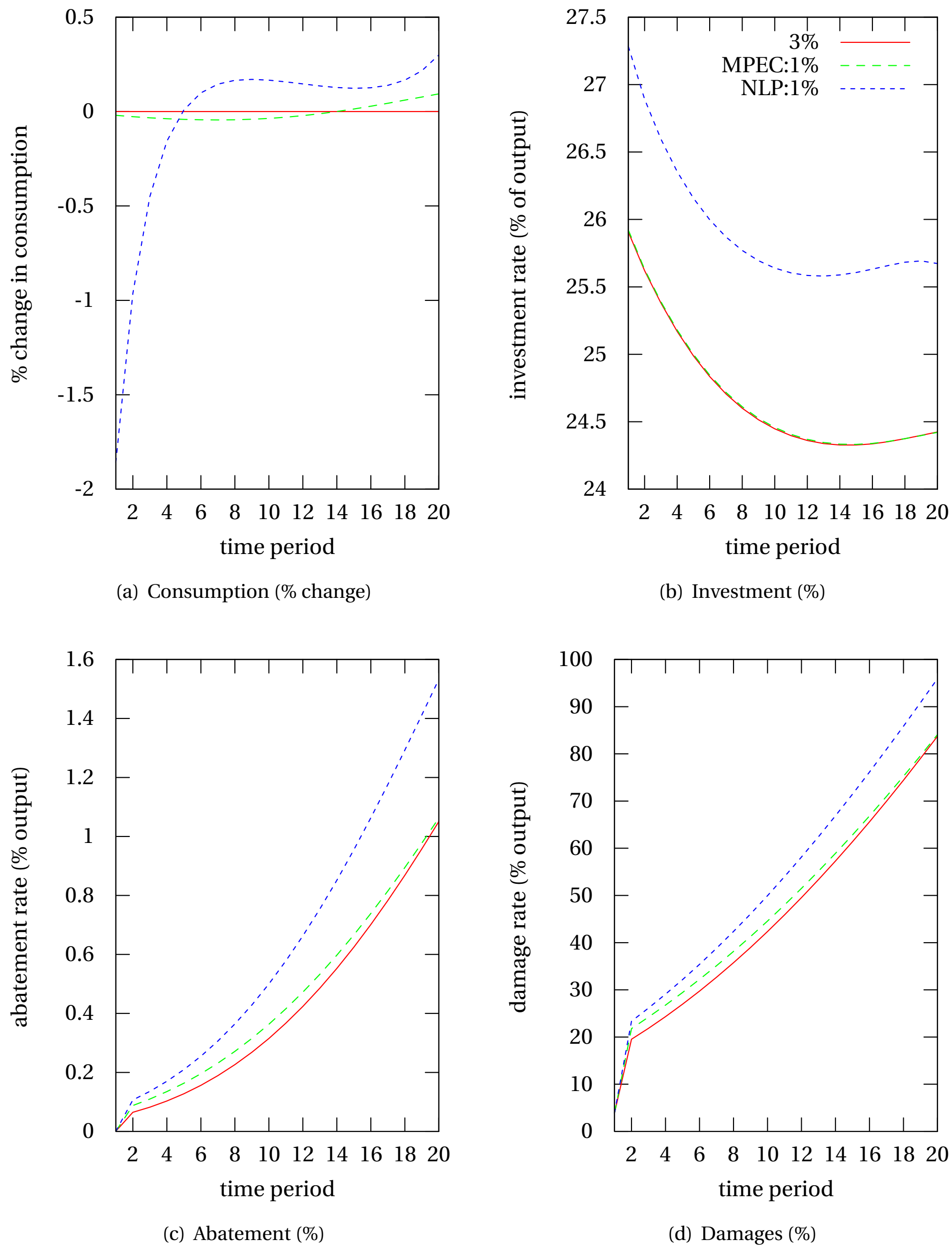
(a) Consumption (% change)

(b) Investment (%)

(c) Abatement (%)

(d) Damages (%)

Figure 1: Optimal Policy Comparisons in Dice 1994



(a) Consumption (% change)

(b) Investment (%)

(c) Abatement (%)

(d) Damages (%)

Figure 2: Optimal Policy Comparisons in Dice 2013

3 Calibration with Nonstationary Discounting

The DICE model works with an instantaneous *per-capita* utility function of the form:

$$U(c) = \frac{c^{1-a} - 1}{1-a} \tag{1}$$

in which parameter a is defined as *elasticity of the marginal utility of consumption.*[1]

Intertemporal welfare in this Ramsey model is defined on the basis of population and an exogenous (and constant) discount rate, δ:

$$W = \sum_t \frac{\mathscr{P}_t}{(1+\delta)^t} \; U(c_t) \tag{2}$$

When the population growth rate in year t equals n_t, an index of population in time t is given by:

$$\mathscr{P}_t = \prod_{\tau=0}^{t-1} (1+n_t). \tag{3}$$

Let Γ_t represent future-value GDP in year t, a parameter which reflects growth of both population and labor productivity:[2]

$$\Gamma_t = \bar{\Gamma} \prod_{\tau=0}^{t-1} (1+\gamma_t) \tag{4}$$

Per-capita GDP in time t is simply $\Gamma_t/\mathscr{P}_t$.

If we convert to aggregate consumption as a decision variable, we define:

$$C_t = \mathscr{P}_t c_t \tag{5}$$

We can then define $\tilde{W}$ as a scaled and recentered welfare index, defined in terms of aggregate consumption:

$$\tilde{W} = \sum_t \frac{\mathscr{P}_t^{\,a}}{(1+\delta)^t} \; U(C_t) \tag{6}$$

Optimal choices are invariate with respect to scaling or shifting of W, hence maximization of $\tilde{W}$ is equivalent to maximization of W. The consumers intertemporal choice problem can be written as:

$$\max \tilde{W}$$

s.t.

$$\sum_t p_t C_t = M$$

in which M represents value of current and future income flow – aggregate societal wealth, e.g. $M = \sum_t p_t \Gamma_t$.

[1] This parameter corresponds to the inverse of the intertemporal elasticity of substitution, σ_T. In the limit as $\sigma_T \to 1$, $a \to 1$ and we have the conventional Cobb-Douglas utility function:

$$U(c) = \ln(c)$$

[2] The productivity growth rate is roughly $\gamma_t) - n_t$.

Optimization of intertemporal welfare involves equating the marginal rate of substitution to the price ratio, that is:

$$
\begin{aligned}
\frac{p_{t+1}}{p_t} &= \frac{\partial \tilde{W}/\partial C_{t+1}}{\partial \tilde{W}/\partial C_t} \\
&= \left(\frac{\mathscr{P}_{t+1}}{\mathscr{P}_t}\right)^a \frac{(1+\delta)^t}{(1+\delta)^{t+1}} \frac{\partial U(C_{t+1})/\partial C_{t+1}}{\partial U(C_t)/\partial C_t} \\
&= \frac{(1+n_t)^a}{1+\delta}\left(\frac{C_t}{C_{t+1}}\right)^a \qquad (7)
\end{aligned}
$$

The interest rate in period t, r_t is defined by the ratio of the present value prices:

$$r_t = p_t/p_{t+1} - 1$$

Likewise the consumption growth rate in year t is defined as:

$$g_t = C_{t+1}/C_t - 1$$

The first order condition (7) then reduces to:

$$\left(\frac{(1+g_t)}{(1+n_t)}\right)^a = \frac{1+r_t}{1+\delta} \qquad (8)$$

Nordhaus reconciles population growth, the discount rate and the base year interest rate through calibration of the intertemporal elasticity of substitution, thus accounting for growth in per-capita consumption. The economy is initially on a non-stationary growth path as population growth, n_t, declines and per-capita income rises. Equation (8) can be applied to calibrate the elasticity of the marginal utility of consumption to base year statistics: the interest rate, consumption growth rate and population growth rate, i.e.[3]

$$a = \log\left(\frac{1+\delta}{1+r_0}\right) / \log\left(\frac{1+n_0}{1+g_0}\right) \qquad (9)$$

The resulting demand system can be interpreted as describing either how interest rates must evolve to assure that per-capita consumption growth follows per-capita GDP, or how consumption growth departs from GDP should interest rates remain unchanged.

The textbook version of the intertemporal consumer model is based on only three parameters: the population growth rate n_t, the discount rate, δ, and the elasticity of the marginal utility of consumption, a. The empirical application of the model faces challenges, particularly in cases where the growth rate of GDP departs from the rate of population growth. In these cases, optimal intertemporal choice may induce high rates of lending or borrowing, depending on the extent to which the discount rate departs from the interest rate (the marginal rate of economic substitution). Faced with the challenge of reconciling the Ramsey model with empirical regularities (e.g., relatively low rates of international capital flows), Manne and Rutherford [1994] proposed an alternative approach to

[3] For x close to zero, $\log(1+x) \approx x$, hence:

$$a \approx \frac{r_0 - \delta}{g_0 - n_0}$$

This is a positive value when $r_0 > \delta$ and $g_0 > n_0$.

multiregional growth models with demographic transition. They modify the Ramsey maximand to include *non-stationary* discounting. Intertemporal welfare is then:

$$W = \sum_t \frac{\mathscr{P}_t}{\Delta_t} \; U(c_t) \tag{10}$$

where the discount factor to year t is:

$$\Delta_t = \prod_{\tau=0}^{t-1} (1+\delta_t).$$

The values of δ_t can then be selected to approximate an arbitrary time path of per-capita consumption and interest rates. Solving for δ in (8) with $g_t = \gamma_t$ (consumption grows with GDP), we have:

$$\delta_t = (1+r_0)\left(\frac{1+n_t}{1+\gamma_t}\right)^a - 1 \tag{11}$$

Optimal growth in per-capita consumption then matches growth in per-capita GDP along the baseline growth path with a constant interest rate, r_0.[4] This approach offers considerable flexibility, as the resulting model can be calibrated to an arbitrary growth path in per-capita consumption.

4 A Single Consumer Model

As a concrete illustration of the issues, consider an economy in which a base year interest rate is 7% per annum, the base year GDP growth rate is 8% and the base year population growth rate is 2%. Following Nordhaus, we assume that the pure rate of social time preference is fixed at 1% per annum, and use (9) to calibrate the elasticity of the marginal utility of consumption to the growth rate of per-capita consumption. This implies that $a \approx 2$, coincidentally consistent with independent empirical estimates.

The economy is assumed to follow a non-stationary growth path in which population growth rates decline from 2% to 0% and GDP growth rates declines from 8% to 1%. These model *inputs* are displayed in Figure 3(a).

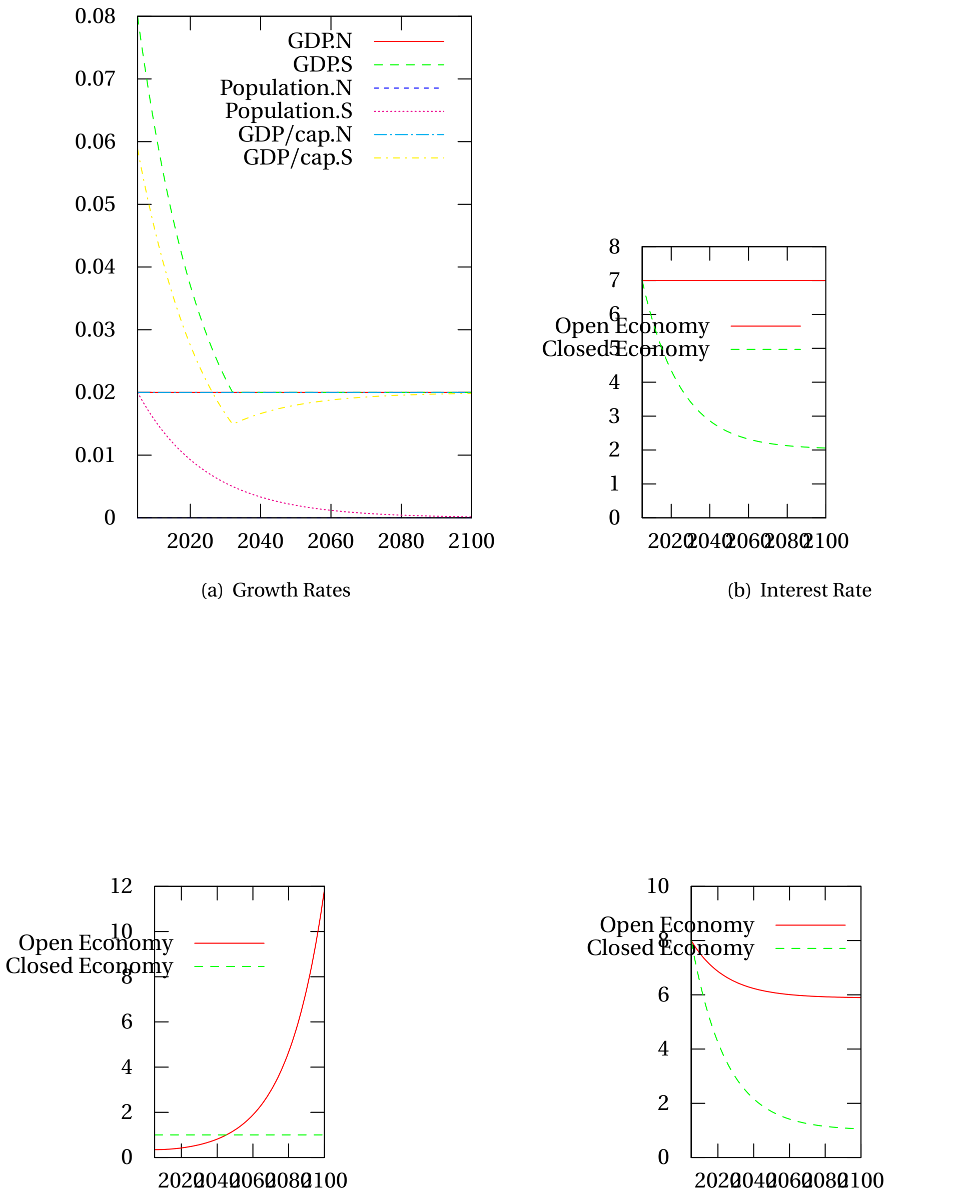
The macroeconomic consequences of these changes in population and per-capita GDP depend crucially on whether we assume that the economy is open or closed. If this is a small open economy, the interest rate is exogenous, i.e. $r = r_0$, and the consumption growth path moves independently of GDP, and the economy exploits international credit markets to move purchases from the present into the future.

If we model a closed economy, then the intereste rate is endogenous, and the consumption growth more or less follows potential GDP growth. If consumption is forced to follow GDP exactly, then based on (7) with $g_t = \gamma_t$), we find that the domestic interest rate declines over time:

$$r_t^* = (1+\delta)\left(\frac{1+\gamma_t}{1+n_t}\right)^a - 1$$

The interest rate assumptions for the open and closed economy are displayed in Figure 3(b). When $r = r_0$, the interest rate remains high and future consumption becomes relatively cheaper than current consumption. As a consequence, current consumption is reduced to finance future consumption, as illustrated in Figures 3(c) and 3(d).

[4] If we replace r_0 by r_t in (11), a baseline growth can we constructed with an exogenous exogenous time profile in the market interest rate.



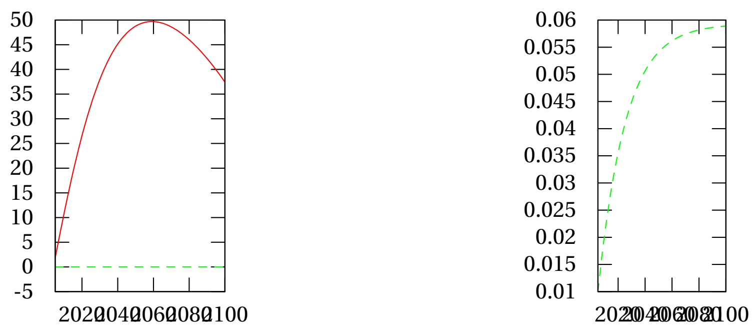
(a) Growth Rates

(b) Interest Rate

(c) Consumption/GDP Index

(d) Consumption Growth Rates

Figure 3: Constant Discount Rate: Open and Closed Economies



(a) Asset Balance / Current Consumption

(b) GDP-Consistent Discount Rate

Figure 4: Constant Discount Rate: Open and Closed Economies

Capital flows in this model are calculated on the basis of the present value of income less consumption:

$$A_{t+1} = A_t + p_t(\Gamma_t - C_t)$$

The asset profile for the case of a constant discount rate, measured as a multiple of current consumption, is illustrated in Figure 4(a). When the discount rate is 1% while the international interest rate is 7%, the optimal accumulation of international debt reaches a level of 50 times current consumption in year 2060.

5 A Multi-Consumer Model

The previous section focused on the extreme cases of closed and small open economies. In the first case, consumption growth follows GDP growth and interest rates adjust accordingly. In the second case the interest rate is fixed by the international credit markets and consumption moves independent of GDP, depending on variations in the per-capita GDP and discounting.

When we move to a model with two consumers, we have a intermediate case. Interest rates are determined through interaction of growth and discounting profiles of the economic agents. We can illustrate the key idea in a single commodity Ramsey model with two consumers. Unlike the previous section which focused solely on demand, this section considers a fully specified general equilibrium model. Output in year t is determined by capital and labor:

$$Y_t = \phi L_t^{\alpha} K_t^{1-\alpha}$$

in which α corresponds to the capital value share and ϕ is a scale factor.

Output in year t is allocated to investment and consumption:

$$Y_t = I_t + \sum_h C_{ht}$$

where C_{ht} is consumption by agent h in year t.

Capital endowments in period $t = 0$ are exogenous:

$$K_0 = \sum_h \bar{k}_h,$$

and subsequent capital stocks are determined by the decisions of profit-maximizing investors. Period t invesment (I_t) provides capital in period $t+1$:

$$K_{t+1} = \lambda K_t + I_t$$

where λ represents the capital survival share (one minus the depreciation rate).

Finally, labor demand in each period equals labor supply with full employment and inelastic supplies:

$$L_t = \sum_h \ell_{ht}$$

Note to self: Need to present a table of data on which the model calibration is based.

Consider then an instance of this model in which we have two consumers, North and South. We specify that the North agent has uniform GDP growth of 2% over the infinite horizon with zero population growth. South has an base year GDP growth rate of 8% and population growth rate of 2%. These both decline to long-run steady-state over the next fifty years, following the time path presented in Figure 3(a).

In all models variants we calibrate the base year interest rate to 7%. Unlike the single consumer case examined in the previous section, in this model the time path of interest rates is determined through the interaction of competing claims for current and future output.

The transition path in the reference equilibrium here depends on two factors: discount rates and relative size of the two consumers. First and foremost, the baseline growth path depends on whether the calibration procedure follows Nordhaus [2007] or Manne and Rutherford [1994]. In Figure 5, results displayed on the left are based on a model in which the elasticity of the marginal utility of consumption is calibrated to the base year interest rate. Results on the right are based on a model with non-stationary discount rate calculated with (11), assuming that the reference interest rate is equal to the base year value.

When the model is calibrated using (9), holding the long-run discount rate equal to 1%, the relative size of the North and South agents have a substantial impact on the resulting growth path. When South is small relative to North (in scenario 02, South is 2% of base year GDP), the model "mimics" the open economy growth path described above. In this case, the interest rate is stable and South consumption increases more rapidly than GDP. When South is large relative to North (scenario 64, South is 64% of base year GDP), the model resembles the closed economy case, as consumption follows GDP closely, the interest rate falls, and the consumption growth rate falls.

In contrast, when intertemporal preferences are calibrated to GDP growth rates in both regions and a constant interest rate, the interest rates remain relatively stable (5(d)), consumption levels follow GDP (5(b)) and consumption growth rates closely follow GDP, resulting in only small levels of lending and borrowing.

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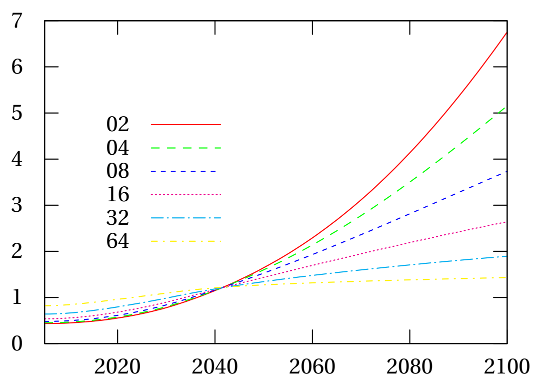
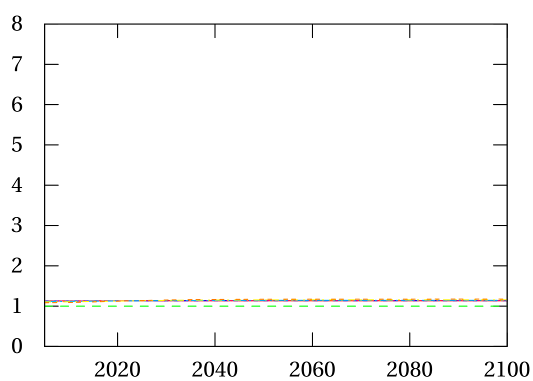
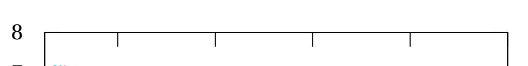
Calibrated a, $\delta = 1\%$ **Calibrated δ_t, a=0.5**



(a) Index of South C/GDP – Constant Discounting



(b) Index of South C/GDP: Variable Discounting



Appendix A: Basic Growth (Figure 1)

```
$title   Calibration Strategies Illustrated

*       A growth model calibrated to a base year interest
*       and population growth is used to simulate the transition
*       to a long-run steady-state.  Population and GDP per capita
*       both evolve gradually from a non-stationary path.
*
*       The calibration procedure can either take the intertemporal
*       discount rate as given and calibrate the intertemporal
*       elasticity to match the base year interest and growth
*       rates, or it can take the intertemporal elasticity of
*       substitution as given and calibrate the discount rate
*       to match the transition path.  Alternative calibration
*       strategies produce significant differences in the
*       marginal product of capital through the transition.

$if not set emuc  $set emuc 1
$if not set prstp $set prstp 0.01

$if not set g0    $set g0 0.08
$if not set n0    $set n0 0.02

*       Transition speed to the terminal growth path:

$if not set speed $set speed 0.05

$set horizon 2150

set     t               Time periods /2005*%horizon%/,
        tp(t)           Time periods to plot /2005*2100/
        tl(*)           /2020,2040,2060,2080,2100/
        tlast(t)        Last time period /%horizon%/
        tfirst(t)       First time period /2005/;

$setglobal domain tp
$setglobal labels tl

parameter

        emuc    Elasticity of the marginal utility of consumption /%emuc%/
        prstp   Pure rate social time preference /%prstp%/
        sigmaT  Intertemporal elasticity
        r       Baseline interest rate / 0.07/
        g0      Base year GDP growth rate /%g0%/,
        n0      Base year population growth rate /%n0%/,
        gterm   Terminal GDP growth rate /0.01/,
        nterm   Terminal population growth rate /0 /,
        speed   Speed of convergence to steady state path  /%speed%/,

        g(t)    GDP growth rate at time t
        n(t)    Population growth rate at time t
        L(t)    Labor supply (efficiency units)
        P(t)    Population
        kinit   Initial capital stock
        cinit   Initial consumption level /1/
        kterm   Terminal capital stock
        gamma   Reference growth path
        beta    Discount factor
        dfactor Discount factor
        pvprice Present value price index,
```

```
        pvw(t)  Present value of wealth;


*       Assume that the growth rate starts at g0
*       and falls to 1% over time:

g(t) = g0; n(t) = n0;
loop(t,
        g(t+1) = (1-speed) * g(t) + speed * gterm;
        n(t+1) = (1-speed) * n(t) + speed * nterm;
);
L(t) = 1; P(t) = 1;
loop(t,
        L(t+1) = (1 + g(t)) * L(t);
        P(t+1) = (1 + n(t)) * P(t);
);

pvprice(t) = power(1/(1+r), ord(t)-1);
dfactor(t) = power(1/(1+prstp),ord(t)-1);
gamma(t) = 1;
beta(t) = 10 * pvprice(t) * L(t);

sigmaT = log( (1+n0)/(1+g0) ) / log( (1+prstp)/(1+r) );

VARIABLES
        UTILITY         Maximand;

VARIABLES
        C(t)            Consumption trillion US dollars,
        A(t)            Asset level (present value);

EQUATIONS
        UTIL            Objective function,
        BUDGET          Intertemporal budget constraint (consumer model)
        ASSETS          Asset profile;

UTIL..          UTILITY =E= SUM(t, beta(t) * (
        ( LOG(C(t)/gamma(t)) )$(sigmaT=1) +
        ( (  (C(t)/gamma(t))**(1-1/sigmaT)-1)/(1-1/sigmaT))$(sigmaT<>1)));

budget..        sum(t, pvprice(t)*C(t)) =e= sum(t,pvw(t));

assets(t)..     A(t) =e= A(t-1) + pvw(t) - pvprice(t)*C(t);

*       Marginal utility:

*       mu(t) = beta(t) * C(t)**(-1/sigmaT) (1/gamma(t))

C.L(t) = 1;  C.LO(t) = 0.01;

parameter       consum  Consumption in successive iterations,
                cpath   Aggregate consumption,
                grate   Growth rate of aggregate consumption,
                irate   Interest rate (marginal product of capital)
                alevl   Asset level;

model consumer / util, budget, assets /;

*       Calibrate marginal rate of technical substitution
*       to a value consistent with the base year interest
*       rate:
```

```
sigmaT = log( (1+n0)/(1+g0) ) / log( (1+prstp)/(1+r) );
emuc = 1/sigmaT;
display emuc;

*       Assign a price profile which is consistent with th
*       exogenously specified growth rate of GDP:

beta(t) = 10 * dfactor(t) * P(t);
gamma(t) = P(t);
loop(t,
    pvprice(t+1) = pvprice(t) * (beta(t+1)/beta(t)) * (gamma(t)/gamma(t+1)) *
                ( gamma(t+1)/gamma(t) * L(t)/L(t+1) )**(1/sigmaT);
);
pvw(t) = pvprice(t)*cinit*L(t);
solve consumer maximizing UTILITY using NLP;

parameter       mu(t)   Marginal utility;

mu(t) = beta(t)*(C.L(t)/gamma(t))**(-1/sigmaT)/gamma(t);

irate(t,"Closed Economy")$mu(t+1) = 100 * (mu(t)/mu(t+1) - 1);
grate(t,"Closed Economy") = 100 * (C.L(t+1)/C.L(t)-1);
cpath(t,"Closed Economy") = C.L(t)/(cinit*L(t));
alevl(t,"Closed Economy") = A.L(t)/(pvprice(t)*C.L(t));

*       Next, compute the consumption path if the
*       interest rate remains constant:

pvprice(t) = power(1/(1+r), ord(t)-1);
pvw(t) = pvprice(t)*cinit*L(t);
solve consumer maximizing UTILITY using NLP;

mu(t) = beta(t)*(C.L(t)/gamma(t))**(-1/sigmaT)/gamma(t);

irate(t,"Open Economy")$mu(t+1) = 100 * (mu(t)/mu(t+1) - 1);
grate(t,"Open Economy") = 100 * (C.L(t+1)/C.L(t)-1);
cpath(t,"Open Economy") = C.L(t)/(cinit*L(t));
alevl(t,"Open Economy") = A.L(t)/(pvprice(t)*C.L(t));

parameter       grates  Growth rates;
grates(t,"GDP") = 100 * g(t);
grates(t,"Population") = 100 * n(t);
```

Appendix B: Two-Agent Growth (Figure 2)

```
$title

*       A growth model calibrated to a base year interest
*       and population growth is used to simulate the transition
*       to a long-run steady-state.  Population and GDP per capita
*       both evolve gradually from a non-stationary path.
*
*       The calibration procedure can either take the intertemporal
*       discount rate as given and calibrate the intertemporal
*       elasticity to match the base year interest and growth
*       rates, or it can take the intertemporal elasticity of
*       substitution as given and calibrate the discount rate
*       to match the transition path.  Alternative calibration
*       strategies produce significant differences in the
*       marginal product of capital through the transition.

$if not set emuc  $set emuc 1
$if not set prstp $set prstp 0.01
$if not set sgdp  $set sgdp 0.2

$eval ngdp 1-%sgdp%

$if not set g0 $set g0 0.08
$if not set n0 $set n0 0.02

*       Transition speed to the terminal growth path:
$if not set speed $set speed 0.05

$set horizon 2120

set     t               Time periods /2005*%horizon%/,
        tlast(t)        Last time period /%horizon%/
        tfirst(t)       First time period /2005/,
        a               Agents /N, S/;

parameter
        prstp           Pure rate social time preference /%prstp%/
        kvs             Capital value share /0.3/
        delta           Capital depreciation rate /0.07/
        r               Baseline interest rate / 0.07/

        emuc(a)         Elasticity of the marginal utility of consumption
                        /n 0.5, s %emuc%/
        sigmaT(a)       Intertemporal elasticity
        y0(a)           Base year GDP index / N %ngdp%, S %sgdp%/
        g0(a)           Base year GDP growth rate /N 0.02, S %g0%/,
        n0(a)           Base year population growth rate /N 0, S %n0%/,
        gterm           Terminal GDP growth rate /0.02/,
        nterm           Terminal population growth rate /0 /,
        c0(a)           Base year consumption level
        w0(a)           Present value consumption
        pl0             Base year wage rate
        speed           Speed of convergence to steady state path  /%speed%/,

        g(t,a)          GDP growth rate at time t
        n(t,a)          Population growth rate at time t
        L(t,a)          Labor supply (efficiency units)
        P(t,a)          Population
        ke0(a)          Initial capital endowment
```

```
        phi(a)          Production scale faster
        k0(a)           Initial capital stock
        gamma           Reference growth path
        beta            Discount factor
        dfactor         Discount factor
        pvprice         Present value price index,
        pvw             Present value of wealth;

$if set bmkchk g0(a) = gterm; n0(a) = nterm;

*       Assume that the growth rate starts at g0
*       and falls to 1% over time:

g(t,a) = g0(a);  n(t,a) = n0(a);
loop(t,
        g(t+1,a) = max((1-speed) * g(t,a), gterm);
        n(t+1,a) = max((1-speed) * n(t,a), nterm);
);

parameter       grates  Growth rates;
grates(t,"GDP",a) = g(t,a);
grates(t,"Population",a) = n(t,a);

L(t,a) = 1; P(t,a) = 1;
loop(t,
        L(t+1,a) = (1 + g(t,a)) * L(t,a);
        P(t+1,a) = (1 + n(t,a)) * P(t,a);
        grates(t,"GDP/cap",a)$P(t+1,a) = ((L(t+1,a)/P(t+1,a))/(L(t,a)/P(t,a))-1);
);

k0(a) = y0(a) * kvs / (r + delta);
c0(a) = y0(a) - k0(a) * (g0(a)+delta);
pvprice(t) = power(1/(1+r), ord(t)-1);
dfactor(t) = power(1/(1+prstp),ord(t)-1);
phi(a) = y0(a) / (kvs / (r + delta))**kvs;
pl0 = (1-kvs);
gamma(t,a) = 1;

set     mdl     Models corresponding to different /"0.5","1","2", dice/;

parameter       invest  Investment in successive iterations,
                consum  Consumption in successive iterations,
                pivotdata       Pivot report data
                refval  Reference growth paths
                cpath   Aggregate consumption,
                ipath   Investment
                ypath   Output
                grate   Growth rate of aggregate consumption,
                irate   Interest rate (marginal product of capital);

refval("GDP",t,a,"rate")  = L(t+1,a)/L(t,a)-1;
refval("C",  t,a,"level") = L(t,a)*c0(a);

$ontext
$model:ramseymge

$sectors:
        C(t,a)  ! Consumption
        Y(t)    ! Production
        K(t)    ! Capital
        I(t)    ! Investment
```

```
$commodities:
        PC(t,a) ! Shadow price of consumption
        PY(t)   ! Price of output
        PK(t)   ! Purchase price of capital
        RK(t)   ! Rental price of capital
        PL(t)   ! Wage rate
        PKT     ! Terminal capital stock

$consumer:
        RA(a)

$auxiliary:
        KT      ! Terminal capital stock

$prod:Y(t)  s:1
        O:PY(t)         Q:(sum(a,y0(a)))
        I:PL(t)         Q:(sum(a,y0(a))) P:pl0
        I:RK(t)         Q:(sum(a,k0(a))) P:(r+delta)

$prod:K(t)
        O:PK(t+1)       Q:(1-delta)
        O:PKT$tlast(t)  Q:(1-delta)
        O:RK(t)         Q:1
        I:PK(t)         Q:1

$prod:I(t)
        O:PK(t+1)       Q:1
        O:PKT$tlast(t)  Q:1
        I:PY(t)         Q:1

$prod:C(t,a)
        O:PC(t,a)
        I:PY(t)

$demand:RA(a)  s:sigmaT(a)
        D:PC(t,a)       Q:gamma(t,a)    P:(beta(t,a)/gamma(t,a))
        E:PL(t)         Q:(y0(a)*L(t,a))
        E:PK(tfirst)    Q:k0(a)
        E:PKT           Q:(-k0(a))              R:KT

$constraint:KT
        sum(tlast(t),   K(t)/K(t-1) - (1+gterm)) =e= 0;


$offtext
$sysinclude mpsgeset ramseymge

Y.L(t) = L(t,"N");
K.L(t) = sum(a,k0(a))*L(t,"N");
I.L(t) = sum(a,k0(a)*(g0(a)+delta))*L(t,"N");

PY.L(t) = pvprice(t);
PK.L(t) = (1+r)*pvprice(t);
PKT.L = sum(tlast,PY.L(tlast));
RK.L(t) =  (r+delta) * pvprice(t);
PL.L(t) = pvprice(t)*pl0;

KT.L = power(1+g0("N"),card(t));

loop(mdl,
```

```
  if (sameas(mdl,"DICE"),
    sigmaT(a) = log( (1+n0(a))/(1+g0(a)) ) / log( (1+prstp)/(1+r) );
    beta(t,a) = 10 * dfactor(t) * P(t,a);
    gamma(t,a) = P(t,a);
  else
    sigmaT(a) = mdl.val;
    gamma(t,a) = c0(a) * L(t,a);
    beta(t,a) = pvprice(t)*gamma(t,a);
  );

$include RAMSEYMGE.GEN
  solve ramseymge using mcp;

  pivotdata("path_bau",t,a,mdl)   = C.L(t,a);
  pivotdata("path_bau",t,"Y",mdl) = Y.L(t);
  pivotdata("path_bau",t,"K",mdl) = K.L(t);
  pivotdata("path_bau",t,"I",mdl) = I.L(t);
  pivotdata("grate_bau",t,a,mdl) = C.L(t+1,a)/C.l(t,a)-1;
  pivotdata("irate_bau",t,a,mdl)$PY.L(t+1) = PY.L(t)/PY.L(t+1) - 1;
);

refval("GDP",t,a,"rate")  = L(t+1,a)/L(t,a)-1;
refval("C",  t,a,"level") = L(t,a)*c0(a);
pivotdata("relpath",t,a,mdl) = pivotdata("path_bau",t,a,mdl)/(L(t,a)*c0(a));
pivotdata("path_bau",t,a,"ref") = L(t,a)*c0(a);
pivotdata("irate_bau",t,a,"ref") = r;
```