Exhaustible resources in the long run

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Abstract

We develop a tractable and flexible general equilibrium model which explains and predicts the long-run development of exhaustible resource markets when resource stocks are inhomogeneous, highlighting the relationship between extraction costs, the scarcity rent, and the nature of resource stocks. We apply the model in two cases, copper and petroleum. The extraction process for a typical resource passes through up to three phases, in which price declines slowly, turns upwards, and finally—if substitutes are expensive—rises at the interest rate. The final phase is a long way off for the resources studied.

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1. Introduction

What is the likely long-run evolution of global markets for mineral and energy resources, given physical constraints on supply? To answer this question we need a model in which both the demand for the resource and its supply are modelled. The quantity of the resource supplied at a given time is equal to the rate of extraction (if we ignore above-ground storage), and ideally we would like to model this as a function of the quantity of extraction inputs used, their productivity, and the nature of the deposits from which the resources are extracted; furthermore, the prices of extraction inputs, their productivities, and demand for the resource should all be linked to developments in the overall economy. We develop and test a tractable and flexible model of this type.

Hotelling (1931) is the seminal paper concerning extraction costs, the scarcity rent, and the nature of resource stocks. Hotelling showed that—in the absence of market power—price is the sum of extraction cost and scarcity rent, and that the rent associated with a given unit of resource kept in the ground should rise at the overall rate of interest in the economy. If the resource stock is homogeneous then the rents associated with each unit of the stock must all be equal, hence the rent of the marginal unit of resource must also rise at the interest rate. If there are multiple resource stocks differing in quality then there will be a different shadow price associated with each stock, and the higher quality resources will—ceteris paribus—be extracted first. On the other hand, if there is a single inhomogeneous stock—i.e. a stock in which quality varies continuously—then Solow and Wan (1976) argue that it is convenient to reframe the problem in terms of 'degradation cost' associated with extraction of high-quality resources, where the degradation cost is due to the fact that current extraction implies higher extraction costs at all future times due to the degradation of resource quality. See also Heal (1976).

Subsequent to Solow and Wan (1976) and Heal (1976) there has been a debate in the theoretical literature concerning the paths of scarcity rent[1] and resource price when stocks are inhomogeneous, with different results emerging under different assumptions. Thus Levhari

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[1] We use the term 'scarcity rent' throughout, even in the case of inhomogeneous stocks.

and Liviatan (1977) and Hanson (1980) show declining rent, Krulce (1993) derives sufficient conditions for increasing rent, and Farzin (1992) argues that rent may also follow a non-monotonic path. Livernois and Martin (2001) show that the results depend crucially on the nature of the instantaneous net benefit function $\pi(x_t, n_t)$ where x is the extraction rate and n is cumulative extraction.[2] For 'clean' results—in which the scarcity rent is always increasing, and approaches the interest rate—Livernois and Martin show that the function should be jointly concave in x and n. But in practice the shape of the function will depend on the nature of stocks, and there exist highly plausible stock characteristics—such as lower-grade resources being more abundant than higher-grade resources—which imply that the function will not be concave. So we are left without much general help regarding the relationship between the inhomogeneity of stocks and the expected paths of price and scarcity rent.

The inability to make concrete predictions can be attributed to a reluctance to impose structure on the models. Consider for instance extraction. The norm in the papers above is to state a function for total extraction costs C, such as $C(x,n)$ where x is the rate of extraction and n is cumulative extraction (Levhari and Liviatan, 1977), or $C(x,b)$ where b is the depth of the marginal resource (Cairns and Quyen, 1998), or $C(x,n,z)$ where z is technology (Farzin, 1992). If the properties of this function are not specified then it is hardly surprising that a range of different results is possible. This point is related to that made by Livernois and Martin (2001) [p.840] who state that their findings can easily be reconciled with 'any kind of behaviour for scarcity rent and price over time', simply by introducing exogenous trends or shocks in variables such as extraction productivity.

A natural way to impose structure is to put the extraction sector into a general equilibrium context. This is done, for instance, in the DHSS[3] model, which is based on the neoclassical growth model with a resource input added to the production function. In the baseline version the production function is Cobb–Douglas, and the resource is extracted from a known non-renewable stock at zero cost. An immediate consequence of the Cobb–Douglas is that the resource factor share is constant, which fits remarkably well with observations, as shown in Figure 1. On the other hand, the assumption of zero extraction costs leads directly to the prediction that resource price should rise at the interest rate while the consumption rate declines exponentially. This is totally contrary to the evidence, which is that prices are remarkably constant while consumption rates tend to track long-run GDP (see Figure 1 again). Returning to the assumptions of the baseline model, the DHSS literature has frequently focused on the theoretical problem of maintaining consumption under conditions of zero technological progress and zero capital depreciation, a set of assumptions which is contrary to the accepted idea that key to long-run economic development is technological change rather than capital accumulation.

The DHSS model exemplifies a general equilibrium framework without extraction costs. If extraction costs are to be included in such a framework it must be via the inclusion of an extraction function rather than an extraction *cost* function. The extraction function is an equation for the rate of extraction as a function of the use of inputs, their productivities, and the nature of the stock from which resources are to be extracted. Having written down such a function it is immediately obvious—as Livernois (2009) points out—that unit extraction costs will only decline if the productivity of extraction inputs rises faster than their price. It follows that technological progress will not automatically drive down resource prices, no more than it drives down the prices of other goods in the economy on average; technological progress will drive up output in the extraction sector for given inputs, but if the productivities of the inputs also rise in the other sectors of the economy then input prices will rise at the same rate as productivity, and unit costs will be unchanged; unit costs will only be driven down if technological progress is faster in the extraction sector than it is in the overall economy.

Another reason to put the resource sector into a general equilibrium framework is demand.

[2] Note that Livernois and Martin actually express the model in terms of remaining stock, but concavity of π is not affected by the switch.

[3] DHSS: Dasgupta–Heal–Solow–Stiglitz, named after the seminal papers published in 1974: Dasgupta and Heal (1974), Solow (1974), Stiglitz (1974); see also the classic paper of Hartwick (1977), and more recent extensions to the model such as Grimaud and Rouge (2003) and Groth and Schou (2007).

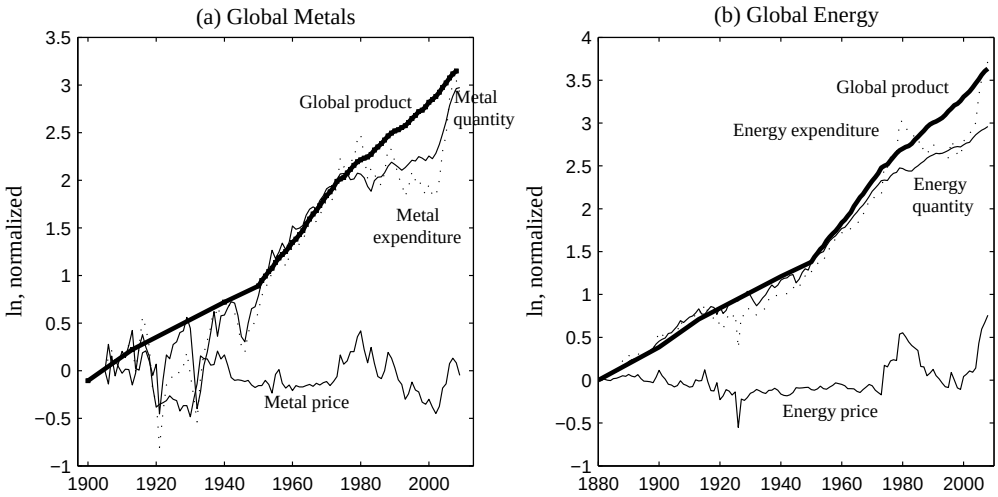

Figure 1: Long-run growth in consumption and prices, compared to growth in global product, for (a) Metals, and (b) Primary energy from combustion.

Note: Global product data from Maddison (2010). Metals: Al, Cr, Cu, Au, Fe, Pb, Mg, Mn, Hg, Ni, Pt, Rare earths, Ag, Sn, W, Zn. All metals data from Kelly and Matos (2012). Energy: Coal, oil, natural gas, and biofuel. Fossil quantity data from Boden et al. (2012). Oil price data from BP (2012). Coal and gas price data from Fouquet (2011); note that these data are only for average prices in England; we make the (heroic) assumption that weighted average global prices are similar. Biofuel quantity data from Maddison (2003). Biofuel price data from Fouquet (2011); again, we assume that the data are representative for global prices, and we extrapolate from the end of Fouquet's series to the present assuming constant prices. The older price data is gathered from historical records and is not constructed based on assumptions about elasticities of demand or similar. Sensitivity analysis shows that the assumptions are not critical in driving the results.

Papers in the literature on extraction costs, scarcity rent, and prices—such as those cited above—typically either assume a constant demand function, or assume nothing at all about demand, while the empirical evidence (e.g. Figure 1) shows that the prices of exhaustible resources have typically remained constant, while quantities have risen very rapidly (in line with global product) over the last century and more. Thus the demand function for resources is clearly shifting upwards over time. If we are interested in long-run price trends, and extraction costs rise in cumulative extraction, then it is clearly relevant to know long-run demand trends; without specifying these trends in some way it will be hard to explain or predict prices.

We develop a general equilibrium model of the global economy with very simple models of resource demand and directed technological change—models which are broadly in line with historical data and our understanding of the key processes—and focus our attention on the extraction cost function. Resource owners hire labour inputs in order to extract the resource, and the rate of extraction per unit of labour depends on the productivity of the input, plus the depth of the marginal resource. The latter is a measure of the physical inaccessibility of the marginal resource, an increasing function of cumulative extraction. The alternative employment for labour is in final-good production; there is thus an arbitrage condition between the sectors. Labour productivity in the two sectors grows exogenously at rates that may differ. We specify a rather flexible functional form linking depth to cumulative extraction, and show that given this specification the dynamic equilibrium path can be solved in closed form. Economic reasoning then leads to the conclusion that the optimal dynamic path for a typical resource will start with a very low extraction rate and zero scarcity rent, and that the resource price will decline at a rate equal to the difference between the growth rates of labour productivity in extraction and final-good production.[4] Over time, however, the extraction rate increases and in the baseline case the economy approaches a new balanced growth path (b.g.p.) along which the decline in resource price is moderated by increasing depth, and the result may be constant or slowly rising prices; the characteristics of the resource deposits are a key factor affecting the growth rates along this 'mature' b.g.p. Extraction continues to

[4]This result presupposes that the factor share of the resource is low. For the full result see equation 34.

rise rapidly, and if and when resource exhaustion looms large the rate of price increase may accelerate towards (in the limit) the interest rate, if the price of substitutes to the resource is sufficiently high; at the other extreme, the rent may approach zero if extraction costs approach the backstop price while resources remain in the ground. We show how the model can also encompass more complex resource deposits, and apply it to explaining and predicting the evolution of global markets for copper and petroleum: the case of copper illustrates the power of the model, whereas the case of petroleum also highlights some limitations.

In the case of copper the cross-sectional area of the representative resource deposit increases with depth initially, and then declines. When extraction is from the upper part of the stock (up to approximately 2050) price declines slowly and the scarcity rent makes up only a very small part of the price, but once extraction moves to the deeper stocks for which cross-sectional area declines with depth, the price starts to rise and the scarcity rent moves towards 40 percent of the price. In the case of petroleum stocks are funnel-shaped. Again, the scarcity rent is initially low, but once extraction moves onto the 'pipe' part of the funnel (around 2030) the scarcity rent is around 20 percent of the price. Exhaustion occurs around 2125, and the price path in the final decades depends almost entirely on the price of the backstop.

The paper in the literature closest to ours in terms of the optimization problem faced by the representative resource owner is Farzin (1992). However, in comparison to Farzin we impose structure on the nature of resource stocks, and insert the extraction problem into a general equilibrium context. This allows us to solve the problem in closed form, and apply the model to the explanation and prediction of the evolution of global resource markets. On the other hand, there are also papers close to this one in the sense that they assess the nature of global resource stock, linking these assessments to heuristic predictions of future price trends. However, in these papers there is no economic model of demand or scarcity rent, hence no quantitative prediction of price or scarcity rent over time. Recent studies of this type include Rogner (1997) assess global hydrocarbon resources (cumulative stock) as a function of the predicted extraction cost per barrel of oil equivalent at the time of extraction, and Mercure and Salas (2012) perform a similar operation plotting resources against current extraction cost. A famous study is that of Goeller and Weinberg (1976), who consider the chemical composition of 'demandite' and 'avalloy', which are weighted averages of non-renewable resources and metal alloys respectively. They find—with the exception of fossil fuels and phosphorus—that the major components of demandite and avalloy are effectively inexhaustible, and that there is a great deal of substitutability between alternative elements in most cases. Nevertheless they predict that 'present patterns of use will persist for the next 30 to 50 years' (implying a change anytime now given the date of publication), followed by a transition phase which might last several hundred years, and finishing in the very long run in the 'Age of Substitutability', a steady state based on abundant materials and recycling. An extended version of our model would allow these conclusions—which are more-or-less pulled out of a hat by Goeller and Weinberg—to be probed more deeply.

Our main focus is the prediction of the development of resource prices over time; however, the size of the scarcity rent itself is also important for a number of reasons, of which we highlight two (see Hart and Spiro, 2011, for further discussion). Firstly, any policy measure which reduces the scarcity of a resource will cause a fall in the scarcity rent, reducing the price of the resource and raising the extraction rate. The famous result that a constant *ad valorem* tax has no effect on the extraction path of Hotelling resource[5] falls into this category. (For the result see Dasgupta and Heal, 1979; for extensions see Ulph and Ulph (1994) and Sinclair (1994).) Secondly, any policy that leads to expectations of lower future demand—such as the announcement of future taxes—will also lower the scarcity rent and hence boost resource consumption in the short run. This is the 'green paradox' of Sinn (2008); see also for instance Di Maria et al. (2012) and Van der Ploeg and Withagen (2012).

The remainder of the paper is organized as follows. In Section 2 we present the model. In Section 3 we solve the infinite-horizon problem in which resource depth is unbounded, and in Section 4 we solve explore solutions to more general types of problem in which there is a bound on depth, or the nature of resource deposits is significantly more complex than the

[5]By 'Hotelling resource' we mean a resource with zero extraction cost, i.e. where price equals scarcity rent.

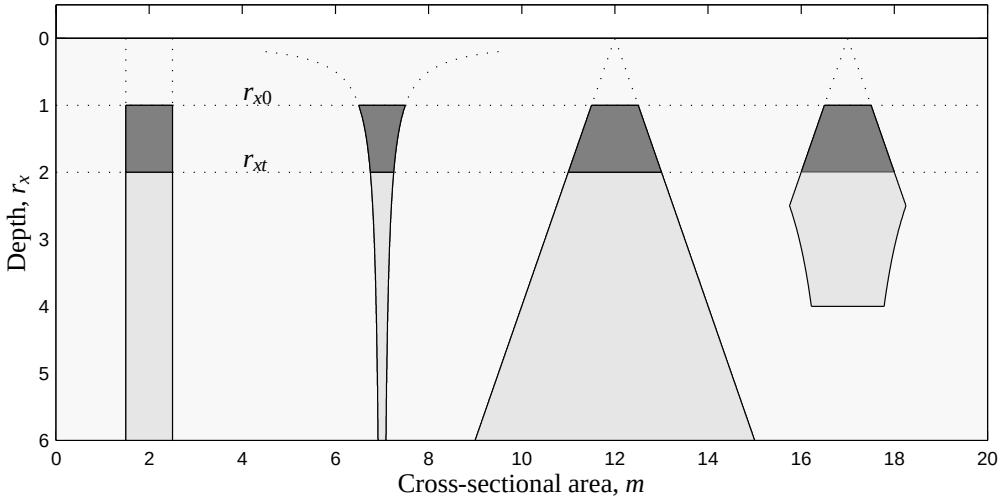

Figure 2: The relationship between depth r_x and cross-sectional area m for four alternative resource stocks. From the left we have (i) $\phi = 1$, (ii) $\phi = 0$, (iii) $\phi = 2$, and (iv) $\phi = 2$ for $r_x < 2.5$, and $\phi = 0$ for $r_x > 2.5$. In each case $r_{x0} = 1$, $r_{xt} = 1$, and $m_0 = 1$; for the second part of stock (iv) we have $r_{x0} = m_0 = 2.5$. The dark shading represents cumulative extraction at time t, n_t, and the lighter shading represents remaining resources at time t. Resource stocks in the first three cases are infinite, but in (iv) there is an upper bound on r_x and the stock is finite.

baseline model allows. In Section 5 we parameterize the model for the cases of copper and petroleum. Section 6 concludes.

2. The model

We begin by explaining the nature of resource stocks in the model. We then move on to the economic model, which hinges on the optimization problem of the representative resource holder.

2.1. Resource stocks

There is a unit continuum of resource owners, each of whom owns identical stocks of the resource; we can thus consider the representative resource owner and her stock. The representative owner's stock is a contiguous mass, underground, which extends in space according to the following function:

$$\frac{m}{m_0} = \left(\frac{r_x}{r_{x0}}\right)^{\phi-1}. \tag{1}$$

Here r_x is depth underground (r for *remoteness*), m is the cross-sectional area of the resource, r_{x0} and m_0 are the depth and surface area at $t = 0$, and ϕ is a parameter which may take any (real) value.

Note that in reality the difficulty of extraction different stocks of the same resource may be a function of several variables, of which the depth of the stock is one. Others may include the grade (i.e. the percentage of the resource by weight in the rock), the size of the deposit (larger deposits will typically have lower unit extraction costs), etc. In Section 5 we give two examples of how r_x can be related to physical variables such as depth or grade in empirical studies. Depth r_x is resource-specific, but also varies along the extraction path since the resource stocks are inhomogeneous.

The function is fairly flexible, as shown in Figure 2, where four different cases are illustrated. In the first three cases we simply choose different values of ϕ, allowing the cross-sectional area of the resource to either be constant, increase, or decrease with depth. In the fourth case we show how a more complex stock may be approximated by building up the total stock piecewise from substocks, and by putting a lower bound on depth.

We denote cumulative extraction at t as n_t, hence

$$n_t = \int_{r_{x0}}^{r_{xt}} m \, \mathrm{d}r_x.$$

Substitute for m from equation 1 to obtain

$$n_t = \frac{m_0 r_{x0}}{\phi} \left[\left(\frac{r_{xt}}{r_{x0}} \right)^{\phi} - 1 \right]. \tag{2}$$

Note that when $\phi \geq 0$ then $n \to \infty$ when $r_x \to \infty$, so if we want the stock n to be bounded we must impose a limit on b beyond which there are no further stocks; on the other hand, when $\phi < 0$ then $r_x \to \infty$ when $n \to -(m_0 r_{x0})/\phi$, hence the total available stock at $t = 0$ is $-(m_0 r_{x0})/\phi$.[6] Finally, note the special case when $\phi = 0$, $r_x/r_{x0} = \exp(n/(m_0 r_{x0}))$.

2.2. The representative resource owner's problem

There is a constant population[7] and utility is a linear function of consumption of the global aggregate good Y:

$$U = \int_0^{\infty} e^{-\rho t} Y_t \mathrm{d}t.$$

Therefore the representative resource owner faces a constant interest rate, and maximizes the discounted sum of net revenue from extracting and selling the resource.

Resources are sold to firms producing final goods, total quantity Y. There is a unit continuum of such firms, which are therefore price takers, and the representative firm's production, denoted y, is a Cobb–Douglas function of labour and resource inputs, as follows:

$$y = a_y l_y^{1-\alpha} q_r^{\alpha}. \tag{3}$$

Here α is a parameter between 0 and 1, a_y is total factor productivity, and l_y and q_r are the respective quantities of labour and resources used by the representative firm in production. We thus abstract from capital, which simplifies the model at little cost in terms of explanatory power since there is perfect foresight and the interest rate is constant; if capital were included its growth rate would be equal to the growth rate in Y, the capital share would be constant, and the model results would not change in essence.

The flow of resource inputs q_r is equal to the flow of resource output x from the representative extraction firm.[8] This firm owns the extraction rights to a representative resource deposit, and the extraction flow x is a function of labour inputs in extraction l_x, the productivity of that labour a_x, and economic depth b_x, which is a constant-elasticity function of depth r_x: $b_x = r_x^{\chi}$. Thus we have

$$q_{rt} = x_t = l_{xt} a_{xt} / b_{xt}. \tag{4}$$

The productivity indices a_y and a_x grow exogenously, and total labour L is fixed.

$$\begin{aligned} \dot{a}_y / a_y &= \theta_{ay} \\ \dot{a}_x / a_x &= \theta_{ax} \\ L &= l_x + l_y. \end{aligned} \tag{5}$$

[6] To give some perspective on the relevance of resource limits consider the following calculation. The current physical extraction rate of minerals is of the order of 10^{10} tonnes per year globally. Assume that this rate continues to grow at the same rate as it has done over the past 100 years, i.e. approximately 3 percent per year. Then extraction would be multiplied by a factor 20 each century, and in 700 years we would be mining and using minerals roughly equal to the entire earth's crust *every year*, based on a figure of 2×10^{19} tonnes for the Earth's crust. Extraction data from Kelly and Matos (2012).

[7] This assumption can easily be relaxed, and we do so in the empirical application.

[8] The same as the representative resource owner. Since there are constant returns to scale in extraction the size of firms makes no difference, as long as they are competitive.

Now take equation 2 of the previous section, and use the following definitions—

$$b_x = r_x^{\chi}, \quad \psi = \phi/\chi, \quad F_0 = m_0 r_{x0}/\chi \tag{6}$$

—to obtain the following relationship between economic depth b_x and cumulative extraction. Furthermore, equation 8 follows by definition.

$$\frac{b_{xt}}{b_{x0}} = \left(1 + \frac{\psi n_t}{F_0}\right)^{1/\psi}; \tag{7}$$

$$\dot{n} = x. \tag{8}$$

We now return to the representative resource owner, whose problem is to maximize discounted net revenue subject to market clearing and the constant exogenous growth rates of a_y and a_x. Define p_x as the price of extracted resource, and p_l as the wage, then take first-order conditions on 3, and finally eliminate l_y using 5 to show that

$$p_l = (1-\alpha)y/(L - l_x), \tag{9}$$

$$p_x = \alpha y/x, \tag{10}$$

where

$$y = a_y x^{\alpha}(L - l_x)^{1-\alpha} \tag{11}$$

and

$$l_x = x b_x / a_x. \tag{12}$$

So—since a_x and a_y are exogenously given—l_x, y, p_l and p_x are determined by the representative resource owner's choice of x. We are therefore in a position to define equilibrium in the economy as follows.[9]

Definition 1. *The economy is on an equilibrium path when the representative resource holder chooses an extraction path $[x_t]_{t=0}^{\infty}$ in order to maximize the net present value of revenue minus cost,*

$$\pi = \int_0^{\infty} e^{-\rho t}(p_x x - p_l l_x)dt,$$

subject to the evolution of the stock (equations 7 and 8 above) and market clearing (equations 9–12 above).

3. Solving the model in the baseline case

Having defined equilibrium we now set about solving in the most straightforward case, i.e. the infinite-horizon problem in which economic depth b_x is unbounded, unique values of ψ, d_{x0} and m_0 apply to the entire stock, and there is no substitute for the resource. In the next section we tackle more general cases. Our solution strategy in both cases is based on Acemoglu (2009), Theorem 7.14, as follows. First we set up the current-value Hamiltonian with n—cumulative extraction—as the state variable. We then find the (necessary) first-order conditions in x and n, and the transversality condition, and find a solution consistent with these conditions. Then we note that the value of the Hamiltonian along the optimal path (in Acemoglu's notation, $M(t,n,\lambda)$) is simply equal to zero; this follows from the first-order condition in x, because of the linearity of the Hamiltonian in x. Since $M \equiv 0$ it follows that M is concave in n for all t. Then since the set of allowed values for n is convex, and if there are no discontinuities, the solution must achieve the global maximum.

[9] Note that the symmetric equilibrium is optimal. To see this, divide the representative resource owner's resources into two equal parts, and assume that the owner has extracted to a greater depth on one part than on the other. In the presence of discounting this can never be optimal, since costs are borne unnecessarily early along the asymmetric path. Since the resource owners have no market power, what holds for the representative owner must also hold across owners; it can never be optimal for one owner to have extracted to a greater depth than the other owners. That is, they extract symmetrically.

3.1. The Hamiltonian, and first-order conditions in x and n

We set up the representative resource holder's problem as a current-value Hamiltonian in which n is the state variable. To write down the Hamiltonian, start with the profit function in Definition 1, and substitute for l_x using equation 12, for b_x using equation 7, and for $\dot{n}$ using equation 8. Hence

$$\mathcal{H} = \left\{ p_x x - p_l x b_{x0} \left[1 + \psi n / F_0\right]^{1/\psi} / a_x \right\} + \lambda x, \tag{13}$$

where $-\lambda$ is the current value scarcity rent. (Note that λ is negative because it is the shadow price of cumulative extraction.)

The first-order condition in x gives

$$-\lambda = p_x - p_l b_x / a_x, \tag{14}$$

which simply states that the scarcity rent is equal to the difference between price and unit cost; (to see this recall from equation 12 that $b_x/a_x = l_x/x$). The first-order condition in n yields

$$\rho = \frac{\dot{\lambda}}{\lambda} - \frac{p_x}{\lambda} P^* \frac{x}{F_0 (b_x/b_{x0})^{\psi}}, \tag{15}$$

where

$$P^* = p_l l_x / (p_x x) = (p_l/p_x)(b_x/a_x). \tag{16}$$

Now use equation 14 to show that

$$\frac{\dot{\lambda}}{\lambda} = \frac{P^*}{1-P^*} \left[\frac{\dot{p}_x}{p_x} \frac{1}{P^*} + \theta_{ax} - \frac{\dot{p}_l}{p_l} - \frac{\dot{b}_x}{b_x} \right], \tag{17}$$

and equation 7 to show that

$$\frac{\dot{b}_x}{b_x} = \frac{x}{F_0 (b_x/b_{x0})^{\psi}}. \tag{18}$$

Put together 14–18 to eliminate λ and b_x and obtain

$$\dot{p}_x / p_x = (1-P^*)\rho + P^*(\dot{p}_l/p_l - \theta_{ax}), \tag{19}$$

which shows that the growth rate of the resource price is equal to the interest rate $\times$ the ratio of scarcity rent to revenue, plus the growth rate of productivity-adjusted unit labour costs $\times$ the ratio of labour costs to revenue. It follows that if the resource price is pure rent it grows at the interest rate, whereas if it is pure labour cost then it tracks labour costs.

Next use equations 9–11 and the definition of P^* to show that

$$P^* = \frac{1-\alpha}{\alpha} \frac{l_x}{L - l_x}, \tag{20}$$

and to find expressions for the growth rates of p_l and p_x in terms of x, n, and l_x.[10] Use these expressions to eliminate prices from 19 and derive equation 22. Recalling equation 8 we then have the following necessary conditions on the dynamic evolution of cumulative extraction and the extraction rate:

$$\frac{\dot{n}}{n} = \frac{x}{n}, \tag{21}$$

$$\frac{\dot{x}}{x} = \frac{l_x}{\alpha L}\left(1 - \frac{l_x}{L}\right)\theta_{ax} + \left(1 - \frac{l_x}{L}\right)\left(\frac{l_x}{\alpha L} - 1\right)\frac{\rho - \theta_{ay}}{1-\alpha} - \frac{l_x}{L}\left(\frac{x}{F_0 + \psi n} - \theta_{ax}\right). \tag{22}$$

[10] The expressions are

$$\frac{\dot{p}_l}{p_l} = \theta_{ay} + \alpha \left[\frac{\dot{x}}{x} + \frac{l_x}{L - l_x}\left(\frac{\dot{x}}{x} + \frac{x}{F_0 + \psi n} - \theta_{ax} \right) \right],$$

$$\frac{\dot{p}_x}{p_x} = \theta_{ay} - (1-\alpha) \left[\frac{\dot{x}}{x} + \frac{l_x}{L - l_x}\left(\frac{\dot{x}}{x} + \frac{x}{F_0 + \psi n} - \theta_{ax} \right) \right].$$

3.2. The steady state

Before considering the transversality condition we show that there exists a balanced growth path which is consistent with the necessary conditions above (equations 21 and 22), and that this path may be characterized as a steady state in a new coordinate system. We start by choosing the following coordinates:

$$(x_1, x_2) = \left(\frac{a_x/b_x}{(b_x/b_{x0})^{\psi}}, \frac{x}{a_x/b_x} \right).$$

Note that x_1 is a state variable, whereas $x_2 = l_x$ and is thus a control variable. We then have the following proposition.

Proposition 1. *The following equations represent necessary conditions for optimal extraction by the representative resource holder:*

$$\frac{\dot{x}_1}{x_1} = \theta_{ax} - (1+\psi)x_1 x_2/F_0, \tag{23}$$

$$\frac{\dot{x}_2}{x_2} = \left(1 - \frac{x_2}{L}\right)\left[x_1 x_2/F_0 - \left(1 - \frac{x_2}{\alpha L}\right)\left(\theta_{ax} + \frac{\rho - \theta_{ay}}{1-\alpha}\right)\right]. \tag{24}$$

PROOF. To derive equation 23, differentiate x_1 w.r.t. time to yield $\dot{x}_1/x_1 = \theta_{ax} - (1+\psi)\dot{b}_x/b_x$, then substitute for $\dot{b}_x/b_x$ using equation 18, and finally use the definitions of x_1 and x_2. To derive equation 24, differentiate x_2 w.r.t. time to yield $\dot{x}_2/x_2 = \dot{x}/x + \dot{b}_x/b_x - \theta_{ax}$. Then substitute for $\dot{b}_x/b_x$ using equation 18, and note that $x/(b_x/b_{x0})^{\psi} = x_1 x_2$. Finally, substitute for $\dot{x}/x$ using equation 22, use equation 2 and the definitions in equation 6 to show that $F_0 + \psi n = F_0(b_x/b_{x0})^{\psi}$, and use $x/(b_x/b_{x0})^{\psi} = x_1 x_2$ again.

This leads directly to Proposition 2. Given equations 25 and 26 we can construct the phase diagram and characterize the relationship between the initial choice of extraction labour l_x and the subsequent development of the economy: see Figure 3.[11]

Proposition 2. *For allowed parameter values there is a unique internal stable steady state to the system described by equations 23 and 24. The steady state is given by the crossing point of the following equations.*

$$\dot{x}_1 = 0, \qquad x_1 = \frac{1}{x_2}\frac{1}{1+\psi}\theta_{ax}F_0; \tag{25}$$

$$\dot{x}_2 = 0, \qquad x_1 = \frac{1}{x_2}\left(1 - \frac{x_2}{\alpha L}\right)\left(\theta_{ax} + \frac{\rho - \theta_{ay}}{1-\alpha}\right)F_0. \tag{26}$$

The allowed parameter values are such that both of the following conditions are satisfied. Firstly, $\psi > -1$; secondly,

$$\frac{(\rho - \theta_{ay})/\theta_{ax}}{1-\alpha} > \frac{-\psi}{1+\psi}. \tag{27}$$

PROOF. The proof is straightforward, hence we sketch it here. To find internal the steady state, set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ in equations 23 and 24, while ruling out corner solutions with either no extraction or no production. To find the restrictions on parameter values, first verify that—for $\psi > -1$—the locus of $\dot{x}_1 = 0$ is always steeper than the locus of $\dot{x}_2 = 0$ for given x_1, hence if the lines cross they will do so only once. The second condition is necessary and sufficient for the lines to cross. Given that the lines cross it is straightforward to verify that the steady state is stable, for instance with the help of the phase diagram.

[11] Parameter values: $\psi = 1$, $\phi = 1$, $\alpha = 0.05$, $\rho = 0.05$, $\theta_{ax} = 0.023$, $\theta_{ay} = 0.017$, $L = 1$, and $F_0 = 1$.

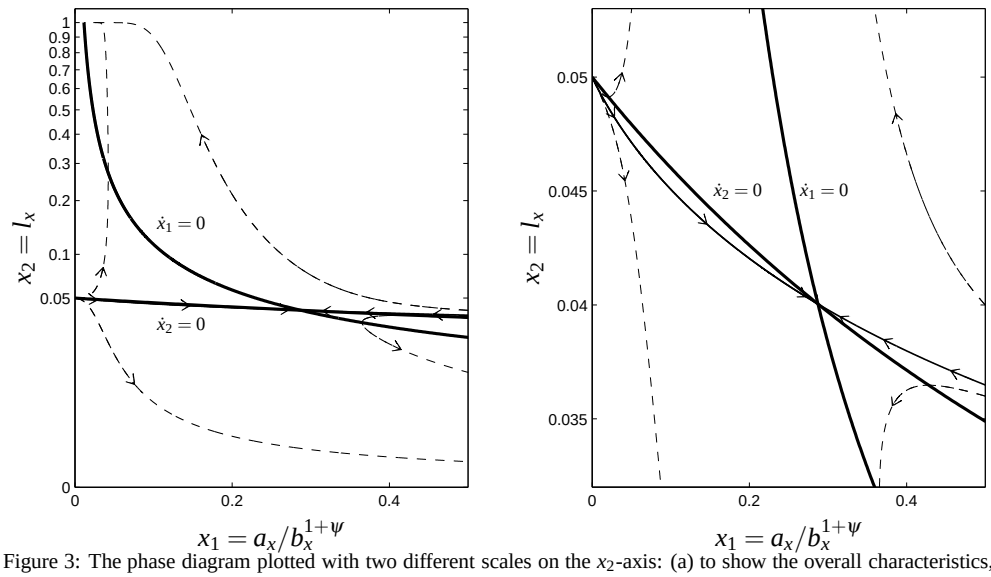

Figure 3: The phase diagram plotted with two different scales on the x_2-axis: (a) to show the overall characteristics, and (b) to show the stable path.

3.3. Transversality

Having characterized the phase diagram we have characterized sets of paths all of which are consistent with the necessary conditions 23 and 24. Of these, one set of paths leads to a situation in which all labour is employed in extraction and none in production, and another to a situation in which no labour is employed in extraction. Finally, there is the unique path which leads—given an initial state $a_x/b_x^{1+\psi}$—to the long-run steady state. Before proceeding we should verify that the path leading to the steady state is optimal. We do this using the transversality condition, which is that

$$\lim_{t\to\infty} e^{-\rho t}\lambda_t n_t = 0.$$

That is, the present value of resources in the ground (the converse of cumulative extraction) must approach zero looking into the very distant future; if that is not the case it must be possible to raise returns by increasing extraction (if the value is positive) or reducing it (if it is negative).

Consider first the corner with $x_2 = 0$ and $x_1 \to \infty$. As we head into this corner P^* approaches zero (equation 20), implying that $-\lambda = p_x$ (from equation 14). So $\dot{\lambda}/\lambda = \dot{p}_x/p_x$. But from 19 we have that $\rho = \dot{p}_x/p_x$ in this corner, so (putting these results together) we have $\dot{\lambda}/\lambda = \rho$. Hence the transversality condition cannot be satisfied; valuable resources are unnecessarily left in the ground. Now to the corner with $x_2 = l_x = L$. Now $P^* \to \infty$ (equation 20), and we can use 14, 17, and 19 to show that $\dot{\lambda}/\lambda = \rho + \dot{b}_x/b_x$, so again the transversality condition is not satisfied. This time, however, the scarcity rent approaches $-\infty$ ($\lambda \to \infty$); loosely speaking, infinitely-priced labour is being devoted to to extracting impossibly deep resources which are then not usable because there is no production labour.

3.4. The economic intuition

Having established that the stable path—as illustrated in Figure 3(b)—is unique and optimal, it is time to characterize it, focusing on the economic intuition. First we focus of the properties of the steady state, after which we turn to the transition path.

In the steady state x_1 and x_2 are constant, implying (from their definitions) that $\dot{b}_x/b_x = \theta_{ax}/(1+\psi)$, and that l_x, and hence also l_y, are constant. Given these results we can quickly derive the following properties of the steady state (which is of course a balanced growth path

in the original coordinate system)[12]

$$\frac{\dot{x}}{x} = \frac{\psi}{1+\psi}\theta_{ax}; \tag{28}$$

$$\frac{\dot{y}}{y} = \frac{\dot{p}_l}{p_l} = \theta_{ay} + \alpha\frac{\dot{x}}{x}; \tag{29}$$

$$\frac{\dot{p}_x}{p_x} = \frac{\dot{\lambda}}{\lambda} = \theta_{ay} - (1-\alpha)\frac{\dot{x}}{x}. \tag{30}$$

$$-\frac{\lambda}{p_x} = \frac{\frac{1}{1+\psi}}{\frac{1+(1-\alpha)\psi}{1+\psi} + \frac{\rho-\theta_{ay}}{\theta_{ax}}} \tag{31}$$

We now discuss each of these equations in turn. (28): If ψ is positive (implying an infinite stock) the b.g.p. implies increasing rates of resource extraction over time, whereas if ψ is negative (finite stock) resource extraction decreases exponentially on the b.g.p., and when $\psi = 0$ resource extraction is constant on the b.g.p.[13] (29): Production growth, and wage growth, are both equal to TFP growth plus the resource factor share $\times$ growth in resource inputs. (30): The resource price, and the scarcity rent, both grow at the rate of TFP growth minus the labour share $\times$ resource input growth. (31): The level of the scarcity rent on the b.g.p. (as a fraction of the resource price) is increasing in α, and decreasing in ψ and $(\rho - \theta_{ay})/\theta_{ax}$. As $\psi \to -1$ the scarcity rent approaches 100 percent of the price on the b.g.p., whereas when $\psi \to \infty$ the rent approaches zero; these results are easily understood in terms of the nature of the resource stocks implied by the value of ψ. The relationship between the b.g.p. rent and the growth and interest rates implies that the higher is the effective discount rate the lower is the scarcity rent. The intuition is that the scarcity rent arises from the fact that today's extraction raises future extraction costs, and the weight of this effect is small when the discount rate is high.[14]

Next we consider the transition path, hence we must identify a plausible starting point for the state variable x_1. Recall that $x_1 = a_x/b_x^{1+\psi}$; hence we need to choose the initial value of the productivity index, a_{x0}, relative to initial economic depth b_{x0}. To find such a starting point we need to think about the economics of the extraction process. Taking a historical perspective we know that if we go back sufficiently far in time then extraction of any given resource falls to zero; in terms of the model, technology a_x is insufficiently developed to allow any extraction at all. We thus postulate a *threshold* level of productivity a_{x0} at which extraction may begin. Furthermore—returning to our historical perspective—we know that at this threshold the extraction rate will typically be very low relative to the available stocks, even if a significant proportion of workers are devoted to the extraction effort; this implies that $a_{x0}/b_{x0} \ll m_0 b_{x0}$ (from equation 18), hence (loosely) that b_x is initially large in comparison to a_x; extraction technology is primitive compared to the difficulties intrinsic in extracting the resource.

Now take the above reasoning to the limit and let x_0 approach zero. It then follows straightforwardly from the representative firm's optimization problem that the economy starts off on a b.g.p., but a different one from the long-run b.g.p. On this *initial* b.g.p. $\dot{b}_x/b_x$ and the scarcity rent are both zero, and p_x is simply equal to the extraction cost. The dynamics are described by the following equations, which correspond to equations 28–31:[15]

$$\dot{x}/x = \theta_{ax} \tag{32}$$

$$\dot{y}/y = \theta_{ay} + \alpha\dot{x}/x \tag{33}$$

$$\dot{p}_x/p_x = \theta_{ay} - (1-\alpha)\dot{x}/x \tag{34}$$

$$-\lambda/p_x = 0. \tag{35}$$

[12]Growth rates of x, y, p_l, and p_x follow directly from equations 9–12. The growth rate of λ is equal to that of p_x since P^* is constant, and $-\lambda/p_x = 1 - P^*$. Equation 31, along with the equation for labour allocation on the b.g.p., is derived in Appendix A.

The remaining equations characterizing the b.g.p. are shown in Appendix A.

[13]Recall that a necessary condition for the existence of a b.g.p. is that $\psi > -1$; for $\psi < -1$ then if the b.g.p. existed b_x would be decreasing, which is not possible.

[14]Note also that, although we chose n as the state variable, n is neither an input nor a productivity index, hence $\dot{n}/n$ is not necessarily constant on the b.g.p., hence we do not present the equation for its b.g.p. value.

[15]In addition, $l_x/L = \alpha$, $x = \alpha L a_x/b_{x0}$, and $p_x = p_l b_{x0}/a_x$.

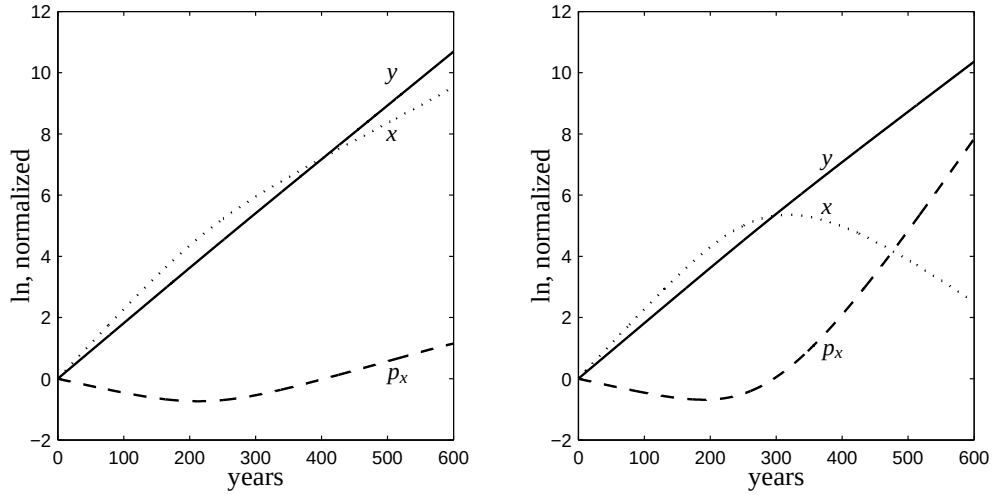

Figure 4: The evolution of p_x and x compared to y when the initial state $x_1 = 10^{-3}$, for alternative values of ψ: (a) $\psi = 1$; (b) $\psi = -0.4$. Other parameters as for Figure 3.

Comparing the equations, the key difference is that resource extraction grows faster on the initial b.g.p. than on the long-run b.g.p., and the size of the effect depends on $\psi/(1+\psi)$, and hence on ψ; when ψ is very large the effect is very small, but when ψ is zero, resource extraction is constant on the long-run b.g.p., and when $\psi < 0$ resource extraction declines expondentially on the long-run b.g.p.

The economy approaches the initial b.g.p. heading *backwards* in time towards a hypothetical starting point at which x_1 is arbitrarily close to zero. However, moving forwards in time from such a starting point the economy gradually moves away from the initial b.g.p., and approaches instead the long-run b.g.p. We denote the two periods—when the economy is close to but moving away from the initial b.g.p., and when the economy is approaching the long-run b.g.p.—as the *initial* and *mature* extraction phases, respectively. The two phases—and the transition from one to the other—are illustrated in Figure 4 for two different values of ψ, one positive and the other negative.

4. Solving the model in more complex cases

Study of the actual nature of resource stocks and associated extraction costs (next section) shows that the case with $\psi > 0$ is empirically relevant; that is, marginal stocks may tend to increase with increasing depth (or decreasing grade) rather than decreasing. However, since total stocks must be finite there must come a point at which marginal stocks decline with depth, or there may be an upper limit on the depth at which resource stocks exist. Furthermore, there may be a limit on the depth at which extraction is practical (possible) irrespective of extraction technology. Finally, because substitutes for the resource may be available, it may be optimal to cease extraction even though stocks remain in the ground. The solution methods for all of these problems are closely related, and we therefore begin with the simplest problem, in which b_x (economic depth) is bounded above. Having solved this problem we briefly consider the more general problem—such as case (iv) illustrated in Figure 2—in which there is some form of discontinuity or kink in the relationship between depth and cross-sectional area of the resource.

4.1. When b_x is bounded

When we put an upper bound on b_x there is no effect on the Hamiltonian, and hence no effect on the necessary conditions for optimization stated in Proposition 1, and hence no effect on the phase diagram. The effect is simply to change the transversality condition, and the result of this change is that the steady state is no longer the long-run optimum; this is to be expected, since the bound on b_x would be violated if the economy stayed in the steady state for indefinitely.

The new transversality condition will depend on the nature of the substitutes available for the resource. The simplest case is that *no* substitute is available, in which case transversality implies that the resource must be used up asymptotically as time approaches infinity; a greater rate of resource use would violate the stock restriction, whereas a lower rate would leave resources in the ground unnecessarily. In the limit—as the extraction rate approaches zero—b_x is again constant, as it is in the limit of the initial extraction phase moving backwards in time.

When b_x is constant while a_x continues to grow, and $x \to 0$ then $l_x \to 0$ (equation 12), and hence $P^* \to 0$ (equation 20), implying (equations 17 and 19) that

$$\dot{\lambda}/\lambda = \dot{p}_x/p_x = \rho.$$

We thus have a *third* phase of resource extraction, in addition to the *initial* and *mature* phases characterized above, which is (in the limit) a simple Hotelling extraction path in which the price is pure rent (hence rising at the discount rate) and extraction costs are zero.[16]

In the more realistic case where there exists a substitute for the resource,[17] then the transversality condition states that the resource price at the point of exhaustion is equal to the price of the substitute.[18] In terms of the phase diagram, the initial level x_2 given the initial state x_1 is chosen such that exhaustion occurs when $p_x = \bar{p}_x$; and when the starting point is determined, the evolution of the system is determined by the dynamic equations. Again, if $\bar{p}_x$ is sufficiently high then the resource price will be almost pure rent in the run-up to exhaustion, hence the price will rise at close to the discount rate and resource extraction will decline. On the other hand, if the price of the substitute is sufficiently *low* then exhaustion will never occur, and instead extraction will cease at the time when the scarcity rent is zero and the price is equal to the backstop price.

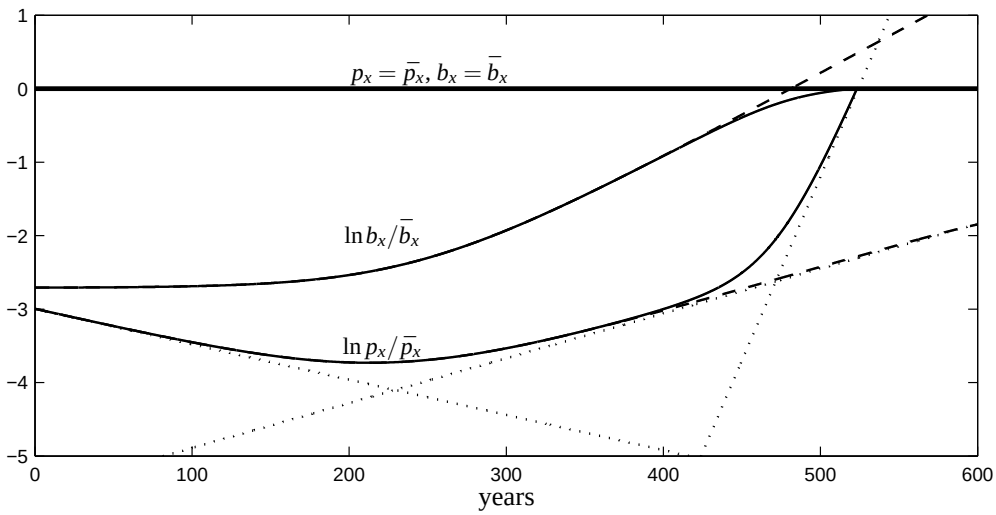

Figure 5: The paths over time of economic depth b_x and resource price p_x (plotted logarithmically and normalized by their final values) when there is a maximum economic depth $\bar{b}_x$ and a constant backstop price $\bar{p}_x$. The dashed lines show the paths of b_x and p_x when depth is unlimited, and the dotted lines have slopes corresponding to the limiting cases of the three phases described above, initial, mature, and exhaustion. Parameters as for Figure 3. Initial state: $a_{x0} = 0.001$, $a_{y0} = 1$, $b_{x0} = 1$. Exhaustion: $\bar{b}_x = 15$, $\bar{p}_x = 1.1627 \times 10^4$.

In Figure 5 we show the paths of price and economic depth using the same parameters as previously but adding a fixed backstop price $\bar{p}_x$ and a limit on economic depth $\bar{b}_x$. Note the three distinct phases—initial, mature, and exhaustion—in accordance with the analyt-

[16]Extraction costs approach zero because b_x approaches a limit whereas a_x/a_y grows without bound.

[17]See the next section for discussion of why this case is more relevant.

[18]The resource price can never be higher than the price of the substitute, whereas if the resource is exhausted with the price lower than the substitute price then resource owners are not extracting the full value of the rent at that point, hence the price path must have been too low.

ical model: the slopes of $\ln p_x$ in the three cases are (i) $\theta_{ay} - (1-\alpha)\theta_{ax}$; (ii) $\theta_{ay} - (1-\alpha)\theta_{ax}\psi/(1+\psi)$; and (iii) ρ.

4.2. When the stock is divided into sections

When the stock is divided into sections, as in Figure 2(iv), the optimization problem is also divided into sections. Because of discounting, it remains true that shallower resources will always be extracted before deeper, and the first-order conditions derived above apply at all times. We are thus left with the problem of transversality. The solution is straightforward, as shown by the following proposition.

Proposition 3. *At the boundary between stocks, as long as m is a continuous function of b_x, the initial values of (x_1, x_2) for the deeper stock are uniquely determined by the time of exhaustion of the upper stock and the values of (x_1, x_2) at that time. The optimal path is then the one for which the initial choice of the control variable x_2 for the uppermost stock leads to satisfaction of the transversality condition for the deepest stock.*

PROOF. Firstly note that the transversality conditions between stocks imply that price must be a continuous function of time; there cannot be a price discontinuity across a stock boundary, since this would imply non-optimal behaviour.[19] This implies that the extraction rate x must also be a continuous function of time, since the extraction rate is a continuous function of time if and only if the price is a continuous function of time.

Now assume that the cross-section m is a continuous function of b_x, including across boundaries between stocks. This implies that extraction employment l_x must also be a continuous function of time, that is extraction labour does not change discontinuously at the boundary between stocks; if l_x did change discontinuously, then the extraction rate would change discontinuously, which is ruled out.

We now consider the transition from one section of stock to the next. Recall that the state and control variables are

$$(x_1, x_2) = \left(\frac{a_x/b_x}{(b_x/b_{x0})^{\psi}}, \frac{x}{a_x/b_x} \right),$$

where x_1 is the state variable and $x_2 = l_x$, the control variable. At the point of transition, b_x and b_{x0} are both known, for both stocks. Furthermore, given the time at which the upper stock is exhausted, a_x is also known. Hence for a given time of exhaustion, the initial value of the state variable is known for the deeper stock. Furthermore, the control variable x_2 is simply extraction labour l_x, which we have just proved is continuous across the boundary.

5. Parameterization

We now turn to the parameterization of the model. The parameterizations are illustrative rather than strictly predictive, the main reason being the great uncertainty concerning many of the assumptions. Nevertheless, the model succeeds in explaining observations from the last 100 years, and makes apparently reasonable predictions for the next several hundred in the cases of oil and copper.

5.1. General observations

We start with some observations which apply whatever the resource for which we are parameterizing the model, in particular the long-run growth rates of TFP, extraction productivity, and the long-run interest rate. For long-run TFP growth we use the estimate of Shackleton (2013) of 1.7 percent per year (1870–2010). Regarding the productivity a_x of extraction labour, it could be argued that this should vary from resource to resource, and should be chosen from case to case in order to fit the data available. However, a more conservative approach

[19] If the price jumps up across the boundary, the upper stock must have been sold too cheaply, whereas if it steps down then the upper stock's price must have been unsustainably high given perfect competition.

is to assume that a_x should in the long run track manufacturing productivity generally, regardless of the extraction industry. Thus we take this figure from the literature, hence reducing the degrees of freedom in the parameterization; this reduces the probability that we are able to achieve a spurious match of the model to observations. We take the value of 2.3 percent per year from Fagerberg (2000). We set the interest rate to 5 percent per year. Since manufacturing productivity grows 0.6 percent faster than TFP we expect extraction costs to decline by 0.6 percent per year, ceteris paribus. Finally, α is simply the factor share of the resource.

If we want to match global extraction data we must also consider population growth. We approximate global population growth by assuming a constant growth rate of 1.3 percent per year from 1900–2037, after which population is assumed to be constant. The expected gradual slow-down towards zero growth over the next 40 years is thus approximately by a kink in the curve after 24 years.[20] The effect of population growth g in the model can be captured by increasing the growth rates of a_x and a_y to $a_x + g$ and $a_y + g$; since labour is the only input, more labour is equivalent to more productive labour.

It remains to find the parameters determining the relationship between cumulative extraction and unit cost for given productivity levels, and the initial values of the variables. How this is problem is tackled varies from case to case, hence we discuss each case in turn.

5.2. Copper

There are two key dimensions along which the quality of copper deposits varies. The first of these is grade (the fraction of copper in the rock, by mass), and the second is depth. We use Kesler and Wilkinson (2008) as our primary source of data concerning copper resources. They build a model of tectonic migration of copper deposits and calibrate it based on data about known deposits. In practice 'known deposits' is almost synonymous with 'deposits part of [whose] vertical thickness is at the surface' (Kesler and Wilkinson, 2008, p256), and the key to estimating long-run resources is the estimation of sub-surface deposits. Based on the model they estimate a recoverable resource of 8.9×10^{10} tons of copper, down to a depth of 3.3 km (below this depth they assume that recovery is not possible). Furthermore, marginal quantity increases with depth, up to 2.8 km. Regarding grade, Gerst (2008) argues that the grade–tonnage density function is log-Gaussian (his equation 7). Finally, Harmsen et al. (2013) estimate that 85 percent of produced copper historically has come from the top 500 metres. We use these four pieces of information—the total quantity up to 3.3 km, the relationship between depth and quantity, the grade–tonnage density function, and the depth of historical extraction—to calibrate our model relating b_x to n.

The first problem is to calibrate equation 1, the relationship between depth r_x and cross-sectional area m. We do this in two stages. First we estimate a curve showing the relationship between r_x and m (without restrictions on the functional form), then we calibrate equation 1 to fit the curve. To estimate the curve we divide the total stock into 7 layers (indexed 1 to 7 with increasing depth) each 471 metres thick, and then use Figure 2 from Kesler and Wilkinson (2008) to divide the total stock of 8.9×10^{10} tons into the fractions shown in Table 1. Regarding grade, we assume that the global grade–tonnage density function in each layer is in accordance with the estimate of Gerst (2008) for global porphyry (his Table 3). Regarding the effect of going down one layer, assume a deposit of grade g, layer l. Then we define the depth (or remoteness) of that deposit as $r_x = 100/(g \cdot 2^{l-1})$. Thus a step down from one layer to the next is equivalent to a halving of the grade, and extraction costs for a deposit of grade 0.5 percent in the top layer are the same as extraction costs for a deposit of grade 1 percent from the second layer. This is consistent with the observation of Harmsen et al. (2013) that extraction is starting from the second layer at the same time as available stocks in the top layer have dropped from 2 percent to 1 percent grade. The matlab program for deriving the relationship between depth and cumulative extraction (and thus also the equations and parameter values) —along with all the other matlab programs used in the production of this paper—is available

[20] This approximation is broadly in line with UN observations and predictions. See for instance United Nations (1999).

on the world wide web.[21]; the relationship itself is shown by the continuous lines in Figure 6, where the left-hand panel focuses on the most accessible stock, whereas the right-hand panel shows the entire stock.

Depth (m)	Fraction of total
0–471	0.10
471–943	0.12
943–1414	0.14
1414–1886	0.15
1886–2357	0.16
2357–2829	0.17
2829–3300	0.16

Table 1: Fraction of total available stock to be found at different depths.

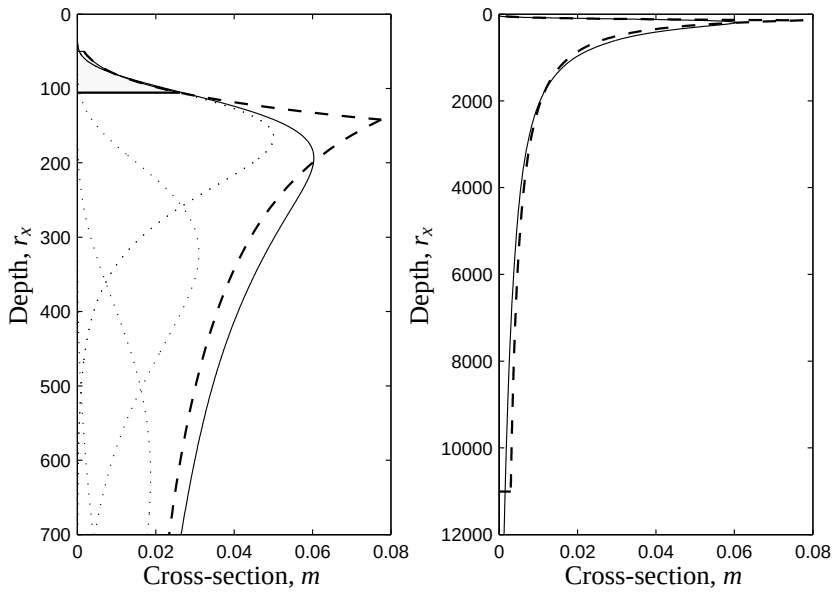

Figure 6: The relationship between the combined measure of depth and grade, r_x, and cross-sectional area m for copper, based on our interpretation of the literature (continuous line), and the parameterization of our economic model (dashed line). Notice the difference in scale on the two panels. The shaded area shows extraction from 1900–2011, 5.75×10^8 tons.

The next task is to calibrate equation 1 in order to fit the curve in Figure 6. In order to match the form of the grade–cumulative stock relationship we divide the stock into two substocks for which different parameters apply: the high-grade substock, consisting of 2.35×10^9 tons of copper, has the following parameters: $r^*_{x0} = 50$, $\phi = 4.66$, and $m_0 = 0.0017$; so in 1900, surface copper at 2 percent grade is available, in according with Gerst (2008). The low-grade substock, consisting of 86.7×10^9 tons of copper, has the following parameters: $r_{x0} = 142$, $\phi = 0.25$, and $m_0 = 0.0776$. As we can see from Figure 6, the economic model stock fits our stock from the physical model quite well. A much better fit could be achieved by dividing the stock into three parts, each with different parameters.

Having fitted these curves, the final task is to parameterize the relationship between r_x and b_x, i.e. to choose the parameter χ such that $\psi = \phi/\chi$. To set the value of χ we assume that the economy was close to the 'mature' b.g.p. in 1900, implying that the growth rate of price should be $\theta_{ay} - (1-\alpha)\psi/(1+\psi)\theta_{ax}$. We approximate the observed rate of price decline

[21] All of the programs used to solve the models and generate the figures (including data) are to be found on the following anonymous site: https://sites.google.com/site/exhaustibleresourcesmatlab/.

over the period as 0.4 percent per year, which gives $\chi = 0.419$, implying that unit costs are relatively insensitive to increasing depth or decreasing grade. (When grade is divided by 2 at constant depth, unit cost rises by a factor 1.34 at constant technology.) Given the observation of real price and extraction rate in 1900, this effectively completes the parameterization; the starting values are fixed by these observations.

Given the change in the growth rate of population in 2037, the simulation must be done in three stages, the first of which is from 1900 to 2037, the second of which is from 2037 to the (endogenous) time at which the switch of stock parameters must be made, 2054, and the third of which continues beyond this point until the depth of 3.3 kilometres is reached and the stock is exhausted. The time of exhaustion is also endogenous, and depends on the assumed backstop price. The program used is reproduced on the website (see Appendix B), and the results are shown in Figure 7.

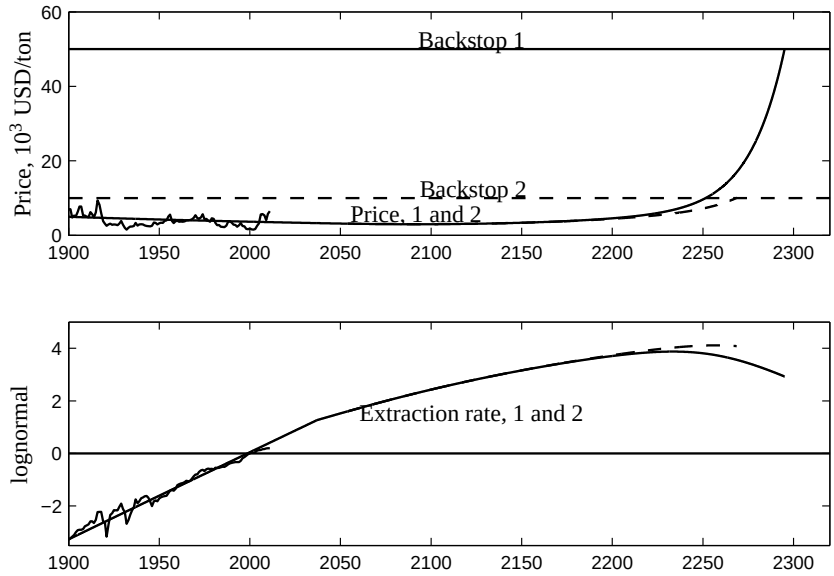

Figure 7: Observed price and extraction rate of copper, and the paths of price and extraction rate—up to the time of exhaustion—predicted by the model. Two model scenarios are shown, which differ in the assumed backstop price. Note that the extraction rate is plotted on a (natural) logarithmic scale, normalized by the rate in 2000. Prices are in 1998 USD.

To help understand Figure 7 we perform two calculations regarding the behaviour of the economy in the limit when it is on each of the two mature balanced growth paths, firstly for the upper portion of the stock with $\psi = 4.66/0.419 = 11.1$, and secondly for the deeper part of the stock for which $\psi = 0.25/0.419 = 0.60$. Firstly we calculate—using equation 30—the growth rate of price on the respective b.g.p.s. Since the resource price grows at rate $\theta_{ay} - (1-\alpha)\psi/(1+\psi)\theta_{ax}$, the growth rate of the price is -0.4 percent per year on the first b.g.p., and 0.8 percent per year on the second b.g.p. Secondly we calculate the scarcity rent as a proportion of the total price, using equation 31: on the first b.g.p. the scarcity rent makes up just 5.3 percent of the total price, whereas on the later b.g.p. (which is approached after extraction has moved on to the narrowing section of the stock), the scarcity rent makes up a much larger 40 percent of the total price. This is a reflection of the much higher degradation cost of extraction when the cross-sectional area of the imputed representative resource deposit is declining with depth rather than expanding.

Turning to the figure, the economy starts close to the first b.g.p. which applies for the initial stock, and price declines by around 0.4 percent per year. Once the initial stock is used up in 2054, the rate of increase in depth b_x increases, and the economy starts moving towards the second b.g.p. on which price rises by 0.8 percent per year. Finally, from around 2200 the scarcity rent starts to rise as exhaustion approaches, at least in the case with a high backstop price. With a low backstop price the scarcity rent hardly rises, and exhaustion occurs a few years earlier. Note the close agreement between the model and observed trends in prices and extraction rates.

5.3. Petroleum

If the copper simulation is an advertisement for the power of the model, the petroleum simulation highlights its weaknesses. There are two key aspects of the petroleum market which the model cannot handle as it stands: firstly, the significance of market power in the petroleum market, and secondly the inextricable links between petroleum and its substitutes, including natural gas, coal, and other energy sources such as nuclear power. Of course, market power and substitutes also exist in the market for copper, but their scale and influence is greater in the oil market. Concerning market power, consider for instance the fact that petroleum extraction occurs simultaneously from deposits for which marginal extraction costs differ by a factor of 5 or more (compare for instance the Ghawar field in Saudi Arabia to the Athabasca oil sands of Alberta). Concerning substitutes, petroleum demand is linked tightly to markets for coal and other energy sources, and strongly affected by technological change. Consider for instance the substitution from coal to oil driven by the development and refinement of the internal combustion engine. Given these problems—which are evident in Figure 9—the model calibration is at best illustrative, showing possible future scenarios and highlighting the effect of backstop energy sources.

The data regarding petroleum resources in the ground are uncertain. Furthermore, the data regarding the cost of extraction of these resources are even more uncertain. The most frequently cited paper on the subject is probably Rogner (1997). However, Rogner's curve relating cumulative extraction to extraction cost (see for instance his Figure 6) shows estimated extraction cost *at the time of extraction*. Its calculation must therefore involve (implicit or explicit) calculations of (i) current extraction costs, (ii) expected decline in extraction costs, and (iii) expected rate of extraction. Since we model the latter two, we need data on the first factor alone, i.e. unit extraction costs for each type of deposit making up the reserves, if full-scale extraction were to be carried out today. This is estimated by the International Energy Agency in their World Energy Outlook 2008 (p.218). The data are very approximate, but can be broadly summarized as follows: considering initial resource stocks, there was a large rather homogeneous stock of easily accessible stocks, approximately 2000 billion barrels at an economic depth of around 18 USD/barrel. Regarding the remaining stocks—about 7 billion barrels—economic depth b_x rises approximately linearly with cumulative extraction, reaching approximately 115 USD/barrel for the deepest stocks. We capture this in the model by assuming an initial stock with low ψ ($\psi = -2.2$), so that the entire near-homogeneous stock is at a depth of 10–20, switching to the deeper stock with $\psi = 1$ from 20–115. The cross-section of the second stock is determined by its size (assumed to be 6.7×10^9 barrels), and the parameters for the first stock are then fixed by the limits on depth (10–20), the size (2.3×10^9 barrels), and the need for m to be continuous over the boundary between the stocks. The result is shown in Figure 8. Note that the curve shows unit extraction costs, in 2008 USD with today's technology, for all petroleum resources including the (hypothetical) current extraction cost of resources already extracted. (Note that we ignore the fact that a significant proportion of cumulative extraction has been from deeper stocks.)

Having fixed the curve in Figure 8, the data on initial price and extraction rate (in 1880) is sufficient to determine the starting value of a_x, given that we assume that the economy starts with a low value of x_1 and hence x_2 close to α. (Recall that α is the share of petroleum in global product, which we set at 3 percent.) Unfortunately, however, this parametrization fails to reflect the fact that the share of petroleum in global product rose rapidly from 1880 to 1970, contrary to the assumption of constant share in the model. The problem is that although the share of *combustibles* in global product was remarkably constant over this period (recall Figure 1), petroleum substituted for other combustibles and hence its share rose.[22] To account for this in the model we raise the productivity growth rates during the period up to 1970, reflecting petroleum's increasing market penetration.

Having parameterized the model, and given the assumption about the total stock of resources, the future development of prices and quantities predicted by the model depends on

[22]The share also rose after 1974, but this was due to short-run inelasticity of demand combined with steeply rising prices.

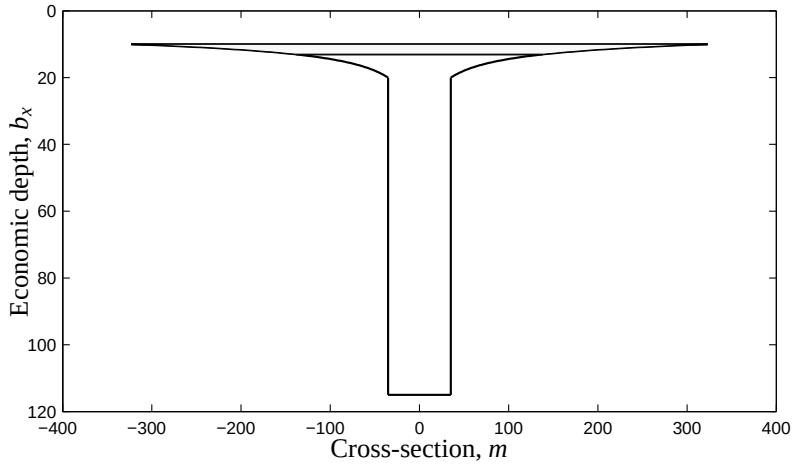

Figure 8: The relationship between economic depth, b_x, and cross-sectional area m for oil, based on our interpretation of the International Energy Agency World Energy Outlook 2008. Depth is measured in 2008 USD, and cross-sectional area in billion barrels per USD. The shaded area shows extraction from 1900–2008, 1100 billion barrels.

what we assume about the price of the backstop (i.e. the substitutes for oil that will take over when oil is exhausted or too expensive). Here we make two alternative assumptions to demonstrate the role played by the backstop resource. In the first case we assume that a backstop is available at a fixed price of 150 US dollars (2011); in the second case we assume that a backstop is available *today* at that price, and that this price will decline at the rate $\theta_{ax} - \theta_{ay}$; that is, the backstop price declines as long as manufacturing productivity growth outstrips TFP growth. The result is that the backstop price is around 65 USD at the time of exhaustion, rather than 150.

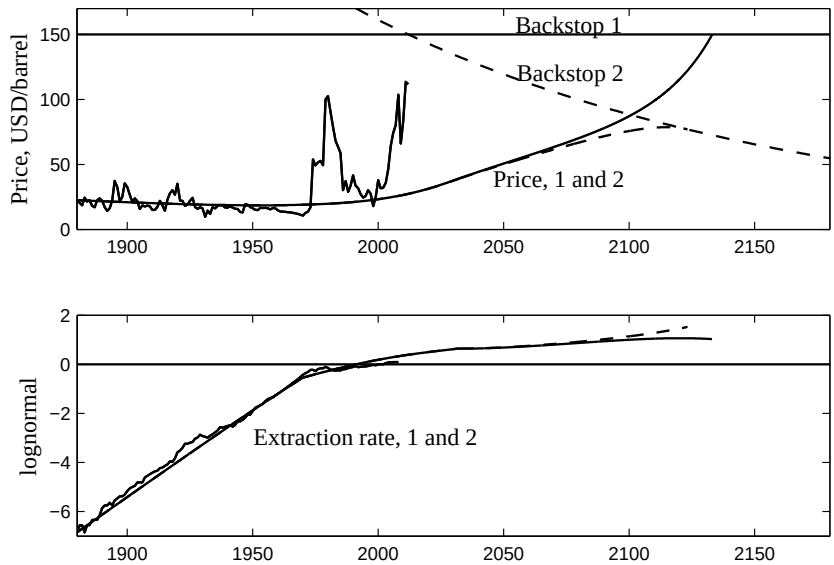

Figure 9: Observed price and extraction rate of petroleum, and the paths of price and extraction rate—up to the time of exhaustion—predicted by the model. Two model scenarios are shown, which differ in the assumed backstop price. Note that the extraction rate is plotted on a logarithmic scale, normalized by the rate in 2000. Prices are in 2012 USD. Scenarios: continuous lines, backstop price 150 USD (year 2012); dashed lines, current backstop price 150 USD, declining at a rate $\theta_{ax} - \theta_{ay}$ per year. Price data from BP (2012), consumption data from Boden et al. (2012), assuming a linear relationship between CO_2 emissions and petroleum consumption.

The results are as follows. Note first that there is no market power in the model economy, hence the results are what the model predicts in an economy similar to the actual global economy but without the exercise of market power by oil producers. Turning now to the results, up to the exhaustion of the upper stock, depth is almost constant, hence (as long as exhaustion

of the entire stock is a long way into the future) the scarcity rent is close to zero and price is expected to decline at a rate equal to the difference between the growth rates of extraction productivity and labour productivity in final-good production, which we set to 0.6 percent per year. (This is in accordance with equations 32–35.) However, as the upper stock nears exhaustion depth starts to rise at a significant rate, and the economy heads back towards the b.g.p. for the lower stock, for which $\psi = 1$; the mature extraction phase. On this b.g.p. we have—from equations 28–31—that the growth rate of extraction is halved, the resource price rises by 0.6 percent per year, and the scarcity rent makes up 21 percent of the price.

Note that after the transition to the deeper stock with $\psi = 1$ the economy approaches the mature b.g.p. for that stock from *above*, i.e. the state variable x_1 is above its level in the steady state. As the economy approaches the new b.g.p. x_1 falls back, which is why prices rise quite steeply throughout the 21st century. In the latter half of the century the price paths of the alternative backstop scenarios diverge significantly: the upper path (high backstop price) is slightly above the b.g.p. price path, while the lower path is below it. Hence when the backstop price is fixed at 150 USD the scarcity rent rises above 21 percent of the total price as exhaustion approaches, whereas given the lower backstop price the rate of price increase slows down as exhaustion approaches, and the rent actually declines as a proportion of the price.

5.4. Sensitivity to assumptions

Clearly the above simulations are sensitive to the assumptions made, the most uncertain of which are those regarding future demand, and future development of extraction productivity. On the one hand, our assumption about future demand is essentially at the upper bound of what is realistic, i.e. that demand continues to grow indefinitely at a similar rate to the rate observed over the last 100 years. At least two factors might be expected to lead to lower future demand: firstly, if global growth slows in the long run, and secondly if there is a transition from 'early' growth based on manufacturing and hence resources, and 'post-industrial' growth based on services and hence labour. The effect of lower demand would be to reduce extraction rates and hence also reduce the growth rate of prices predicted by the model. On the other hand, our assumption regarding future development of extraction productivity is also an upper bound; again, we assume that it continues to increase indefinitely. This is unlikely, not least because in reality resource extraction requires energy, and there are physical limits to the efficiency with which this energy can be used. Since these limits are already coming close in some cases, this implies that even if labour productivity continues to increase, energy productivity will not do so and hence the proportion of the energy cost in the unit cost will rise, and the rise of overall extraction productivity will slow. This assumption therefore biases the results towards lower prices and higher extraction rates than are likely to be observed.

The effect of assuming both lower future demand and lower productivity growth rates in extraction is therefore that the price path is likely to be relatively unchanged, whereas the extraction path will be lower. Furthermore, the proportion of the price accounted for by the scarcity rent will be lower. Given a finite stock, the lower extraction path will lead to later exhaustion, and hence any price spike as exhaustion approaches is also likely to be delayed.

6. Conclusions

We show how the problem of optimal extraction of inhomogeneous resource deposits can be set up in such a way that closed-form solutions for the equilibrium extraction path can be obtained. The model provides an explanation for the observation of slowly declining resource prices—the productivity of the extraction input grows faster than its price, since extraction productivity grows faster than total factor productivity in the final-good sector—and also predicts a more constant or slowly rising price trend in the medium term, as the above effect is supplemented by the effect of the increasing depth from which resources must be extracted. Finally, if exhaustion beckons and substitutes are very expensive there may be a third phase in which price rises at a rate approaching the interest rate while the extraction rate declines. Simulations suggest that this phase is a long way off for the resources studied, copper and petroleum.

The model is highly simplified. Better data about the nature of resource stocks, and a better understanding of long-run demand for resources (in particular whether increasing resource

extraction is primarily driven by relative price effects or by income effects) would allow the model to be refined and increase confidence in the predictions. Another relatively straightforward extension would be to include other inputs—such as capital and energy—into the extraction cost and production functions, which would greatly increase the realism of the extraction cost function, allowing it to be based more directly on empirical evidence. Finally, to account properly for resource markets in which the exercise of market power is a crucial factor—such as the market for petroleum—a major extension to the model would be required. Nevertheless, the model as it stands supports the view that the high price of petroleum is neither due to extraction costs nor scarcity rent, and hence is presumably due to market power.

A. The balanced growth path

To find the levels of the variables on the b.g.p., use the growth rate of b_x on the b.g.p.,

$$\frac{\dot{b}_x}{b_x} = \frac{1}{1+\psi}\theta_{ax},$$

and equation 18, to show that

$$\frac{\theta_{ax}}{1+\psi} = \frac{x}{F_0(b_x/b_{x0})^{\psi}}.$$

Rearrange and substitute for b_x using equation 12 to derive

$$x = \left(\frac{a_x l_x}{b_{x0}}\right)^{\psi/(1+\psi)} \left(\frac{F_0\theta_{ax}}{1+\psi}\right)^{1/(1+\psi)}. \tag{36}$$

The value of l_x (which is just x_2) follows directly from equations 25 and 26:

$$\frac{l_x}{\alpha L} = \frac{\frac{\psi}{1+\psi}\theta_{ax} + \frac{\rho-\theta_{ay}}{1-\alpha}}{\theta_{ax} + \frac{\rho-\theta_{ay}}{1-\alpha}}. \tag{37}$$

Having found l_x, equations 12, 14, 16, and 20 yield the expression for $-\lambda/p_x$ in equation 31.

B. Supplementary material regarding the solutions to the models and the figures

All of the programs used to solve the models and generate the figures (including data) are to be found on the following site: https://sites.google.com/site/exhaustibleresourcesmatlab/.

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