Steady-state properties in a class of dynamic models

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Abstract

We characterize the location, stability and approach-time of optimal steady states in single-state, infinite-horizon, autonomous models by means of a simple function of the state variable, defined in terms of the model's primitives. The method does not require the solution of the underlying dynamic optimization problem. Its application is illustrated in the context of a generic class of resource management problems.

Keywords: infinite horizon, autonomous problems, optimal policy, steady-state, approach time.

JEL classification: C61, Q20, Q30

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1 Introduction

An important property of intertemporal economic systems concerns their long-run stability or, more precisely, whether or not they converge to a steady state. When they do, two questions immediately arise regarding the location of the optimal steady state and the time it takes to approach it. While the conditions needed for the stability of multi-state processes are often hard to specify and/or verify (see Sorger 1989, and references cited there), the conditions ensuring (local) stability of single-state, autonomous processes are rather straightforward. In this work we characterize the steady-states' *location*, *stability* and *approach time* of the latter type of processes by means of a function of the state variable, denoted $L(\cdot)$, defined in terms of the model's primitives, without the need to actually solve the underlying dynamic optimization problem.

The function $L(\cdot)$ was introduced by Tsur and Zemel (2001) who used it to identify the *location* of candidate steady states. The present work extends Tsur and Zemel (2001) in two ways. First, *stable* steady states are identified in terms of the slope of $L(\cdot)$, which is particularly useful when multiple steady states exist. Second, $L(\cdot)$ is used to determine whether the steady-state *approach time* is finite or infinite. Thus, the steady state properties regarding location, stability and approach time are characterized in terms of this function $L(\cdot)$ even when the explicit (closed-form) dynamic solutions are hard to obtain.

The methodology developed here can be applied in a wide variety of economic situations (see the assortment of examples in Kamien and Schwartz 1991, of which many are formulated as single-state, autonomous models), including models with unbounded state domain that can be reformulated in terms of a bounded (normalized) state (e.g., per-capita variables in exogenous growth models). Models of natural resource management provide a particularly fertile ground for demonstrating the application of the L-methodology. In this context, the steady-state location concerns issues such as whether a resource should eventually be depleted (led to extinction) or some finite stock must always be reserved for future use. Alternatively, one can investigate the conditions or regulatory measures that give rise to more conservative exploitation relative to some benchmark, by comparing the location of the corresponding steady states (for examples of this type of analysis, see Tsur and Zemel 2004). The steady-state approach time has initially been examined in the context of nonrenewable resources (e.g., minerals) that are both finite and essential (see Dasgupta and Heal 1974, Salant et al. 1983, and references cited therein), with the main insight that the larger the alternative price of the mineral resource, the longer it should take to deplete it (in the limit, when the alternative price is infinite, depletion occurs asymptotically). In other settings, questions regarding the steady-state approach time are sometimes posed in such concrete terms as "when is it optimal to exhaust a resource in finite time?" (Akao and Farzin 2007) or "when should we stop extracting a nonrenewable resource?" (Schumacher 2011).

The next section lays out the setup for a general infinite-horizon dynamic optimization problem, defines the corresponding $L(\cdot)$ function and briefly summarizes the results of Tsur and Zemel (2001). Section 3 presents the new results – the stability condition and whether the steady-state approach time is finite or infinite. Section 4 applies our methodology to a generic model of natural resource management, considering, in turn, non-renewable and renewable resources. It considers also the problem of resource management under threat of abrupt, catastrophic events, where the event occurrence probability depends on the management policy – a problem for which an explicit dynamic solution is not readily available. Nevertheless, the L-method identifies the stable steady states and whether the approach time (to each) is finite or infinite in much the same way, and at the same level of simplicity, as in the other, simpler examples. Section 5 concludes and proofs are presented in the Appendix.

2 Setup

Let $X(t)$ and $c(t)$ denote, respectively, the state (stock) and control (action flow) of an economic system at time t. The action $c(t)$ affects the state's evolution according to

$$\dot{X}(t) = g(X(t), c(t)) \tag{2.1}$$

and gives rise to the instantaneous benefit $f(X(t), c(t))$. The policy $\{c(t);\ t \geq 0\}$ generates the payoff

$$\int_0^\infty f(X(t), c(t))e^{-\rho t}dt, \tag{2.2}$$

where ρ is the time rate of discount. The instantaneous benefit and the state dynamics functions, $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$, are twice continuously differentiable and satisfy

$$f_c > 0, \quad f_{cc} < 0, \quad g_c \leq -\alpha < 0, \quad g_{cc} \leq 0, \quad f_X \geq 0, \tag{2.3}$$

for all $X \in (\underline{X}, \bar{X})$ and feasible c, where α is a given positive constant and the subscripts X and c denote partial derivatives with respect to these variables.[1]

[1] The assumptions regarding the signs of f_c and g_c entail no loss of generality, as one can always formulate the model in terms of the control $-c$ instead of c, keeping the requirement that $f_c g_c < 0$. The same applies to f_X when X represents a damaging state, e.g., a pollution stock. The condition $|g_c| \geq \alpha > 0$ implies that c is influential in controlling the state evolution. The signs in assumption (2.3) are maintained for the sake of concreteness.

A feasible policy satisfies $X(t) \in [\underline{X}, \bar{X}]$ and $c(t) \in \mathcal{C} \equiv [\underline{c}, \bar{c}]$ for all $t \geq 0$, where $\underline{X} < \bar{X}$ are given state bounds[2] and $\underline{c} < \bar{c}$ are given action bounds. The lower and upper bounds $\underline{X}$ and $\bar{X}$ can be determined by physical or regulatory constraints (e.g., natural resource stocks and produced capital stocks cannot turn negative, or a regulator may impose some positive lower bound on these stocks). Alternatively, these bounds can be induced by the action feasibility constraint $c \in \mathcal{C}$ (e.g., the initial stock of a non-renewable resource or the carrying capacity stock of a renewable biomass resource determine the upper bound $\bar{X}$ if the exploitation rate is restricted to be non-negative). This distinction will turn out to be important in the analysis below.

The optimal policy is the feasible policy that maximizes (2.2) subject to (2.1) given $X(0) = X_0$. We assume that an optimal policy exists and denote the corresponding value function (the payoff under the optimal policy) by $v(X_0)$. Since the (single state) dynamic optimization problem under consideration is infinite-horizon and autonomous, the ensuing optimal state trajectory is monotonic in time (see Hartl 1987).[3] As it is also bounded, the optimal state process must converge to a steady state, which we denote by $\hat{X} \in [\underline{X}, \bar{X}]$.

We further assume that the *constant-state* function $M(X)$, defined by

$$g(X, M(X)) = 0, \tag{2.4}$$

is single-valued and corresponds to a feasible policy for all $X \in [\underline{X}, \bar{X}]$. It

[2]The restriction of the state-space to a finite domain appears to exclude common economic models, such as those describing economic growth. Evidently, if the state variable diverges to infinite values, then a discussion of steady states is not relevant. However, in most cases these models can be recast in a normalized formulation (e.g. by considering per-capita variables) and the normalized processes typically approach a steady state.

[3]Hartl's argument applies only when the problem admits a unique optimal process. However, the same argument can be used to show that in the case of non-uniqueness, at least one optimal process must be monotonic by specifying, for all states admitting more than one optimal control, a consistent selection rule (e.g. always choose the optimal control corresponding to maximal $\dot{X}$). This monotonic process must converge to a steady state.

follows from (2.3)-(2.4) that

$$M'(X) = -g_X(X, M(X))/g_c(X, M(X)) \tag{2.5}$$

is well defined. Adopting the policy $c = M(X)$ leaves the process at the state X indefinitely, yielding the payoff

$$W(X) \equiv f(X, M(X))/\rho \le v(X), \tag{2.6}$$

where the inequality holds as an equality only at the optimal steady state $\hat{X}$. Define the function

$$L(X) \equiv \rho f_c(X, M(X))/g_c(X, M(X)) + \rho W'(X), \tag{2.7}$$

which, noting (2.5), can be expressed as

$$L(X) = \frac{f_c(X, M(X))}{g_c(X, M(X))}[\rho - g_X(X, M(X))] + f_X(X, M(X)). \tag{2.8}$$

We refer to the states where $L(\cdot)$ vanishes as the *roots* of L. The function $L(\cdot)$ serves to formulate the following necessary conditions for the location of the optimal steady state $\hat{X}$ (see Tsur and Zemel 2001):

Proposition 1 (necessary conditions for the location of $\hat{X}$)**.** *(i)* $L(\hat{X}) = 0$ *is necessary for* $\hat{X} \in (\underline{X}, \bar{X})$*; (ii)* $L(\underline{X}) \le 0$ *is necessary for* $\hat{X} = \underline{X}$*; (iii)* $L(\bar{X}) \ge 0$ *is necessary for* $\hat{X} = \bar{X}$*.*

The proofs of all propositions are presented in the Appendix.

To see why an internal steady state must be a root of $L(\cdot)$, consider the case in which the value function $v(\cdot)$ is differentiable. In such cases, it is well known that the derivative $v'(\cdot)$ equals the current-value costate variable. In particular the relation $v'(\hat{X}) = \hat{\lambda}$ (where $\hat{\lambda}$ denotes the steady state value of the costate) holds at the steady state $\hat{X}$. In this state, (2.6) implies both $v(\hat{X}) = W(\hat{X})$ and $v'(\hat{X}) = W'(\hat{X})$ (otherwise, the curves of $v(\cdot)$ and $W(\cdot)$ would cross at $\hat{X}$). It follows that (2.7) specializes at $\hat{X}$ to $L(\hat{X}) = \rho[f_c(\hat{X}, M(\hat{X}))/g_c(\hat{X}, M(\hat{X})) + \hat{\lambda}]$ and the latter expression vanishes when the Maximum Principle condition (see (B.2) in the Appendix) is applied with the constant state control $c = M(\hat{X})$.

Proposition 1 identifies the optimal steady state as either a root of L or one of the state bounds. It determines $\hat{X}$ uniquely in the following cases:

Corollary 1. *(i) If $L(\cdot)$ crosses zero once from above in $[\underline{X}\,, \bar{X}]$, then $\hat{X}$ falls on the unique root of $L(\cdot)$. (ii) If $L(X) > 0$ for all $X \in [\underline{X}\,, \bar{X}]$, then $\hat{X} = \bar{X}$. (iii) If $L(X) < 0$ for all $X \in [\underline{X}\,, \bar{X}]$, then $\hat{X} = \underline{X}$.*

3 Steady state properties

Proposition 1 limits the set of candidate steady states and determines the optimal steady state uniquely only in the cases of Corollary 1. In other cases, when $L(\cdot)$ obtains multiple roots in $[\underline{X}\,, \bar{X}]$, the proposition cannot determine $\hat{X}$ uniquely. In this section we limit further the list of optimal steady states by using properties of $L(\cdot)$ to identify stable steady states. We then determine, for each stable steady state, whether the optimal process it attracts approaches the state at a finite time or asymptotically.

3.1 Stability

The following result employs the slope of $L(\cdot)$ to identify unstable steady states.

Proposition 2. *A root X of $L(\cdot)$ cannot be a stable steady state if $L'(X) > 0$.*

Proposition 2 rules out roots of $L(\cdot)$ in which $L(\cdot)$ crosses zero from below. It implies that processes initiated away from such a root will either not evolve toward it or pass through it on their way to a stable steady state. Only if the initial state falls exactly on the unstable root can the latter be a candidate optimal steady state, and even this singular possibility can be ruled out in most cases: If the unstable root falls within the domain of attraction of another, stable, root, then an optimal process will not stay at the unstable state even if initiated at this state. Thus, the unstable steady state can be optimal only if it happens to form the boundary point between the domains of attraction of two other stable roots (see Wirl and Feichtinger 2005, for a discussion of unstable steady states).

The relation between the slope of $L(\cdot)$ and the stability of the steady state is not coincidental, as the following property reveals (see Appendix for the derivation):

$$\operatorname{sign}(L'(\hat{X})) = \operatorname{sign}(\det(J)), \tag{3.1}$$

where J is the Jacobian matrix of the dynamic system corresponding to the underlying dynamic optimization problem (see equation (C.4)) and $\det(J)$ denotes its determinant, evaluated at $X = \hat{X}$. As is well known, stability conditions are usually formulated in terms of the signs of the eigenvalues associated with J. In particular, for a single-state problem $\det(J) > 0$ excludes the possibility of a pair of eigenvalues of opposite signs which is associated with the saddle-point stability of optimal trajectories proceeding towards a stable steady state. Viewed from this angle, Proposition 2 and the sign relation (3.1) establish an interesting link between the L-function and the standard tool to characterize the stability of optimal dynamic processes.[4]

[4]We are grateful to a reviewer for suggesting to consider this relation.

3.2 Time of approach

We turn now to study whether the steady-state approach time is finite or infinite. To that end, we distinguish between steady states at which $L(\cdot)$ vanishes and those at which $L(\cdot)$ does not. We refer to the former as *unconstrained* steady states and to the latter as *constrained* steady states. Note that unconstrained steady states (with $L(\hat{X}) = 0$ and $L'(\hat{X}) < 0$) can fall anywhere in $[\underline{X}, \bar{X}]$, including the upper and lower bounds, whereas constrained steady-states must fall on one of the bounds ($\underline{X}$ or $\bar{X}$). The modifier "unconstrained" indicates that the steady-state $\hat{X}$ would remain optimal even if the constraint $X(t) \in [\underline{X}, \bar{X}]$ were slightly relaxed, whereas the "constrained" modifier indicates that a change in the relevant bound ($\underline{X}$ or $\bar{X}$) would entail a different steady state. Let T denote the time it takes the optimal state process to approach the steady-state. The following result characterizes T for unconstrained steady states:

Proposition 3 (approach time to unconstrained steady states)**.** *Under assumption (2.3) and $X(0) \neq \hat{X}$, the approach to unconstrained steady states is asymptotic, i.e., $T = \infty$.*

Roughly speaking, this property stems from the continuity of the optimal action (control) process. As the state process approaches the steady state, the control approaches the constant-state rate $M(\hat{X})$, so the rate of further change in the state becomes small. Note the importance of the curvature properties of f and g in assumption (2.3) for this characterization. Indeed, if both f and g are linear in the control, a most rapid approach path (Spence and Starrett 1975) can bring the process to $\hat{X}$ within a minimal (finite) time T, followed by a discontinuous jump of $c(t)$ to the constant-state control $M(\hat{X})$ at $t = T$.

Constrained steady states (where $L(\hat{X}) \neq 0$) must fall on one of the bounds $\underline{X}$ or $\bar{X}$ (see Proposition 1), and the approach time to such steady states depends on the following two classifications: (1) whether the feasibility constraint $c \in \mathcal{C}$ is binding at the steady state (it is *not* binding if $M(\hat{X})$ lies in the interior of $\mathcal{C}$); and (2) whether the stock is *essential* at $\hat{X}$ (the stock is essential at $\hat{X}$ if $f_c(X, M(X)) \to \infty$ as $X \to \hat{X}$).[5] Typically, the latter property is relevant only at the lower bound, where $\hat{X} = \underline{X}$ and $M(\underline{X})$ is too small to meet essential needs (e.g., non-renewable resources that are being depleted or regenerating resources on the way towards extinction). Indeed, if $\hat{X} = \bar{X}$, the stock is not essential at $\hat{X}$. To see this, recall that $f_{cc} < 0$ hence the divergence of f_c implies that $M'(X) < 0$ in the vicinity of $\bar{X}$. Thus, $g_X(\bar{X}, M(\bar{X})) < 0$ (see (2.5)) hence $\rho - g_X(\bar{X}, M(\bar{X})) > 0$ and (2.8) implies $L(\bar{X}) = -\infty$, violating the condition in Proposition 1 for $\bar{X}$ to be an optimal steady state. At the other bound, where $\hat{X} = \underline{X}$, we refer to a stock as *essential* or *non-essential* depending on whether $f_c(\underline{X}, M(\underline{X}))$ is infinite or finite, respectively.

The properties of T for constrained steady states are summarized in:

Proposition 4 (approach time to constrained steady-states)**.** *Suppose* $\hat{X} \neq X_0$ *and* $L(\hat{X}) \neq 0$ *(so* $\hat{X}$ *must fall on either the upper or the lower state bound):* (i) *If the feasibility constraint* $c \in \mathcal{C}$ *is not binding at* $\hat{X}$*, then* $T = \infty$ *or* $T < \infty$ *depending on whether the stock is essential or non-essential, respectively.* (ii) *If the feasibility constraint* $c \in \mathcal{C}$ *is binding at* $\hat{X}$ *and* $|g_X(\hat{X}, M(\hat{X}))| < \infty$*, then* $T = \infty$*.*

The asymptotic steady-state approach ($T = \infty$) in case (ii) owes to the

[5] This terminology borrows from resource economics, though our results apply for more general contexts. Nevertheless, the condition bears a direct intuitive appeal when applied to resource exploitation so we adopt the adjective 'essential' rather than using a more general but less informative term.

feasibility constraint on the action c near the steady state. In case (i) this constraint is not binding, but for essential stocks the divergence of the marginal benefit near the steady state acts as an effective c-constraint, giving rise again to asymptotic steady-state approach.

Figure 1 summarizes the steady state properties of Propositions 1-4. The list of steady state candidates includes the roots of $L(\cdot)$, the lower bound $\underline{X}$ if $L(\underline{X}) \leq 0$ and the upper bound $\bar{X}$ if $L(\bar{X}) \geq 0$ (cf. Proposition 1). Thus, in the left panel the lower and upper state bounds are excluded from the list of optimal steady states whereas in the right panel they are included. The roots at which $L(\cdot)$ crosses zero from above admit saddle-point stability (cf. Proposition 2 and (3.1)), which includes also the lower state bound of the right panel. A stable root is unconstrained. A feasible steady state is constrained if it falls on the lower or upper boundary with $L(\cdot)$ strictly negative or strictly positive, respectively. The approach to stable-unconstrained steady states is asymptotic. The approach time to constrained steady states can be finite or infinite, as characterized in Proposition 4.

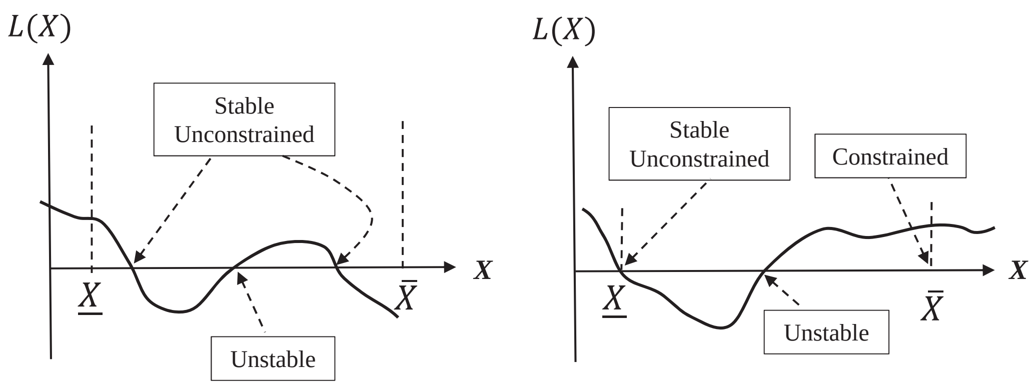

Figure 1: Summary of steady state properties.

4 Application to resource management

The above properties are particularly useful in the context of problems that do not admit a closed-form solution, where identifying the list of stable steady states and the ensuing approach time by standard methods is not trivial. This is now demonstrated by applying the L-methodology to natural resource management models, where $X(t)$ represents the remaining resource stock and $c(t)$ is the rate of exploitation (mining, extraction, harvesting) at time t. The net benefit function takes the form $f(X, c) = u(c) - Z(X)c$, where $u(\cdot)$ is an increasing and strictly concave utility function and $Z(\cdot) \geq 0$ is a non-increasing and convex unit extraction cost. The resource dynamics is specified as

$$g(X, c) = R(X) - c \tag{4.1}$$

where $R(\cdot)$ is the recharge (growth, regeneration) function. For non-renewable resources, $R(X) = 0$ for all X, whereas for renewable resources $R(X)$ is positive over some interval $(0, \bar{X})$. The resource exploitation policy $\{c(t),\ t \geq 0\}$ is feasible if $c(t) \geq 0$ and $X(t) \geq 0$ for all $t \geq 0$. This policy generates the payoff

$$\int_0^\infty [u(c(t)) - Z(X(t))c(t)]e^{-\rho t}dt \tag{4.2}$$

and the optimal policy is the feasible policy that maximizes (4.2) subject to the state dynamics constraint (4.1) given the initial stock X_0.

This formulation is widely used in the resource economics literature and many properties of the ensuing optimal policies are well known (see, e.g., Clark 1976, Dasgupta and Heal 1979). It therefore serves well the purpose of demonstrating the use of the above analysis for locating the stable steady states and determining the associated approach times without actually solving the dynamic optimization problems. We discuss non-renewable and renewable resources in turn.

4.1 Non-renewable resources

With a vanishing $R(X)$, (2.1) specializes to

$$\dot{X}(t) = -c(t). \tag{4.3}$$

The state process is non-increasing hence the upper bound is set as $\bar{X} = X_0$. The constant-state function $M(X)$ vanishes identically for all X and $L(\cdot)$ of (2.8) specializes to

$$L(X) = \rho[Z(X) - u'(0)]. \tag{4.4}$$

Consider first a non-essential resource with $u'(0) < \infty$. The function $L(\cdot)$ decreases and Proposition 1 identifies the unique optimal steady-state

$$\hat{X} = \begin{cases} 0 & \text{if } Z(0) - u'(0) \leq 0 \\ Z^{-1}(u'(0)) & \text{if } Z(0) - u'(0) > 0 \text{ and } Z(X_0) - u'(0) < 0 \ , \\ X_0 & \text{if } Z(X_0) - u'(0) \geq 0 \end{cases} \tag{4.5}$$

thereby determining whether the resource will be depleted ($\hat{X} = 0$), exploited but not depleted ($\hat{X} \in (0, X_0)$) or not exploited at all ($\hat{X} = X_0$).

From (4.4) and (4.5), we see that if $u'(0) \in (Z(X_0), Z(0)]$, then $\hat{X} = Z^{-1}(u'(0))$ is unconstrained ($L(\hat{X}) = 0$). Thus, the approach to $\hat{X}$ is asymptotic, with $T = \infty$ (cf. Proposition 3). If $u'(0) > Z(0)$ then $\hat{X} = 0$ is constrained ($L(\hat{X}) < 0$) and according to Proposition 4(i), depletion occurs at a finite time. In this case, the marginal benefit $u'(0)$ is sufficiently large to justify early depletion, but the resource is nonessential (since $u'(0) < \infty$). If $u'(0) \leq Z(X_0)$, the extraction cost exceeds the marginal benefit at all states and the resource does not admit profitable exploitation ($c = 0$, $\hat{X} = X_0$ and $T = 0$).

When $u'(c) \to \infty$ as $c \downarrow 0$, $L(X) = -\infty$ at all $X \in [0, X_0]$ and Proposition 1 implies depletion ($\hat{X} = 0$). Since $f_c(\hat{X}, M(\hat{X})) = u'(0) - Z(0) = \infty$, the resource is essential and Proposition 4(i) implies asymptotic depletion ($T = \infty$).

The special case in which the extraction cost is constant, say $Z(X) = z$, corresponds to Hotelling's (1931) model. In this case $L(X) = \rho[z - u'(0)]$ is constant and Proposition 1 implies that $\hat{X} = X_0$ or $\hat{X} = 0$ depending on whether $z \geq u'(0)$ or $z < u'(0)$, respectively. In the former case the resource does not admit profitable exploitation. In the latter case the steady state $\hat{X} = 0$ is constrained (since $L(0) < 0$) and according to Proposition 4, the steady state will be approached at a finite time or asymptotically depending on whether the resource is non-essential ($u'(0)$ is finite) or essential ($u'(0) = \infty$).

4.2 Renewable resources

The stock of a renewable resource evolves according to

$$\dot{X}(t) = R(X(t)) - c(t), \tag{4.6}$$

where the recharge function $R(\cdot)$ varies across resource types. In all cases we assume the existence of some state $\bar{X} > 0$ (corresponding to the resource maximal volume or carrying capacity) with $R(\bar{X}) = 0$ and $R(X) \leq 0$ for all $X > \bar{X}$. With exogenous recharge, e.g. precipitation feeding water sources, $R(\cdot)$ is typically decreasing and concave over $[0, \bar{X}]$. When the recharge is due to growth (regeneration) as in biomass resources (e.g. a fishery or forests) $R(\cdot)$ initially increases and reaches a peak at the maximum-sustainable-yield stock, X_{MSY}, and decreases thereafter to vanish at the carrying capacity stock $\bar{X}$.

The management problem entails finding the feasible policy $\{c(t), t \geq 0\}$ that maximizes (4.2) subject to (4.6), given the initial stock X_0. The constant-stock function is $M(X) = R(X)$ and equation (2.8) specializes to

$$L(X) = -[\rho - R'(X)][u'(R(X)) - Z(X)] - Z'(X)R(X)\,. \tag{4.7}$$

We discuss water and biomass resources in turn.

4.2.1 Water resources

Suppose that the recharge function is decreasing and concave, as is typically the case for water resources (see Tsur and Zemel 2004, and references cited therein). The state $\bar{X}$ represents a full reservoir, with $R(\bar{X}) = 0$ and $R'(\bar{X}) > -\infty$. Differentiating (4.7) gives

$$\begin{aligned} L'(X) &= R''(X)[u'(R(X)) - Z(X)] - \\ &\{[\rho - R'(X)][u''(R(X))R'(X) - Z'(X)] + Z''(X)R(X) + Z'(X)R'(X)\} \end{aligned} \tag{4.8}$$

The expression inside the curly brackets is positive. Setting $L(\hat{X}) = 0$ in (4.7) gives

$$u'(R(\hat{X})) - Z(\hat{X}) = \frac{-Z'(\hat{X})R(\hat{X})}{\rho - R'(\hat{X})} \geq 0\,, \tag{4.9}$$

hence the first term of (4.8) is negative at $X = \hat{X}$, ensuring that $L'(\hat{X}) < 0$ at any root $\hat{X}$. It follows that $L(\cdot)$ can have at most one root in $[0, \bar{X}]$, in which case the optimal steady state is uniquely identified by Proposition 1 as follows:

$$\hat{X} = \begin{cases} 0 & \text{if } L(0) < 0 \\ L^{-1}(0) & \text{if } L(0) \geq 0 \text{ and } L(\bar{X}) \leq 0 \\ \bar{X} & \text{if } L(\bar{X}) > 0 \end{cases}.$$

Denoting

$$\Lambda(X) \equiv \frac{-Z'(X)R(X)}{\rho - R'(X)} \geq 0\,, \tag{4.10}$$

we specify the steady state in terms of the model's primitives

$$\hat{X} = \begin{cases} 0 & \text{if } \; u'(R(0)) > Z(0) + \Lambda(0) \\ L^{-1}(0) & \text{if } u'(R(0)) \le Z(0) + \Lambda(0) \text{ and } u'(0) \ge Z(\bar{X}) \\ \bar{X} & \text{if } u'(0) < Z(\bar{X}) \end{cases} . \tag{4.11}$$

When $u'(R(0)) \le Z(0) + \Lambda(0)$ and $u'(0) > Z(\bar{X})$, the steady-state is unconstrained and, according to Proposition 3, is approached asymptotically, with $T = \infty$. When $u'(R(0)) > Z(0)+\Lambda(0)$, the steady-state $\hat{X} = 0$ is constrained ($L(0) < 0$) and the resource is nonessential (since $R(0) > 0$ and $Z(0) > 0$ ensure that $u'(R(0)) - Z(0) < \infty$). Thus, according to Proposition 4(i), depletion occurs at a finite time. Finally, when $u'(0) \le Z(\bar{X})$, the unit extraction cost exceeds the highest price the resource can obtain and the resource does not admit profitable exploitation. The constant-state rate $M(\bar{X}) = 0$ lies at the boundary of $\mathcal{C}$ hence, although the resource is nonessential at $\bar{X}$, the feasibility constraint $c \ge 0$ implies that the water reservoir is filled to its full capacity $\hat{X} = \bar{X}$ asymptotically (unless $X(0) = \bar{X}$, see Proposition 4(ii)).

The vanishing of $L(\cdot)$ at unconstrained steady states bears a simple economic interpretation. Writing (4.9) as

$$u'(R(\hat{X})) = Z(\hat{X}) + \Lambda(\hat{X}), \tag{4.12}$$

we recall that optimal management requires that at each point of time the marginal value of extraction equals the full cost of the resource, which consists of the unit extraction cost $Z(\cdot)$ plus the shadow price (scarcity rent, royalty, *in situ* value) of the resource. The second term on the right-hand side of (4.12), defined in (4.10), is the shadow price of the resource at the steady state: Increasing the stock by dX decreases the unit extraction cost by $-Z'(\hat{X})dX$ and the total extraction cost by $-Z'(\hat{X})R(\hat{X})dX$. The present value of this change in cost flow is imputed at the effective discount rate $\rho - R'(\hat{X})$ (which accounts also for the change in recharge rate $R(\cdot)$ due to the change in stock).

4.2.2 Biomass resources

For biomass resources, the function $R(\cdot)$ represents the natural growth rate which increases from $R(0) = 0$ at all $X \in [0, X_{MSY})$, reaches a peak at the maximum-sustainable-yield state X_{MSY} and decreases over $X \in (X_{MSY}, \bar{X}]$ until reaching zero again at the carrying capacity state $\bar{X}$. Over its decreasing domain, the function $R(\cdot)$ is concave.

If $L(\bar{X}) \geq 0$ then $u'(0) < Z(X)$ for all $X \in (0, \bar{X})$ and the resource does not admit profitable exploitation. To study harvesting policies, we suppose that $L(\bar{X}) < 0$. If $L(X) < 0$ for all $X > 0$ then, according to Proposition 1, the unique steady state is $\hat{X} = 0$, where the biomass resource is brought to extinction. If, in addition, $L(0) = 0$ then, according to Proposition 3, extinction occurs asymptotically. If $L(0) < 0$ then, according to Proposition 4-(i), extinction occurs asymptotically or at a finite time depending on whether the resource is essential (i.e., $u'(0)$ is unbounded) or nonessential ($u'(0)$ is finite).

When $L(0) > 0$ the function $L(\cdot)$ obtains at least one root in $(0, \bar{X})$, at which (4.12) holds. According to Propositions 1-2, only roots of $L(\cdot)$ at which $L'(\cdot) \leq 0$ are legitimate candidates for stable steady states. The approach to such (unconstrained) states is asymptotic (unless $X(0) = \hat{X}$). If only one such root exists, it is the steady state to which the optimal stock process converges from any $X(0) \in (0, \bar{X}]$. If multiple roots exist, depending on the specifications of $u(\cdot)$, $Z(\cdot)$ and $R(\cdot)$, each (legitimate) root may or may not have a nonempty basin of attraction. In some cases, a global maximum exists, which attracts the process from any initial state. In other cases, the optimal steady state varies with the initial stock. In such cases, the optimal root must be determined by evaluating the objective associated with each of the legitimate steady states, since the local analysis embodied in the $L-$function formalism cannot provide global information (see the discussion in Tsur and Zemel 2001).

4.3 Event uncertainty

The examples discussed above illustrate the application of the L-function in a simple and transparent manner. As these examples are well known, one may wonder whether the L-function method provides any value added beyond conventional methods. The following example shows that it does. It considers a renewable resource under risk of abrupt catastrophic event. This resource situation is considerably more involved than the examples discussed above, yet the optimal steady states and their sensitivity to the catastrophic threat, as well as whether the steady-state approach time is finite or asymptotic, are derived via the L-function method in much the same way and at the same level of simplicity.

Consider again a renewable resource but suppose that the exploitation policy may be interrupted at some future uncertain time T^e by a damaging, possibly catastrophic, event. Upon occurrence (at time T^e when the resource state is $X(T^e)$), the calamity inflicts the damage $\psi(X(T^e))$ and exploitation continues under the risk of another future occurrence and so on.[6] While it is impossible to prevent the event occurrence with certainty, the occurrence

[6] This situation was termed "recurrent events" in Tsur and Zemel (1998). Other catastrophic events, e.g., events that occur only once following which the resource may or may not admit profitable exploitation, can be analyzed with minor modifications.

probability can be controlled by the stock level X. Formally, take the hazard rate $h(\cdot)$ corresponding to T^e to be a function of X. The survival function $S(t) \equiv Prob\{T^e > t\}$ is related to the hazard rate process according to

$$S(t) = \exp\big[-\int_0^t h(X(\tau))d\tau\big]. \tag{4.13}$$

Let $v(X)$ denote the resource value at the initial time, given the initial state X. At time T^e the calamity hits, inflicting the damage $\psi(X(T^e))$, and the continuation value is $v(X(T^e)) - \psi(X(T^e))$. The payoff at $t = 0$ is

$$\int_0^{T^e} [u(c(t)) - Z(X(t))c(t)]e^{-\rho t}dt + e^{-\rho T^e}\left[v(X(T^e)) - \psi(X(T^e))\right].$$

Taking expectation with respect to the random occurrence time T^e gives the expected payoff

$$\int_0^{\infty} [u(c(t)) - Z(X(t))c(t) + h(X(t))(v(X(t)) - \psi(X(t)))]S(t)e^{-\rho t}dt. \tag{4.14}$$

The optimal policy is the feasible policy $\{c(t),\, t \geq 0\}$ that maximizes the expected payoff subject to (4.6), given the initial stock $X(0)$. This is a non-trivial optimization problem because the objective contains the value function, which is only implicitly defined.[7] Another difficulty arises due to the presence of $S(t)$ in the objective. This endogenous term, which depends on the exploitation policy up to the current time (see equation (4.13)), renders the problem non-autonomous or, alternatively, can be viewed as a second state variable satisfying $\dot{S}(t) = -h(X(t))S(t)$. Either way, the model does not formally belong to the class of models considered so far, namely single-state

[7] The value function satisfies

$$v(X(0)) = \max \int_0^{\infty} [u(c(t)) - Z(X(t))c(t) + h(X(t))(v(X(t)) - \psi(X(t)))]S(t)e^{-\rho t}dt$$

subject to (4.6), given $X(0)$, where the maximization is over all feasible policies. It is assumed that a solution exists and the corresponding value function is differentiable.

autonomous problems. Nevertheless, the corresponding L-function can be obtained and used to characterize properties of the optimal policy. The reason is that $L(\cdot)$ is derived by considering only slight variations from the constant state policy (see the proof of Proposition 1 in the Appendix) and under such policies the hazard h is constant and $S(t)$ reduces to a simple exponential, which resembles the standard discount factor, hence does not disturb the autonomous property in the region of interest. Thus, the following analysis not only shows how the method is applied to characterize the steady states of a challenging problem, but it also suggests that its applicability extends beyond the single-state, autonomous framework.

Let $W^{uc}(X)$ denote the expected payoff under the constant state policy $c = R(X)$ for some arbitrary feasible state X. Under this (not necessarily optimal) policy the survival function, defined in (4.13), reduces to the exponential $S(t) = \exp[-h(X)t]$. Moreover, to yield $W^{uc}(X)$ this constant state policy should hold both before *and after* occurrence, hence the continuation value corresponding to this policy is $W^{uc}(X) - \psi(X)$. Using these specifications in (4.14) gives

$$W^{uc}(X) = \frac{u(R(X)) - Z(X)R(X) + h(X)[W^{uc}(X) - \psi(X)]}{\rho + h(X)}. \tag{4.15}$$

Solving (4.15) for $W^{uc}(X)$ we find

$$W^{uc}(X) = W(X) - h(X)\psi(X)/\rho, \tag{4.16}$$

where $W(X) = [u(R(X)) - Z(X)R(X)]/\rho$ is the steady state value associated with the risk-free problem considered above. (Note the modification of the penalty $\psi(\cdot)$ by $h(X)/\rho$ – the number of occurrences over the infinite horizon, adjusted according to the discount rate ρ such that the contribution of each delayed occurrence is diminished by the corresponding discount factor.)

Let $L^{uc}(X)$ denote the L-function associated with the uncertain-event problem while $L(X)$ of (4.7) corresponds to the event-free problem. Applying (2.7) with W^{uc} replacing W, $f(X, c) = u(c) - Z(X)c + h(X)[W^{uc}(X) - \psi(X)]$ and $g(X, c) = R(X) - c$ gives

$$L^{uc}(X) = L(X) - \varphi'(X), \tag{4.17}$$

where

$$\varphi(X) \equiv h(X)\psi(X) \tag{4.18}$$

is the expected immediate catastrophic damage (or restoration cost). The list of optimal steady-state candidates is identified by applying Propositions 1 and 2 with $L^{uc}(\cdot)$ serving as the L-function. Likewise, the finite or asymptotic nature of the steady-state approach time is determined (for each candidate) by Propositions 3 - 4. Both tasks are carried out in exactly the same way as in the event-free case.

It is seen that the effects of uncertainty can be measured by the difference between the roots of $L(\cdot)$ and $L^{uc}(\cdot)$. For example, if the expected damage $\varphi = h\psi$ is independent of the stock, the two L-functions coincide and uncertainty does not affect the optimal steady state, no matter how large the expected penalty might be. In more realistic scenarios one expects the occurrence hazard to increase as the resource dwindles, i.e., $h'(X) < 0$. In such cases, retaining a stock-independent ψ implies $L^{uc}(X) > L(X)$ for all X. Recalling that $L(\cdot)$ decreases at its respective root, we conclude that $\hat{X}^{uc} > \hat{X}$. Thus, uncertainty implies a more conservative extraction policy. This effect is enhanced when $\psi(X)$ is decreasing and fades, possibly even reversed, when $\psi(X)$ is increasing.

5 Concluding comments

Two questions come up in any intertemporal planning problem: the first concerns the location of the final destination; the second deals with how long it takes to get there. The straightforward way to address these questions is to solve the underlying optimization problem and examine the long-term behavior of the state processes under the optimal policy. This approach may be quite cumbersome as it requires the parametric specification of all functions involved and often (when closed-form solutions are not available) resorts to numerical techniques. The alternative approach developed here addresses these questions for single-state processes via a simple algebraic method which avoids the solution of the underlying dynamic optimization problem.

The proposed method is based on the function $L(\cdot)$ of the state variable, which is used to formulate necessary conditions for optimal steady states. In many cases of interest, the necessary conditions narrow down the list of candidate steady states to a singleton, and the optimal state process converges to this unique state from any initial point. If several candidates exist, the slope of $L(\cdot)$ narrows further the list by identifying the stable (in the local sense of saddle-point stability) steady states. If more than one stable steady states exist, the optimal policy may depend on the initial state and the final choice is determined by comparing the corresponding objectives.

The value of $L(\cdot)$ at an optimal steady state determines whether the corresponding time of approach is finite or infinite. Steady states at which $L(\cdot)$ vanishes are always approached asymptotically. Otherwise, the time of approach depends on whether the constraints on the action (control) are binding at this state and on whether the stock is essential (i.e., the marginal benefit and $L(\cdot)$ diverge at the steady state) or non-essential.

Applying the method to a generic class of non-renewable (minerals) and renewable resource models, we show how it can locate the candidates for an optimal steady state and characterize the time of approach under the optimal policy. We also apply the L-methodology to the problem of resource management under threat of triggering an abrupt, catastrophic event. This latter problem is considerably more involved than the risk-free examples and it violates the restrictions imposed by the strict single-state autonomous framework. Yet, the L-function identifies the optimal steady state candidates, and the approach time to each candidate, in much the same way as in the other, simpler examples. The procedure can be similarly applied to other intertemporal planning models and is particularly useful in problems that do not easily lend themselves to a full dynamic characterization using standard methods.

To allow the derivation of sharp results and for the sake of clarity of presentation we introduce a rather restrictive set of assumptions in this work. In fact, a broader class of problems can be considered. An example of a linear control model is discussed in Tsur and Zemel (2001), where it is shown that the steady states can be located using the L-function, which can be interpreted in this case as the derivative of an equivalent utility function that depends only on the state X. The conditions for a stable steady state, then, imply the maximization of the equivalent utility. With linear control, however, the characterization of the approach times does not hold. Other possible extensions may follow the direction of the uncertain event problem considered here which suggests that the restriction to a single state might be relaxed in some cases. Investigating such extensions remains a challenge for future work.

Appendix: Proofs

This appendix presents the proofs of Propositions 1-4. It also illustrates the optimal constrained steady-state arrival times, characterized in Proposition 4, by means of simple examples of the resource management problem of Section 4 for which explicit solutions are readily available. These examples illuminate the underlying factors determining whether the steady-state approach time is finite or infinite.

A Proof of Proposition 1

For the sake of completeness, we reproduce here the variational argument used by Tsur and Zemel (2001) to prove Proposition 1:

Proof of Proposition 1. For any feasible state X we compare the constant-state value $W(X)$ obtained from the policy $c = M(X)$ with the value obtained from a small feasible variation from this policy. If the variation policy yields a value that exceeds $W(X)$, then the constant-state policy is not optimal at X and this state does not qualify as an optimal steady state. Choose the arbitrarily small constants $h > 0$ and δ and consider the following variation policy:

$$c^{h\delta}(t) = \begin{cases} M(X) + \delta/g_c(X, M(X)) & \text{if } \ t \leq h \\ M(X(h)) & \text{if } \ t > h \end{cases}.$$

For the short period $t \leq h$, this policy deviates slightly from the constant state policy, then it enters a steady state at $X(h)$. During the first period, $\dot{X} = \delta + o(h\delta)$ which brings the state at $t = h$ to $X(h) = X + h\delta + o(h\delta)$. The contribution of this period to the objective is evaluated (up to $o(h\delta)$) as[8]

$$\int_0^h f(X(t), M(X)+\frac{\delta}{g_c}) \exp(-\rho t) dt = \int_0^h \rho W(X) \exp(-\rho t) dt + \frac{f_c(X, M(X))}{g_c(X, M(X))} h\delta.$$

[8] $o(h\delta)$ indicates terms such that $o(h\delta)/(h\delta) \to 0$ as $h\delta \to 0$.

The contribution of the constant-state policy $c = M(X(h))$ during the period $t > h$ is approximated up to $o(h\delta)$ terms as

$$\int_h^\infty f(X(h), M(X(h))) \exp(-\rho t)dt = \int_h^\infty \rho W(X(h)) \exp(-\rho t)dt$$
$$= \int_h^\infty \rho W(X) \exp(-\rho t)dt + W'(X)h\delta.$$

Summing the contributions of the two periods gives the value $V^{h\delta}(X)$ obtained with $c^{h\delta}$. Recalling (2.8), we find

$$V^{h\delta}(X) - W(X) = L(X)h\delta/\rho + o(h\delta).$$

The sign of δ can be freely chosen, while $h > 0$. Now, if $L(X) \neq 0$ we can set sign(δ) =sign($L(X)$) which gives $V^{h\delta}(X) > W(X)$ and X is not an optimal steady state. Thus, only the roots of $L(\cdot)$ qualify as legitimate candidates for $\hat{X}$. The only possible exceptions are the bounds $\underline{X}$ and $\bar{X}$. Choosing $\delta > 0$ is not feasible at $\bar{X}$ because this policy would lead the process outside the feasible domain. It follows that $\bar{X}$ cannot be excluded as an optimal steady state if $L(\bar{X}) > 0$. A similar argument holds for the lower bound $\underline{X}$ if L is negative at this state. $\square$

B Preliminary derivations and definitions

We introduce some general derivations and definitions that are used in the following proofs. A steady state $\hat{X}$ at which the constraint $c \in \mathcal{C} \equiv [\underline{c}, \bar{c}]$ is binding (i.e., $M(\hat{X}) \in \partial\mathcal{C} \equiv \{\underline{c}, \bar{c}\}$) is referred to as c-constrained. A steady state $\hat{X}$ is called s-constrained if it falls on one of the bounds, $\underline{X}$ or $\bar{X}$, which is a result of a physical or a regulatory constraint (see discussion below assumption (2.3)). With $\lambda(t)$ denoting the current-value costate, the current-value Hamiltonian corresponding to the problem of maximizing the objective (2.2) subject to the dynamic constraint (2.1), given the initial state X_0, is

$$\mathcal{H} = f(X, c) + \lambda g(X, c). \tag{B.1}$$

The necessary conditions for (an interior) optimum include:

$$f_c(X, c) + \lambda g_c(X, c) = 0 \tag{B.2}$$

and

$$\dot{\lambda} - \rho\lambda = -[f_X(X, c) + \lambda g_X(X, c)]. \tag{B.3}$$

The arguments below consider local properties in the immediate vicinity of the steady states. Moreover, we shall be interested in such steady states in which the constant-state policy ($c = M(X)$) strictly outperforms any other policy, which excludes indeterminate states (such as those identified by Skiba 1978) as possible steady states, even if such indeterminacy is allowed away from the steady states. In the neighborhood of such steady states, the optimal control of infinite-horizon, autonomous problems can be expressed as a single-valued function of the state, say $c(t) = C(X(t))$. Since the Hamiltonian is strictly concave in c (assumption (2.3)), we show that $C(\cdot)$ is continuous and

$$C(\hat{X}) = M(\hat{X}). \tag{B.4}$$

We begin by verifying (B.4), using the property that the optimal state process converges to the steady state hence the objective (2.2) can be written as

$$\max_{\{T, c(t)\}} \int_0^T f(X(t), c(t)) e^{-\rho t} dt + e^{-\rho T} W(\hat{X}),$$

where it is recalled that T is the steady state entrance time and the steady state value $W(X) = f(X, M(X))/\rho$. The transversality condition corresponding to the free choice of T is

$$e^{-\rho T}\left[\mathcal{H}(\hat{X}, c(T), \lambda(T)) - f(\hat{X}, M(\hat{X}))\right] = 0.$$

When T is finite, the term inside the squared brackets above vanishes and a straightforward Taylor expansion around $c = M(\hat{X})$, using (B.1), (B.2), $g(\hat{X}, M(\hat{X})) = 0$ and the strict concavity of $\mathcal{H}$ in c, implies that $c(T) = M(\hat{X})$, verifying (B.4). When $T = \infty$, the properties that $X(t) \to \hat{X}$ monotonically and $\dot{X}(t) = g(X(t), c(t)) \to 0 = g(\hat{X}, M(\hat{X}))$ imply, recalling that $g(\cdot, \cdot)$ is continuous in both arguments, that $c(t) \to M(\hat{X})$, which again gives (B.4). Next, recall that all other states along the optimal trajectory are arrived at some finite time t. Moreover, the strict concavity of $\mathcal{H}$ in c ensures that $c(\cdot)$ is continuous in time at any finite t (see Fleming and Rishel 1975, Theorem 6.2, p. 76), hence the corresponding process $C(\cdot)$ is continuous in X.

We turn now to consider the case in which the steady state $\hat{X}$ is not c-constrained, i.e., $M(\hat{X})$ lies in the interior of $\mathcal{C}$ and, noting (B.4), $C(X) \notin \partial\mathcal{C}$ for all X in some vicinity of $\hat{X}$. In this vicinity, the feasibility constraints on c can be ignored and the interior optimum is obtained from the necessary conditions (B.2)-(B.3).

Define the functions

$$A(X) = g_c(X, C(X)) f_{cc}(X, C(X)) - f_c(X, C(X)) g_{cc}(X, C(X)), \tag{B.5}$$

$$B(X) = g_c(X, C(X)) f_{cX}(X, C(X)) - f_c(X, C(X)) g_{cX}(X, C(X)) \tag{B.6}$$

and

$$\psi(X, c) = -(\rho - g_X(X, c)) f_c(X, c)/g_c(X, c) - f_X(X, c). \tag{B.7}$$

Then, (B.2)-(B.3) imply

$$\dot{\lambda} = \psi(X, C(X)), \tag{B.8}$$

while (2.8) reduces to

$$L(X) = -\psi(X, M(X)). \tag{B.9}$$

Taking the time derivative of (B.2) and using (B.8) to eliminate $\dot{\lambda}$, we find

$$C'(X)\frac{A(X)}{g_c^2(X, C(X))} + \frac{B(X)}{g_c^2(X, C(X))} + \frac{\psi(X, C(X))}{g(X, C(X))} = 0. \tag{B.10}$$

Equation (B.10) is a first order differential equation, which together with (B.4) defines $C(X)$ for all X in the relevant neighborhood. Indeed, for $X \neq \hat{X}$ the coefficient of $C'(X)$ is positive and finite while the other two terms of (B.10) are finite, hence the derivative $C'(X)$ is well defined. A difficulty with its evaluation at $\hat{X}$ arises because the function $g(\cdot, \cdot)$, appearing at the denominator of the last term, vanishes at $\hat{X}$. In order to address this problem, we distinguish between unconstrained steady states, where $L(\hat{X}) = 0$, and constrained steady states, where $L(\hat{X}) \neq 0$. We show that in the former case $C'(\hat{X}) < \infty$ while in the latter case this derivative may indeed diverge even though $C(\cdot)$ approaches the finite limit $M(\hat{X})$ (see (D.4) and the discussion following it).

C Unconstrained steady states

C.1 Stability

Proof of Proposition 2. In an unconstrained steady state, $L(\hat{X}) = 0$ and the singularity of the last term of (B.10) at $\hat{X}$ is removed because $\psi(\hat{X}, C(\hat{X})) = \psi(\hat{X}, M(\hat{X})) = -L(\hat{X}) = 0$ (cf. (B.9)) This term, then, can be evaluated using l'Hôpital's rule. Using (B.9), we find

$$\frac{d\psi(\hat{X}, C(\hat{X}))}{dX} = -L'(\hat{X}) + \psi_c(\hat{X}, C(\hat{X}))[C'(\hat{X}) - M'(\hat{X})],$$

while (2.5) implies

$$\frac{dg(X,C(X))}{dX} = g_X(X,C(X)) + g_c(X,C(X))C'(X) = g_c(X,C(X))[C'(X) - M'(X)].$$

It follows that

$$\lim_{X\to\hat{X}} \left\{ \frac{\psi(X,C(X))}{g(X,C(X))} \right\} = \frac{1}{g_c(\hat{X},C(\hat{X}))} \left(\frac{-L'(\hat{X})}{C'(\hat{X}) - M'(\hat{X})} + \psi_c(\hat{X},C(\hat{X})) \right).$$

The last term on the right hand side is obtained by taking the derivative of (B.7) with respect to c,

$$\psi_c(X,C(X)) = -A(X)\frac{\rho - g_X(X,C(X))}{g_c^2(X,C(X))} - \frac{B(X)}{g_c(X,C(X))},$$

which reduces (B.10) in the limit $X \to \hat{X}$ to

$$\frac{A(\hat{X})}{g_c(\hat{X},C(\hat{X}))} \left(C'(\hat{X}) - M'(\hat{X}) - \frac{\rho}{g_c(\hat{X},C(\hat{X}))} \right) + \frac{-L'(\hat{X})}{C'(\hat{X}) - M'(\hat{X})} = 0.$$

Denoting

$$\Delta(X) \equiv C'(X) - M'(X), \qquad \text{(C.1)}$$

we obtain the quadratic equation

$$\Delta^2(\hat{X}) - \frac{\rho}{g_c(\hat{X},C(\hat{X}))}\Delta(\hat{X}) - \frac{g_c(\hat{X},C(\hat{X}))L'(\hat{X})}{A(\hat{X})} = 0. \qquad \text{(C.2)}$$

To determine which of the solutions of (C.2) corresponds to the stable steady-state slope-difference $\Delta(\hat{X})$, observe that the state $\hat{X}$ is attractive only if $\Delta(\hat{X}) \geq 0$. To see this, consider a state just below the steady state, say $X_\varepsilon = \hat{X} - \varepsilon$. To approach $\hat{X}$ from below requires $\dot{X} = g(X_\varepsilon, C(X_\varepsilon)) > 0$. Recalling that $g(X_\varepsilon, M(X_\varepsilon)) = 0$ and $g_c < 0$, this implies $C(X_\varepsilon) < M(X_\varepsilon)$, while $C(\hat{X}) = M(\hat{X})$. Thus, $C'(\hat{X}) \geq M'(\hat{X})$ and $\Delta(\hat{X}) \geq 0$.

Next, we write the solutions of (C.2) as

$$\Delta(\hat{X}) = \frac{\rho}{-2g_c(\hat{X},C(\hat{X}))} \left(-1 \pm \sqrt{1 + \frac{4L'(\hat{X})g_c^3(\hat{X},C(\hat{X}))}{\rho^2 A(\hat{X})}} \right). \qquad \text{(C.3)}$$

Since $g_c < 0$ and $A(\hat{X}) > 0$, the argument of the square-root operator above does not fall short of unity only if $L'(\hat{X}) \leq 0$. In this case, we have one non-negative solution for $\Delta(\hat{X})$ which can provide the boundary value $C'(\hat{X}) = M'(\hat{X}) + \Delta(\hat{X})$ for the differential equation (B.10). In contrast, if $L'(\hat{X}) > 0$, the argument falls short of unity and the two solutions of (C.3) are either negative or complex, hence (B.10) does not yield a solution that converges to $\hat{X}$. This rules out the possibility that $L'(\hat{X}) > 0$ at a stable steady state, verifying Proposition 2. □

We now verify equation (3.1), thus establishing the relation between the slope $L'(\hat{X})$ and the saddle-point stability of $\hat{X}$.

Proof of Equation (3.1). Expressing the value of the control obtained from (B.2) as $c(X, \lambda)$, the state and costate equations (2.1) and (B.3) obtain the canonical form

$$\dot{X} = g(X, c(X, \lambda)),$$

$$\dot{\lambda} = \rho\lambda - \mathcal{H}_X(X, c(X, \lambda), \lambda),$$

where $\mathcal{H}$ is the current value Hamiltonian given in (B.1). The Jacobian matrix associated with the above canonical pair is

$$J = \begin{pmatrix} g_X - g_c \frac{\mathcal{H}_{cX}}{\mathcal{H}_{cc}} & -\frac{g_c^2}{\mathcal{H}_{cc}} \\ \frac{\mathcal{H}_{cX}^2}{\mathcal{H}_{cc}} - \mathcal{H}_{XX} & \rho - g_X + g_c \frac{\mathcal{H}_{cX}}{\mathcal{H}_{cc}} \end{pmatrix} \tag{C.4}$$

(see Wirl and Feichtinger 2005, Equations (8-9)) with the determinant

$$\det(J) = \frac{\mathcal{H}_{cc} g_X (g_X - \rho) - \mathcal{H}_{cX} g_c (2g_X - \rho) + \mathcal{H}_{XX} g_c^2}{-\mathcal{H}_{cc}} . \tag{C.5}$$

While the derivation of (C.5) is naturally carried out in terms of X and λ, it is convenient to evaluate the partial derivatives of $\mathcal{H}$ that show up in this equation in terms of the arguments X and c, using the fact that in the vicinity of the steady state (hence away from any Skiba point) we can express both λ and c as functions of the state X alone. Thus, we use (B.2) again to write $\lambda = -f_c(X, C(X))/g_c(X, C(X))$ which gives the partial derivatives of $\mathcal{H}$ in terms of the functions $A(\cdot)$ and $B(\cdot)$ of (B.5)-(B.6) and the associated function

$$Q(X) = g_c(X, C(X))f_{XX}(X, C(X)) - f_c(X, C(X))g_{XX}(X, C(X)) \tag{C.6}$$

as

$$\mathcal{H}_{cc} = \frac{A(X)}{g_c(X, C(X))}\,; \quad \mathcal{H}_{cX} = \frac{B(X)}{g_c(X, C(X))}\,; \quad \mathcal{H}_{XX} = \frac{Q(X)}{g_c(X, C(X))}\,. \tag{C.7}$$

Thus, omitting the arguments of the derivatives of g for the sake of brevity,

$$\det(J) = \frac{A(X)g_X(g_X - \rho) - B(X)g_c(2g_X - \rho) + Q(X)g_c^2}{-A(X)}. \tag{C.8}$$

Next, we use (2.5) to evaluate the derivative of $L(\cdot)$ as expressed in (2.8), which gives (after some tedious but straightforward algebraic manipulations)

$$L'(\hat{X}) = \frac{A(\hat{X})g_X(g_X - \rho) - B(\hat{X})g_c(2g_X - \rho) + Q(\hat{X})g_c^2}{g_c^3}. \tag{C.9}$$

Comparing with (C.8), using $X = \hat{X}$ and $C(\hat{X}) = M(\hat{X})$ in the arguments of the various functions, we find

$$L'(\hat{X}) = \frac{A(\hat{X})}{-g_c^3}\det(J). \tag{C.10}$$

Recalling that $A(\cdot) > 0$ and $g_c(\cdot,\cdot) < 0$ (see assumption (2.3)) verifies (3.1). $\square$

C.2 Time of approach

Proof of Proposition 3. We now show that an unconstrained steady state $\hat{X}$, at which $L(\hat{X}) = 0$ and $L'(\hat{X}) \leq 0$, cannot be approached at a finite time, i.e., $T = \infty$ (except, of course, for the special case where $X(0) = \hat{X}$ which gives

$T = 0$). Suppose to the contrary, that T is finite. Using the solution $C(\cdot)$ of (B.10), the optimal state trajectory $X(t)$ can be obtained implicitly for any $t \in [0, T]$ from the solution of (2.1):

$$T - t = \int_{X(t)}^{\hat{X}} \frac{dx}{g(x, C(x))} \ . \tag{C.11}$$

Assume, for the sake of concreteness, that $X(t) < \hat{X}$, so the state process increases toward $\hat{X}$, i.e., $\dot{X}(s) = g(X(s), C(X(s))) > 0$ during $s \in [t, T)$. Since $X(t) \to \hat{X}$, for every $\varepsilon > 0$ there exists some time t_ε such that $\hat{X} - X(t) < \varepsilon$ for all $t_\varepsilon \le t \le T$. Denote $X_\varepsilon = X(t_\varepsilon)$ and consider the case $\Delta(\hat{X}) > 0$. Then, for all $X \in [X_\varepsilon, \hat{X}]$

$$\begin{aligned} g(X, C(X)) &= g(\hat{X}, C(\hat{X})) + [g_X(\hat{X}, C(\hat{X})) + g_c(\hat{X}, C(\hat{X}))C'(\hat{X}) + O(\varepsilon)](X - \hat{X}) \\ &= g(\hat{X}, M(\hat{X})) + [-g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X}) + O(\varepsilon)](\hat{X} - X) \\ &\le -2g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})(\hat{X} - X), \end{aligned}$$

where the last inequality follows when ε is chosen to be sufficiently small so that[9] $O(\varepsilon) < -g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})$. Thus

$$T - t_\varepsilon > \frac{1}{-2g_c(\hat{X}, M(\hat{X}))\Delta(\hat{X})} \int_{X_\varepsilon}^{\hat{X}} \frac{dx}{\hat{X} - x} \ . \tag{C.12}$$

The integral on the right side of (C.12) diverges for every $X_\varepsilon < \hat{X}$, contradicting the assumption that T is finite.

When $\Delta(\hat{X}) = 0$, we have $g(X, C(X)) = O(\varepsilon)(X - \hat{X})$, hence $T - t_\varepsilon$ is proportional to $1/\varepsilon$ times the integral of (C.12) which obviously diverges. The case of decreasing state processes, with $X(t) > \hat{X}$ and $g < 0$ is treated in a similar manner. $\square$

[9] $O(\varepsilon)$ denotes terms such that $O(\varepsilon)/\varepsilon$ is bounded when $\varepsilon \to 0$.

D Constrained steady states

When the optimal steady state is constrained, i.e, $L(\hat{X}) \neq 0$ and the steady falls on one of the corners ($\underline{X}$ or $\bar{X}$), the above derivation of T cannot be applied because the term $\psi(\hat{X}, C(\hat{X}))/g(\hat{X}, C(\hat{X}))$ of (B.10) diverges since g vanishes while $L = -\psi$ does not. It follows that $C'(\hat{X})$, hence also $\Delta(\hat{X})$, diverges so the right hand side of (C.12) does not necessarily yield an infinite value. Indeed, if all the functions and derivatives in (2.3) are continuous and bounded at the constrained steady state, then T obtains a finite value. If, however, f_c diverges at the steady state, then T may diverge as well.

Proof of Proposition 4(i): essential stocks. Recall that the stock can be essential at a constrained steady state only when the latter falls on the lower bound $\underline{X}$. Suppose that $\hat{X} = \underline{X}$, and $f_c(\underline{X}, M(\underline{X})) = \infty$. Since $L(\underline{X})$ must be negative at this steady state, we find from (2.8) that $L(\hat{X}) = -\infty$, so that $m \equiv \rho - g_X(\underline{X}, M(\underline{X})) > 0$. Write, recalling (B.2) and (B.7),

$$\frac{\psi}{\lambda} = \rho - g_X + \frac{f_X g_c}{f_c}$$

and observe that the first two terms on the right-hand side approach the constant m while the third term shrinks to 0 when $X(t) \to \underline{X}$. It follows that close enough to the steady state, $\dot{\lambda}/\lambda \approx m$ hence $\lambda(t) \approx \tilde{\lambda}\exp(mt)$, where $\tilde{\lambda}$ is some positive constant. Fast as this exponential growth may be, it cannot take the $\lambda(\cdot)$ process to its target value $\hat{\lambda} = -f_c(\underline{X}, M(\underline{X}))/g_c(\underline{X}, M(\underline{X})) = \infty$ within a finite period of time. We conclude that $T = \infty$ in this case. $\square$

As an example, consider the simplest nonrenewable resource management problem obtained under the specifications $f = c^\beta$ with $0 < \beta < 1$ and $g = -c$, subject to $X(t) \geq 0$, given $X(0) = X_0$. In this problem, $M(X) = 0$ for all X, so $f_c(X, M(X)) = \beta M(X)^{\beta-1} = \infty$ and the resource is essential. Moreover, $L(X) = -\rho\beta M(X)^{\beta-1} = -\infty$ for all X. Thus, the lower bound $\underline{X} = 0$ is the unique steady state. Let $\alpha \equiv \rho/(1-\beta) > 0$. It is easy to verify that the optimal processes for this problem are given by

$$c(t) = \alpha X_0 e^{-\alpha t}$$

$$X(t) = X_0 e^{-\alpha t}\,.$$

Although this solution converges to the corner state with $L(0) \neq 0$, we have $T = \infty$ in this case, due to the divergence of $f_c(0,0)$. Indeed, this solution implies $C(X) = \alpha X$, which is consistent with the value $C'(X) = \alpha$ obtained from (B.10) with the specifications $A = \beta(1-\beta)C^{\beta-2}$, $B = 0$, $\psi = \rho\beta C^{\beta-1}$, $g = -C$ and $g_c = -1$. With $M'(0) = 0$, $\Delta(0)$ is finite and the argument based on (C.12) establishes the asymptotic approach to the s-constraint steady state.

Proof of Proposition 4(i): non-essential stocks. Consider, first, the case in which the upper bound $\bar{X}$ is a steady state where $L(\bar{X})$ is positive but finite. We show that T is finite in this case. Suppose otherwise, that T is infinite. Since all the functions listed in assumption (2.3) are continuous and bounded as $X(t) \to \bar{X}$ monotonically, then for any $\varepsilon > 0$ there exists some time t_ε such that $|\psi(X(t), C(X(t))) - \psi(\bar{X}, C(\bar{X}))| \leq \varepsilon$ for all $t > t_\varepsilon$. Recalling (B.9) and $C(\bar{X}) = M(\bar{X})$, we find $\psi(X(t), C(X(t))) \leq \varepsilon - L(\bar{X})$. Choosing $\varepsilon = L(\bar{X})/2 > 0$, equation (B.8) implies $\dot{\lambda} \leq -L(\bar{X})/2$ for all $t > t_\varepsilon$. Such a constant decrease in $\lambda(\cdot)$ cannot continue indefinitely because it would bring the shadow price process below the finite steady state value $\hat{\lambda} = -f_c(\bar{X}, M(\bar{X}))/g_c(\bar{X}, M(\bar{X}))$ in a finite time period, so the end conditions would be violated. We conclude that T must be finite.

The treatment of the case in which the lower bound $\underline{X}$ is a steady state where $L(\underline{X})$ is negative but finite is similar. If $T = \infty$, one finds that $\dot{\lambda}$ is larger than the positive constant $-L(\underline{X})/2$ for all $t > t_\varepsilon$, implying that $\lambda(\cdot)$ exceeds the finite steady state value $\hat{\lambda} = -f_c(\underline{X}, M(\underline{X}))/g_c(\underline{X}, M(\underline{X})$ after a finite time, violating the end conditions. The steady-state entrance time T, then, must be finite also in this case. □

As an explicit example for the characterization of a non-essential resource, consider again the nonrenewable resource with $g = -c$ but change the specification of the benefit function to $f = \beta c - c^2/2$, where $\beta > 0$ is a given constant. We find again $M(X) = 0$ hence $f_c(X, M(X)) = \beta < \infty$ and the resource is not essential. For any state X, $L(X) = -\rho\beta < 0$ hence only the lower bound $\underline{X} = 0$ can be an optimal steady state. The optimal policy in this case is given by

$$c(t) = \begin{cases} \beta - \lambda_0 e^{\rho t} & \text{if } \ t < T \\ 0 & \text{if } \ t \geq T \end{cases}, \tag{D.1}$$

where the constants λ_0 and T are determined by the conditions

$$\lambda_0 e^{\rho T} = \beta \quad \leftrightarrow \quad c(T) = 0; \qquad X_0 - \beta T + \lambda_0 \int_0^T e^{\rho t} dt = 0 \quad \leftrightarrow \quad X(T) = 0.$$

Eliminating λ_0, we determine T implicitly from the equation

$$e^{-\rho T} = 1 + \frac{\rho X_0}{\beta} - \rho T \tag{D.2}$$

which admits a unique finite solution $0 < T < X_0/\beta + 1/\rho$, while

$$\lambda_0 = \beta + \rho X_0 - \beta\rho T. \tag{D.3}$$

The $C(\cdot)$ function is expressed implicitly as

$$\rho[X(t) - X_0] = C(X_0) - C(X(t)) - \beta \log\left[\frac{\beta - C(X(t))}{\beta - C(X_0)}\right],$$

where $C(X_0) = c(0) = \beta - \lambda_0$. Taking the derivative with respect to X we find

$$C'(X)\frac{C(X)}{\beta - C(X)} = \rho. \tag{D.4}$$

It follows that $C'(0) = \infty$ because $C(0) = c(T) = 0$. Thus, the argument based on (C.12) to establish an asymptotic approach is not valid in this case.

Proof of Proposition 4(ii). We consider the case in which $\hat{X}$ is c-constrained, with $L(\hat{X}) \neq 0$ and $C(X) = M(\hat{X})$ for all X in some vicinity of $\hat{X}$ and $M(\hat{X}) \in \partial\mathcal{C}$ is determined by the c-constraint, while $|g_X(\hat{X}, M(\hat{X}))| < \infty$. We assume, without loss of generality, that $\hat{X} = \bar{X}$ so that the process increases towards $\hat{X}$ with $g(\cdot,\cdot) > 0$. Thus, $g_X(\hat{X}, M(\hat{X}))$ is negative and finite in this case. Consider the vicinity $\hat{X} - X < \varepsilon$ for some $\varepsilon > 0$ and write

$$g(X, C(X)) = g(X, M(\hat{X})) = g(\hat{X}, M(\hat{X})) + g_X(\tilde{X}, M(\hat{X}))[X - \hat{X}]$$

for some $\tilde{X} \in (X, \hat{X})$, hence

$$0 < g(X, C(X)) = [g_X(\hat{X}, M(\hat{X})) + O(\varepsilon)][X - \hat{X}] < -2g_X(\hat{X}, M(\hat{X}))[\hat{X} - X].$$

Repeating the arguments used to derive (C.12), we obtain

$$T - t_\varepsilon > \frac{1}{-2g_X(\hat{X}, M(\hat{X}))}\int_{X_\varepsilon}^{\hat{X}} \frac{dx}{\hat{X} - x},$$

and the integral diverges for every $X_\varepsilon < \hat{X}$, hence $T = \infty$. The case $\hat{X} = \underline{X}$ with $g(\cdot,\cdot) < 0$ and $g_X(\cdot,\cdot) > 0$ is treated in the same manner. □

Consider, for example, the water resource management problem of the previous section with the specification $R(X) = \bar{X}^2 - X^2$ and $Z(\bar{X}) > u'(0)$. As shown above, the high unit cost of water extraction leaves no room for profitable exploitation and the stock approaches the upper bound $\bar{X}$ with

$c = 0$ hence the steady state is c-constrained. Under this policy, the state evolution is governed by $\dot{X} = \bar{X}^2 - X^2$ which is readily integrated, yielding

$$X(t) = \bar{X}\frac{(\bar{X} + X_0)\exp(2\bar{X}t) - (\bar{X} - X_0)}{(\bar{X} + X_0)\exp(2\bar{X}t) + (\bar{X} - X_0)},$$

hence $T = \infty$ in agreement with the proposition. Observe that $C'(\bar{X}) = 0$ even though the resource is not essential at this c-constrained steady state, in contrast to the divergence of $C'(\cdot)$ for non-essential resources at their s-constrained boundaries. This difference underlies the different characterization of the corresponding arrival times.

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