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Abstract

Several contributions have recently reconsidered the role of the time to build assumption in explaining some relevant stylized facts. In this paper, the similarities and differences which may emerge when the time to build structure of capital is introduced in a continuous or discrete time framework are studied and enlightened. The most striking difference lies in the *dimensionality* of the two frameworks, which is always finite in discrete but infinite in continuous time. Then, the deterministic version of the traditional time to build model developed by Kydland and Prescott is presented, and it is shown how the typical time to build model setup in continuous time can be obtained. Moreover, the richest dynamics in continuous time is investigated and, more importantly, it is shown that the predictions in terms of capital, output, and consumption behavior are not significantly different from its discrete version once the economy is calibrated properly.
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1 Introduction

Several contributions have recently reconsidered the role of the time to build assumption in explaining some relevant stylized facts. In a real business cycle framework, for example, Casares [12] shows how the introduction of multiple types of capital, each of them with different time to build structure, can reproduce the responses of the main variables of the economy to a monetary policy shock. Gomme et al. [17] are able to solve two typical puzzles rising in household production models through the introduction of a time to build structure of capital. Again, time to build is critical in Edge [14] to generate empirically plausible liquidity effect in a sticky price monetary business cycle model.

Also in the economic growth literature, the time to build hypothesis has relevant implications as shown by Bambi [4], and earlier by Asea and Zak [2]. It has been shown indeed that the presence of gestation lag may induce (deterministic) cycles through a Hopf bifurcation both in exogenous and endogenous growth models. Moreover, some empirical evidences, like the negative correlation between mean output growth and output growth volatility, are explained quite well even through a very simple AK model.

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However, the time to build structure of capital is introduced in the real business cycle and in the economic growth literature in different ways. This is probably due to the different choices of time characterizing them. Gestation lag in investment is the key that Kalecki [20] uses to open the door of continuous time models to time to build. On the other hand, Kydland and Prescott [23] suggest multi-period investment projects as the main way to introduce time to build in a discrete time framework. Time to build is also introduced for different purposes: it is a nonstandard source of aggregate fluctuation in growth theory, while a way to enhance the persistence of the cycle (in a similar way as adjustment cost) in the Kydland and Prescott contribution. Observe that the word "time to build" is used to describe two concepts so different that Kydland and Prescott never refer or quote the Kalecki's contribution in their paper.

This difference is even more magnified when we look at the mathematics behind the two frameworks. In continuous time, the capital accumulation equation is a delay differential equation while the intertemporal consumers choices are described by a Euler - type equation which is an advanced differential equation. Then, the dimensionality of the problem is always *infinite*, independently by the specification of the time to build parameter. On the other hand, the dimensionality of the model in discrete time is exactly equal to twice the value of the time to build parameter, which is in any case *finite*. As a consequence, the optimal equilibrium path of the aggregate variables in the two frameworks have different functional forms.

This paper proposes a unified approach to models with time to build in production. Several striking connections emerge by a parallel comparison of these two worlds. In particular, the pure investment lag in production model developed by Kydland and Prescott [23] has a continuous time representation, which is exactly the baseline model studied in Asea and Zak [2], and in Bambi [4]. Then, a clear relation between gestation lag and multi-period investment projects is detected.

Once proved this connection, the deterministic versions of both models are studied and the implications of the time to build assumption on the short run and long run equilibrium properties of the economy are fully developed. It emerges that the most relevant difference is in the transitional dynamics since a Hopf bifurcation may rise in the continuous time framework while it has a zero probability measure to appear in a discrete time world. Keeping in mind this difference, it is shown that capital, output and consumption don't display neither qualitative nor relevant quantitative differences in the two contexts, once the economy is properly calibrated. Then this paper suggests that in absence of any kind of market distortions and independently by the different dimensionality of the two problems, it exists a clear connection between time to build model in discrete time and continuous time very similar to that one identified by Turnovsky [31] in models without time to build.

The paper is organized as follows. Section 2 describes the time-to-build economy as suggested by Kydland and Prescott [23]. In section 3 we prove the connection between the pure investment lag economy in discrete time and the time to build and gestation lag in production in continuous time. Section 4 propose a steady state analysis, Section 5 compares the two framework in term of their transitional dynamics while in Section 6 the economy dynamics is presented once the parameters are calibrated. Section 7 concludes.

2 Baseline model

The model presented here is the purely deterministic version of the Kydland and Prescott [23] model. The time to build hypothesis is introduced by considering various investment technologies, which differ in the amount of time they take to build the capital good.

Thus, the representative agent seeks to maximize his lifetime utility:

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \tag{1}$$

subject to the technology constraint and the information structure

$$\begin{aligned} Y_t &= AF(K_t, N_t) && (2)\\ Y_t &= C_t + I_t && (3)\\ S_{j,t+1} &= S_{j+1,t} && (4)\\ I_t &= \sum_{j=1}^{J} b_j S_{j,t} && (5)\\ K_{t+1} &= (1-\delta_K)K_t + S_{1,t} && (6) \end{aligned}$$

$u(.)$ and $F(.)$ are concave, increasing, and twice continuously differentiable functions of consumption, C_t and leisure L_t, and capital K_t and labor input N_t, respectively. $\beta \in (0,1)$ is the utility discount factor, while $\delta \in (0,1)$ the depreciation rate of capital. Moreover, $\sum_{j=1}^{J} b_j = 1$, and the initial conditions are K_0 and $S_{j,0}$ with $j = 1, ..., J-1$. After observing that

$$I_t = \sum_{j=1}^{J} b_j S_{j,t} = \sum_{j=1}^{J} b_j S_{1,t+j-1} = \sum_{j=1}^{J} b_j \left[K_{t+j} - (1-\delta_K)K_{t+j-1}\right] \tag{7}$$

it is possible to sum up all the constraints in one, and rewriting the optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \tag{8}$$

subject to

$$\begin{aligned} \sum_{j=1}^{J} b_j \left[K_{t+j} - (1-\delta_K)K_{t+j-1}\right] &= AF(K_t, L_t) - C_t && (9)\\ K_0, ..., K_{J-1} &\geq 0 \quad given && (10) \end{aligned}$$

After doing that, we can construct the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(C_t, L_t) + \mu_t \left(AF(K_t, L_t) - \sum_{j=1}^{J} b_j \left[K_{t+j} - (1-\delta_K)K_{t+j-1}\right] - C_t\right)\right] \tag{11}$$

with $\lambda_t = \beta^t \mu_t$, Lagrangian multipliers. Taking the first order conditions respect to C_t, L_t, and K_{t+J}, and after some algebra (look at the Appendix for derivations), we get an intra-temporal substitution condition between consumption and labor

$$\frac{u_L(C_t, L_t)}{u_C(C_t, L_t)} = AF_L(K_t, L_t) \tag{12}$$

and a forward looking Euler-type equation

$$\frac{u_C(C_t,L_t)}{u_C(C_{t+J},L_{t+J})}=\frac{\beta^J}{b_J}\left[AF_K(K_{t+J},L_{t+J})+b_1(1-\delta_K)-\sum_{j=1}^{J-1}[b_j-b_{j+1}(1-\delta_K)]\frac{u_C(C_{t+J-j},L_{t+J-j})}{\beta^j u_C(C_{t+J},L_{t+J})}\right] \tag{13}$$

Investment projects are usually designed in two alternative ways: a pure investment lag in production or a design proposed by Kydland and Prescott, according to which the fraction of resources, b_j, allocated to the investment project are the same at each stage. These two experiments are summed up in the following conditions:

Condition 1 *Pure investment lag in the investment process implies that $b_j = 0$ for any $j \neq J$ and, without loss of generality, $b_J = 1$.*

Then following Condition 1, equations (9) and (13) become:

$$K_{t+J} = AF(K_t,L_t)+(1-\delta_K)K_{t+J-1}-C_t \tag{14}$$

$$\frac{u_C(C_t,L_t)}{u_C(C_{t+J},L_{t+J})} = \beta^J\left[AF_K(K_{t+J},L_{t+J})+(1-\delta_K)\frac{u_C(C_{t+1},L_{t+1})}{\beta^{J-1}u_C(C_{t+J},L_{t+J})}\right] \tag{15}$$

On the other hand Condition 2 states that

Condition 2 *Kydland and Prescott investment project implies that $b_j = \frac{1}{J}$ for any $j \neq J$.*

In this case we can rewrite equations (9) and (13) as:

$$K_{t+J} = AJF(K_t,L_t)+K_t-\sum_{j=0}^{J-1}\delta_K K_{t+j}-JC_t \tag{16}$$

$$\frac{u_C(C_t,L_t)}{u_C(C_{t+J},L_{t+J})} = \beta^J\left[AJF_K(K_{t+J},L_{t+J})+1-\delta\sum_{j=0}^{J-1}\frac{u_C(C_{t+J-j},L_{t+J-j})}{\beta^j u_C(C_{t+J},L_{t+J})}\right] \tag{17}$$

In the following section we will focus on the relation between the discrete and continuous time specification for the case of pure investment lag in production.

3 Pure investment lag in production

We focus our analysis on the case of pure investment lag in production. Before proceeding with the analysis of the steady state and of the dynamics we present in the following proposition the continuous time version of the model.

Proposition 1 *The continuous time problem which corresponds to the discrete time Kydland and Prescott model with pure investment lag in production is*

$$\max\int_0^\infty U(C(t),L(t))e^{-\rho t}dt \tag{18}$$

subject to

$$\dot{K}(t+J)=AF(K(t),L(t))-\delta K(t+J)-C(t) \tag{19}$$

given the initial condition $K(t)=K_0(t)$ for $t\in[0,J]$.

Proof. Consider Kydland and Prescott version of the model when the period length is h.

$$\max \sum_{s=t}^{\infty} \left(\frac{1}{1+\rho h} \right)^{\frac{s-t}{h}} U(C_s, L_s)$$

subject to

$$K_{t+J} - K_{t+J-h} = AhF(K_t, L_t) - \delta h K_{t+J-h} - hC_t$$

where changes in productivity, production and depreciation are proportionated to period length, as suggested in Obstfeld and Rogoff [26], while the investment project is now described by the relations $S_{j,t+h} = S_{j+h,t}$, and $I_t = S_{J,t}$. Moreover if $h = 1$, and $\beta = \frac{1}{1+\rho}$ we come back to the original setting of Kydland and Prescott.

Observe now that dividing both sides of the resources constraint by h and taking the limit for $h \to 0$, we obtain exactly the continuous time resource constraint described in equation (19). It is worth noting that the time to build structure is preserved by this operation only when the length of period h is reduced but let the frequency of time (quarters, years, etc.) invariant. More precisely suppose a quarterly frequency of time, then h has to be chosen as a smaller and smaller fraction of a quarter.[1]

The Euler equation of the discrete problem with period length, h, is

$$u_C(C_t, L_t) = \left(\frac{1}{1+\rho h} \right) (1 + \delta h) u_C(C_{t+h}, L_{t+h}) + \left(\frac{1}{1+\rho h} \right)^{\frac{J}{h}} u_C(C_{t+J}, L_{t+J}) AF(K_{t+J}, L_{t+J}) \tag{20}$$

which again remains coherent with Kydland and Prescott specification for $h = 1$. On the other hand, if we rearrange the terms and take the limit of $h \to 0$ we have that:

$$\lim_{h \to 0} \frac{u_C(C_{t+h}, L_{t+h}) - u_C(C_t, L_t)}{h} = \lim_{h \to 0} u_C(C_{t+h}, L_{t+h}) \left[\frac{\rho}{1+\rho h} + \frac{\delta}{1+\rho h} - \left(\frac{1}{1+\rho h} \right)^{\frac{J}{h}} \frac{u_C(C_{t+J}, L_{t+J})}{u_C(C_{t+h}, L_{t+h})} AF(K_{t+J}, L_{t+J}) \right] \tag{21}$$

Now using the chain rule on the left hand side of the equation, taking into account the limit properties, and remembering that $\left[\lim_{h \to 0} (1 + \rho h)^{\frac{1}{h}} \right]^{-J} = e^{-J\rho}$, we obtain the continuous time version of the Euler equation

$$\frac{\dot{C}(t)}{C(t)} = - \frac{U_C(C(t), L(t))}{U_{CC}(C(t), L(t)) C(t)} \left[\frac{U_C(C(t+J), L(t+J))}{U_C(C(t), L(t))} AF_K(K(t+J), L(t+J)) e^{-\rho J} - \delta - \rho \right] \tag{22}$$

This is exactly the Euler equation of the continuous time problem (18), (19). In fact, as shown by Kolmanovskii and Myshkis [21], the Hamiltonian associated to (18), (19) is

$$\mathcal{H}(t) = U(C(t), L(t)) e^{-\rho t} + \lambda(t) \left[AF(K(t-J), L(t-J)) - \delta K(t) - C(t-J) \right],$$

with first order conditions:

$$U_C(C(t), L(t)) e^{-\rho t} = \lambda(t+J) \tag{23}$$

$$\lambda(t+J) AF_K(K(t), L(t)) = -\dot{\lambda}(t) + \delta \lambda(t) \tag{24}$$

from which the Euler equation (22) can be easily derived taking into account that $\frac{\dot{\lambda}(t+J)}{\lambda(t+J)} = \frac{U_{CC}(C(t),L(t))\dot{C}(t)}{U_C(C(t),L(t))} - \rho$, and $\frac{\lambda(t+J)}{\lambda(t)} = \frac{U_C(C(t),L(t))}{U_C(C(t-J),L(t-J))} e^{-\rho J}$. ■

[1] Suppose that the reduction of the time length is not frequency invariant and we have multiple types of capital with different time to build parameters as in Casares [12]. In this case when h goes to zero all the delays go to infinity and the difference among capitals disappears.

It is also worth noting that the Kydland and Prescott model with period length h, contains all the developed specifications of time to build. In fact when h shrinks to zero, the stock of capital becomes

$$K(t) = \int_{-\infty}^{t+J} I(s) e^{\delta(s-t-J)} ds$$

which is exactly the same relation used in Asea and Zak [2], and Bambi [4]. On the other hand, gestation lags as defined by Kalecki [20], can be easily exploited by calling $I(t) = \dot{K}(t+J) - \delta K(t+J)$. [2]

4 Steady state analysis

In this section we deal with the properties of the long run equilibrium both in the continuous and in the discrete time setup.

Proposition 2 *The time to build structure affects the steady state level of consumption and capital in the same way, independently by the choice of time.*

Proof. We start with the discrete time case. In equilibrium we have that equations (14), and (15) becomes

$$\begin{aligned} K^* &= AF(K^*, L^*) + (1-\delta)K^* - C^* && (25) \\ 1 &= \beta^J AF_K(K^*, L^*) + \beta(1-\delta) && (26) \end{aligned}$$

and by applying implicit function theorem on (26) we observe that

$$\frac{\partial K^*}{\partial J} = -\frac{\log(\beta) F_K(K^*, L^*)}{F_{KK}(K^*, L^*)} < 0 \quad (27)$$

where the inequality follows by the usual assumption on positive but diminishing marginal product of each factor. Moreover by observing that $\frac{\partial C^*}{\partial J} = \frac{\partial C^*}{\partial K^*} \cdot \frac{\partial K^*}{\partial J}$ it follows that there exists a negative relation between the time to build parameter and the steady state level of consumption and capital.

Now let's consider the capital accumulation equation and the Euler equation in the continuous time setup. The following relations hold in equilibrium:

$$\begin{aligned} AF(K^*, L^*) - \delta K^* - C^* &= 0 && (28) \\ -\frac{U_C(C^*, L^*)}{U_{CC}(C^*, L^*)C^*} \left[AF_K(K^*, L^*) e^{-\rho J} - \delta - \rho \right] &= 0 && (29) \end{aligned}$$

and by applying implicit function theorem we get

$$\frac{\partial K^*}{\partial J} = -\frac{-\rho F_K(K^*, L^*)}{F_{KK}(K^*, L^*)} < 0 \quad (30)$$

For the same considerations used in the discrete case, consumption in steady state reacts negatively to positive variation of the delay parameter. ■

Then this proposition shows that an increase in the time to build parameter, J, influences the long run equilibrium of the economy negatively, by increasing the time needed to produce output. Moreover this negative influence is independent by the choice of time.

[2] More precisely, Kalecki's *"gestation period"* of any investment starts with the investment orders, $I(t-J)$ and finishes with the deliveries of finished industrial equipments, $L(t)$. Finally in Kalecki's terminology $K'(t) = L(t) - U$ where U indicates the demand for restoration of equipments used up per unit of time.

5 Transitional Dynamics

In this section we deal with the transitional dynamics of the economy in the two frameworks: the continuous and the discrete one. In particular we focus first on a particular form of the model, namely the Hansen model [19], with indivisible labor supply, then we come back to the original Kydland and Prescott model. One of the reasons of this choice is the easier analytical tractability in the continuous time case of the Hansen model, since the system of equations describing the economy is now *recursively* solvable. A second, that this model is very well known and it is considered by many the model to beat for its simplicity and its power in explaining relevant stylized facts of the business cycle. In both models we assume a Cobb-Douglas production function and separable utility in consumption and leisure, $U(C(t), N(t)) = \log C(t) + B\frac{1-N(t)^{1-\chi}}{1-\chi}$. Hansen's specification requires $\chi = 0$.

5.1 Dynamics in continuous time

The system of equations describing the local dynamics of the economy around the steady state can be derived by linearizing the first order conditions near the steady state and solving the associated linearized system. Following this procedure we arrive to a system of linear mixed functional differential equations in $\log K(t) = k(t)$, and $\log C(t) = c(t)$:

$$\dot{k}(t) = \frac{(\chi-1)(\delta+\rho)}{b+\chi-1}e^{\rho J}k(t-J) - \delta k(t) - \left[\frac{\delta+\rho}{a}e^{\rho J}\left(\frac{\chi-1}{b+\chi-1}\right) - \delta\right]c(t-J) \quad (31)$$

$$\dot{c}(t) = (\delta+\rho)c(t) - \frac{(\delta+\rho)(\chi-1)}{b+\chi-1}c(t+J) - \frac{(\rho+\delta)b\chi}{b+\chi-1}k(t+J) \quad (32)$$

where we have used small letters to indicate variables in log term.

Proposition 3 *The continuous time version of the Hansen model with time to build has a local dynamics characterized by unique oscillatory convergence of capital and unique monotonic convergence of consumption when* $J \in \left(0, \frac{3a\pi}{2(\delta+\rho)}e^{-\rho J}\right)$.

Proof. The spectrum of roots of the system is given by the union of the spectrum of roots of (31), and (32) which are described respectively by their characteristic equations:

$$h_k(\lambda) = \lambda - \frac{\delta+\rho}{a}e^{\rho J}e^{-\lambda J} + \delta = 0 \quad (33)$$

$$h_c(z) = z + \frac{\delta+\rho}{a}e^{zJ} - \delta - \rho = 0 \quad (34)$$

From the D-Subdivision method (see Appendix), the sufficient condition for $h_c(z)$ to have one negative real root and all the other complex conjugates and with positive real part is $J \in \left(0, \frac{3\pi a}{2(\delta+\rho)}\right)$.

Using the finite Laplace transform method developed in Belmann and Cooke ([6], pp 197-205), it is possible to write any solution of (32) as

$$c(t) = \lim_{\ell\to\infty}\sum_{z_m\in C_\ell} a_m e^{z_m t} \quad (35)$$

where the contour C_ℓ, with $\ell = 1, 2, \ldots$, cuts the negative and positive imaginary axis to the point $-y_\ell$, and y_ℓ, respectively. Moreover a_m is the residue of the root z_m and depends on the boundary (initial) condition of consumption. Then the general convergent solution of consumption can be obtained by ruling out all the roots with positive real part, imposing $a_{z_m} = 0 \; \forall m \neq \tilde{m}$, and then rewriting (35) as

$$c(t) = a_{\tilde{m}} e^{\tilde{z}t} \tag{36}$$

with $\tilde{z}$ the negative real root of $h_c(z)$.

On the other hand, the spectrum of $h_k(\lambda)$ is characterized by one positive real root and all the other complex conjugates and with negative real part when $J \in \left(0, \frac{3\pi a}{2(\delta+\rho)} e^{-\rho J}\right)$.

The general solution of capital (Belmann and Cooke [6], and Bambi [4]) is

$$\begin{aligned} k(t) &= \sum_v n_v e^{\lambda_v t} - \int_0^t c(s) \sum_v \frac{e^{\lambda_v (t-s)}}{h'_{\lambda_v}} ds & (37) \\ &= \sum_v n_v e^{\lambda_v t} - \int_0^t a_{\tilde{m}} e^{\tilde{z}s} \sum_v \frac{e^{\lambda_v (t-s)}}{h'_{\lambda_v}} ds & (38) \\ &= \sum\nolimits_v (n_v + P_{\tilde{m},v}) e^{\lambda_v t} - \sum\nolimits_v P_{\tilde{m},v} e^{\tilde{z}t} & (39) \end{aligned}$$

where $P_{\tilde{m},v} = \frac{a_{\tilde{m}}}{(\tilde{z}-\lambda_v)h'(\lambda_v)}$. Moreover capital convergence requires that $a_{\tilde{m}} = \hat{a}$ with

$$\hat{a} = (\tilde{z} - \lambda_{\tilde{v}}) h'(\lambda_{\tilde{v}}) n_{\tilde{v}} \tag{40}$$

which is the only condition to rule out the divergent root coming from $h_k(z)$ and to make the transversality condition hold. Then, the real unique oscillatory equilibrium path of capital is (see Appendix):

$$k(t) = \left(n_{\tilde{v}} - 2 \sum_{v \neq \tilde{v}} \hat{a} \Phi_{1,v} \right) e^{z_{\tilde{v}} t} + 2 \sum_{v \neq \tilde{v}} \left[(\varsigma_v + \hat{a}\Phi_{1,v}) \cos y_v t - (\omega_v + \hat{a}\Phi_{2,v}) \sin y_v t \right] e^{x_v t} \tag{41}$$

where $\lambda_v = x_v + iy_v$, $n_v = \varsigma_v + i\omega_v$, and $\Phi_{1,v}$ and $\Phi_{2,v}$ are constants. ■

Moreover, it is also worth noting that a Hopf bifurcation emerges when $J = J^* = \frac{3\pi a}{2(\delta+\rho)} e^{-\rho J^*}$ since two conjugate complex roots of the characteristic equation $h_k(\lambda)$ cross the imaginary axis. The local determinacy of the equilibrium is still preserved by the unique specification of the residue $\hat{a}$.

Finally the continuous time specification even with the discrete component represented by the delay, leaves the results of the Proposition above, invariable to different frequency of calibration. Following Aaland and Huang (JEDC 2004), we have indeed verified how changing the frequency of calibration doesn't affect significantly the ratio $\frac{J}{J^*}$.

The Gary Hansen model is completely studied under the time to build assumption. We introduce now the following proposition which described the transitional dynamics when the indivisible labor supply assumption is relaxed.

Proposition 4 *The continuous time version of the Kydland and Prescott model has a local dynamics characterized by unique oscillatory convergence of capital and consumption when* $J \in (0, J^*)$

Proof. The local stability properties of the economy remain the same for continuous variation of the parameters in the Jacobian matrix as underlined by Rustichini [29]. This implies that by increasing continuously the inverse of the elasticity of labor supply, χ, the uniqueness of the optimal equilibrium path of consumption and capital is guaranteed. However, $\chi \neq 0$ implies that the Jacobian matrix is no more lower triangular and as shown in the Appendix, the optimal solution path now writes

$$k(t) = \sum_{z_r \in C_\ell^-} b_{z_r} e^{z_r t} \tag{42}$$

$$c(t) = \frac{1}{\eta_{12}} \sum_{z_r \in C_\ell^-} b_{z_r} \left[z_r - \eta_{11} e^{-z_r J} \right] e^{z_r t} \tag{43}$$

with $\eta_{11} = \left[aA\frac{1-\chi}{a-\chi} - \delta \right] e^{-z_r J}$ and $\eta_{12} = A\frac{1-\chi}{\chi-a} + \delta$, while all the other $z_r \in C_\ell^+$ are ruled out by the transversality condition. This implies an oscillatory convergence for both capital and consumption as appears clear from the trigonometric form of (42), and (43):

$$k(t) = \varsigma_{\tilde{r}} e^{x_{\tilde{r}} t} + 2 \sum_{r \neq \tilde{r}} (\varsigma_r \cos y_r t - \omega_r \sin y_r t) e^{x_r t} \tag{44}$$

$$c(t) = \frac{1}{\eta_{12}} \varsigma_{\tilde{r}} \left(x_{\tilde{r}} - \eta_{11} e^{-x_{\tilde{r}} J} \right) e^{x_{\tilde{r}} t} + \frac{2}{\eta_{12}} \sum_{r \neq \tilde{r}} (\Phi_1 \cos y_r t - \Phi_2 \sin y_r t) e^{x_r t} \tag{45}$$

where $\Phi_1 = \varsigma_r x_r - \omega_r y_r - \eta_{11}(\varsigma r \cos y_r J + \omega_r \sin y_r J) e^{-x_r J}$ and $\Phi_2 = (\varsigma_r y_r + \omega_r x_r) + \eta_{11}(\varsigma r \sin y_r J - \omega_r \cos y_r J) e^{-x_r J}$. ■

5.2 Dynamics in discrete time

Now we focus on the discrete time version of the model. Here it is more difficult to give a general statement on the behavior of the dynamics of the economy since the choice of J determines the dimensionality of the problem itself.[3] The local dynamics can be deduced by a non naive linear quadratic approximation as originally developed by Fleming [15], and recently revised by Benigno and Woodford [7], or by linearizing the first order conditions near the steady state and solving the associated linearized system. In both cases the economy is described by the following system of linear difference equations:

$$c_{t+J} = \frac{\beta(1-\delta)(b+\chi-1)}{[1-\beta(1-\delta)](1-\chi)} c_{t+1} - \frac{b+\chi-1}{[1-\beta(1-\delta)](1-\chi)} c_t + \frac{b\chi}{1-\chi} k_{t+J} \tag{46}$$

$$k_{t+J} = (1-\delta) k_{t+J-1} - \frac{[1-\beta(1-\delta)](1-\chi)}{(b+\chi-1)\beta^J} k_t + \left(\frac{[1-\beta(1-\delta)](1-\chi)}{a(b+\chi-1)\beta^J} + \delta \right) c_t \tag{47}$$

We proceed now to study the stability properties of the economy. Since the Grandmont & al., [18], geometrical method for investigating the stability of a dynamical system cannot be applied when the dimension is higher than two, we introduce the following condition.

Condition 3 *The Blanchard-Kahn condition [9] is always respected when $J \in [1, J^*]$. Moreover the characteristic equation of consumption has only one root inside the unit circle when the system is recursively solvable and J is in the previously specified interval.*

[3] More precisely a time to build J implies a linearized capital accumulation equation and a linearized Euler equation described by two linear difference equations of order J.

This condition guarantees the uniqueness of the optimal equilibrium path, since the number of roots outside the unit circle is equal to the number of predetermined variables. As we will see in the numerical simulation part, this condition is always met for any plausible value of the parameters.

Proposition 5 *Given Condition 3, the discrete time version of the Gary Hansen model with time to build has a local dynamics characterized by unique oscillatory convergence of capital and unique not oscillatory convergence of consumption when* $J \in [3, J^*]$, *while unique not oscillatory convergence in consumption and capital when* $J \in [1, 2]$.

Proof. When labor supply is assumed indivisible, the system can be solved recursively as in the continuous time case. Then we can consider the characteristic equations of (46), and (47) which write respectively

$$h_c(z) \equiv z^J + \frac{a\beta(1-\delta)}{1-\beta(1-\delta)} z - \frac{a}{1-\beta(1-\delta)} = 0 \quad (48)$$

$$h_k(\lambda) \equiv \lambda^J - (1-\delta)\lambda^{J-1} - \frac{1-\beta(1-\delta)}{a\beta^J} = 0 \quad (49)$$

with the properties that $h_c(0) < 0$ and $h_k(0) < 0$ since $\beta < \frac{1}{1-\delta}$ given that $\beta < 1$ and $\delta \in (0, 1)$; $\lim_{z\to\pm\infty} h_c(z) = +\infty$ and $\lim_{\lambda\to\pm\infty} h_k(\lambda) = +\infty$ when J is even, while $\lim_{z\to\pm\infty} h_c(z) = \pm\infty$ and $\lim_{\lambda\to\pm\infty} h_k(\lambda) = \pm\infty$ when J is odd.

When $J = 2$ the first two properties are sufficient to guarantee the presence of four real roots, precisely one negative and one positive raising from each characteristic equation, and then the optimal equilibrium path is not oscillatory and, for condition 3, also unique.

When $J > 2$, other two properties have to be considered. First, $h_c(z)$ has no critical point if J is odd, but a negative one, z^*, if J is even and in this case z^* is a local minimum. Second, $h_k(\lambda)$ has two critical points $\lambda_1^* = 0$ and $\lambda_2^* > 0$, which are respectively a flex and a local minimum, when J is even; and a local maximum and a local minimum,when J is odd.

Taking into account all these properties, it follows immediately that for J even $h_c(z)$ has one real root inside the unit circle and all the others (one real and $J-2$ complex and conjugate) outside it. On the other hand for J odd, $h_c(z)$ has one real root inside the unit circle and all the others $J-1$ complex and conjugate outside it.

The general solution of consumption can be derived by the linearized Euler equation:

$$c_t = \sum_{j=1}^{J} a_j z_j^t \quad with \quad t > J \quad (50)$$

where the a_j are constants whose values depends on the initial condition of consumption $\{c_t\}_{t=0}^J$ which can be arbitrarily chosen. Taking into account Condition 3, consumption convergence over time is possible if and only if $\{a_j\}_{j=2}^J = 0$. But then from the method of the undetermined coefficients, it follows immediately that the solution of consumption becomes

$$c_t = c_0 z_1^t \quad with \quad t > J \quad (51)$$

and then no oscillatory behavior in consumption may rise. Moreover taking into account the solution of consumption we may rewrite the linearized capital accumulation equation as a nonhomogeneous J order difference equation in capital:

$$k_{t+J} = (1-\delta)k_{t+J-1} + \frac{1-\beta(1-\delta)}{a\beta^J}k_t - \left(\frac{1-\beta(1-\delta)}{a^2\beta^J} - \delta\right)c_0 z_1^t \tag{52}$$

It follows that the general solution of (52) is $k_t = k_{hom.} + k_p$, namely

$$k_t = \sum_{j=1}^{J} b_j\lambda_j^t + c_0\Psi z_1^t \quad with \;\; t > J \tag{53}$$

with $\Psi = \frac{\frac{1-\beta(1-\delta)}{a^2\beta^J}-\delta}{z_1^J-(1-\delta)z_1^{J-1}-\frac{1-\beta(1-\delta)}{a\beta^J}}$. Suppose now that J is odd, which implies only one real root of $h_k(z)$, which for Condition 3 has to be outside the unit circle. Applying De Moivre's theorem on the complex roots, we can write

$$k_t = b_1\lambda_1^t + \sum_{j=1}^{\frac{J-1}{2}} r_{2j}^t\left[(b_{2j}+b_{2j+1})\cos tw_{2j} + i(b_{2j}-b_{2j+1})\sin tw_{2j}\right] + c_0\Psi z_1^t \quad with \;\; t > J \tag{54}$$

In order to guarantee a real path of capital, the coefficients have to be complex and conjugate, namely $b_{2j} = x_{2j} + iy_{2j}$, and $b_{2j+1} = \bar{b}_{2j}$ for every j. Then the general solution of capital rewrites

$$k_t = b_1\lambda_1^t + 2\sum_{j=1}^{\frac{J-1}{2}} r_{2j}^t\left[x_{2j}\cos tw_{2j} - y_{2j}\sin tw_{2j}\right] + c_0\Psi z_1^t \quad with \;\; t > J \tag{55}$$

Finally, the optimal general solution of capital can be obtained by ruling out the explosive root λ_1 by imposing $b_1 = 0$, and then using the method of the undetermined coefficients together with the initial condition of capital $k_0 = k_1 = \ldots = k_J = \bar{k}$ in order to find the J coefficients $(\{x_{2j}, y_{2j}\}_{j=1}^{\frac{J-1}{2}}, c_0)$. Then the unique competitive equilibrium of the economy when $t > J$ and J is odd, writes

$$c_t = \tilde{c}_0 z_1^t \tag{56}$$

$$k_t = 2\sum_{j=1}^{\frac{J-1}{2}} r_{2j}^t\left[\tilde{x}_{2j}\cos tw_{2j} - \tilde{y}_{2j}\sin tw_{2j}\right] + \tilde{c}_0\Psi z_1^t \tag{57}$$

where the terms in tilde are found through the method of the undetermined coefficients. Moreover in the case of J even the optimal general solution of capital becomes

$$k_t = \tilde{b}_J\lambda_J^t + 2\sum_{j=1}^{\frac{J}{2}-1} r_{2j}^t\left[\tilde{x}_{2j}\cos tw_{2j} - \tilde{y}_{2j}\sin tw_{2j}\right] + \tilde{c}_0\Psi z_1^t \tag{58}$$

Hence, it is clear that consumption converges to its steady state always monotonically while oscillation in the capital equilibrium path rises for $J > 2$. ■

Once we move to the divisible labor supply case, and assuming Condition 3 still satisfied, the optimal equilibrium path of both capital and consumption is oscillatory as extensively documented by Rouwenhorst [28].

6 Quantitative Comparison after Calibration

The parameters in these simulations are chosen in line with Kydland and Prescott *quarterly* calibration. In particular the capital's share in production, a, is set to 0.36, the depreciation rate of capital to 10 per cent per year while an inverse labor elasticity close to zero (Hansen model). Finally β is chosen to yield a return on capital of 6.5 percent per year. This means a rate of intertemporal preference $\rho \approx 0.01$ when $J = 4$. This calibration of the time to build parameter is in line with Kydland and Prescott [23], but also with Rouwenhorst [28], and more recently with Casares [12]. The empirical contributions of Montgomery [25], and Brooks [11], estimates a time to build between 4-6 and 8 quarters, respectively.

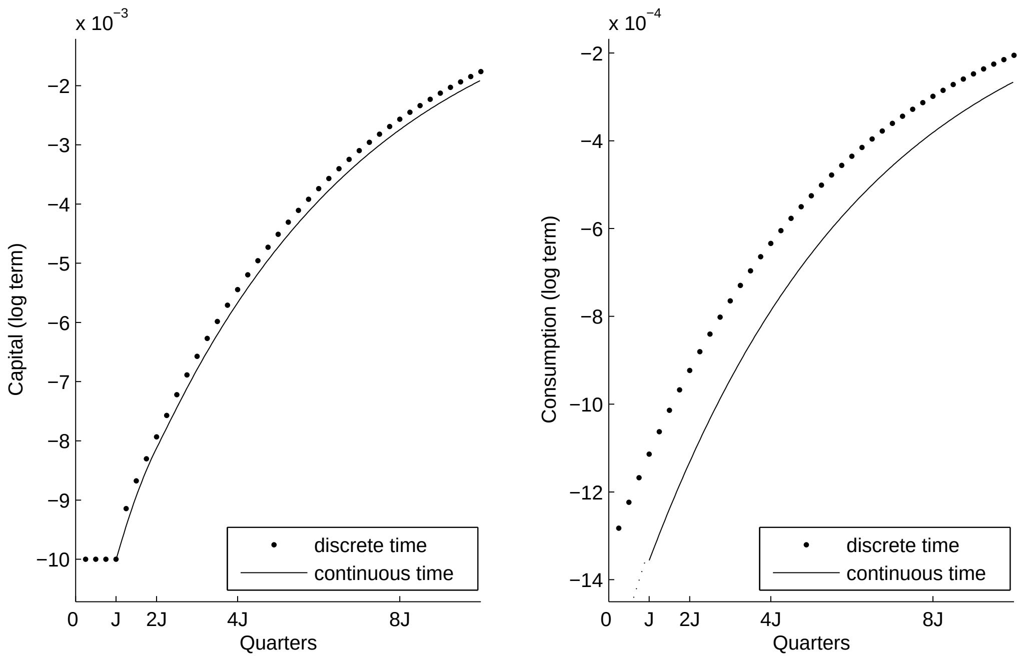

Figure 1: Optimal Equilibrium Path of Capital and Consumption.

Once calibrated the economy, the eigenvalues of the two characteristic equations, $h_c(z)$, and $h_k(\lambda)$ have been computed through a polynomial roots finder algorithm in the discrete time case, and their moduli reported in the Appendix, table 1. It is worth noting that the requirements in Condition 3 are always met when the delay parameter is less than two years. On the other hand, it has also been observed that by increasing it further, local instability may rise even in the discrete time framework. In particular for $J = 36$ the characteristic equation of capital have three roots outside the unit circle. A similar change in stability happens in the continuous time case, for $J = 35.36$, which is the critical value of the delay parameter for an Hopf bifurcation.

Computation of the roots in the case of continuous time has been done using Lambert function since a polynomial roots finder cannot be used here, given the transcendental nature of the characteristic equations.[4]

Once the spectrum of roots has been estimated, we have studied the optimal equilibrium path of capital and consumption in both the frameworks taking into account the results in Proposition 3 and 4. In Figure 1 the results are reported for the choice of $J = 4$. As it appears clear the dynamics behavior of these variables doesn't display significantly relevant differences. In both cases, the dominant real root hides the oscillatory components. Volatility around the dominant root has been also studied by measuring the capital standard deviation in the first 3J periods, both in the continuous and the discrete case. It has been found that $\sigma_k^{cont.} > \sigma_k^{discr.}$, and that the detrended capital dynamics oscillate around its steady state value with variation between 1.27% and −2.55% in the continuous case while between 0.5% and −0.2% in the discrete time case. Very similar results have been found for the other values of the delay parameter suggested by Montgomery [25], and Brooks [11].

7 Conclusion

Time to build is a concept which has attracted a renewed interest among economists in these years. Models in continuous and discrete time have been built up and have succeeded in explaining several interesting stylized facts. However no clear connection among these two types of models has been detected before. This paper has filled this gap by enlightening the similarities among these two apparently different worlds. More precisely, it has been proved the passage from a traditional Kydland and Prescott model to its continuous time version, whose baseline structure is exactly the same used by Kalecki [20] and more recently by Asea and Zak [2], and Bambi [4]. Finally a qualitative and quantitative analysis has shown that the choice of time and then the difference in the dimensionality induced by this choice, doesn't induce any relevant distinction in term of long run and short run dynamics in time to build models when no market imperfection is considered. Relaxing this last assumption is left as future research.

Appendix

First order conditions

$$u_C(C_t, L_t) = \mu_t \tag{59}$$

$$u_L(C_t, L_t) = \mu_t A F_L(K_t, L_t) \tag{60}$$

$$\mu_{t+J} A F_K(K_{t+J}, L_{t+J}) = \sum_{j=1}^{J} b_j \beta^{-j} \mathcal{E}^{J-j} \mu_t \left[1 - (1-\delta_K)\mathcal{E}\beta\right] \tag{61}$$

where $\mathcal{E}$ indicates the shift operator. It is worth noting that these FOC are exactly equals to the standard one when we assume no time lag in the investment process ($J = 1, b_J = 1$). In

[4]More details on Lambert function can be found in Asl and Ulsoy [3]. In this context, we have computed the first $m = 16$ branches and from them the corresponding roots. A higher number of computed branches doesn't induce relevant differences in the results.

order to derive a Euler-type equation we rewrite (61), in the following way:

$$\mu_{t+J}\left[AF_K(K_{t+J},L_{t+J})+b_1(1-\delta_K)\right]-\mu_t b_J\beta^{-J}-\sum_{j=1}^{J-1}[b_j-b_{j+1}(1-\delta_K)]\mathcal{E}^{J-j}\mu_t\beta^{-j}=0 \quad (62)$$

where the sum is well defined for $J>1$ but zero otherwise.

Spectrum of Roots - Continuous Time

In this part of the appendix, we give some details on how to locate the roots of the characteristic equation $h_c(z)$, and $h_k(\lambda)$. Let start with

$$h_c(-w)\equiv w-\frac{\delta+\rho}{a}e^{-wJ}+\delta+\rho=0 \quad (63)$$

By the D-Subdivision method (look at Bambi [4], Figure 1), only one positive real root exists when $-\frac{\delta+\rho}{a}<0$; $\left|-\frac{\delta+\rho}{a}\right|>\delta+\rho$; $-\frac{\delta+\rho}{a}\in\left(-\frac{3\pi}{2J},0\right)$.

The first two requirements are met since $\delta\in(0,1)$, $\rho>0$, and $a\in(0,1)$. The last requirement is satisfied if and only if $J<\frac{3\pi a}{2(\delta+\rho)}$. Following the same procedure we can show the spectrum properties of $h_k(\lambda)$.

How to arrive at the solution of capital (41)

After imposing $a_{\tilde{m}}=\hat{a}$, the solution of capital writes

$$k(t)=\sum\nolimits_{v\neq\tilde{v}}(n_v+P_v)e^{\lambda_v t}-\sum\nolimits_{v}P_v e^{\tilde{z}t} \quad (64)$$

where as shown in Bellman and Cooke [6] the residues are

$$n_v=\frac{\bar{k}+\bar{k}(\lambda_v+\delta)\int_{-J}^{0}e^{-\lambda_v s}ds}{1+J\Lambda e^{-\lambda_v J}}=\frac{\bar{k}\left[1+(\lambda_v+\delta)\left(-\frac{1}{\lambda_v}\right)(1-e^{\lambda_v J})\right]}{1+J\Lambda e^{-\lambda_v J}} \quad (65)$$

with $k(t)=\bar{k}\in\mathbb{R}^+$ in $t\in[-J,0]$, and $\Lambda=\frac{\delta+\rho}{a}e^{\rho J}$.

Now let's call $\varsigma_v=Re(n_v)$, $\omega_v=Im(n_v)$, $\lambda=x+iy$ and $\bar{\lambda}$ its conjugate, we rewrite

$$\sum_v P_v=-n_{\tilde{v}}+\hat{a}\sum_{v\neq\tilde{v}}\left(\frac{1}{(\tilde{z}-\lambda_v)h'(\lambda_v)}+\frac{1}{(\tilde{z}-\bar{\lambda}_v)h'(\bar{\lambda}_v)}\right) \quad (66)$$

which, after some algebra and taking into account the relation $\bar{\lambda}e^{-\bar{\lambda}J}+\lambda e^{-\lambda J}=2[x\cos yJ+y\sin yJ]e^{-xJ}$, becomes

$$\sum_v P_v=-n_{\tilde{v}}+2\hat{a}\sum_{v\neq\tilde{v}}\Phi_{1,v} \quad (67)$$

with

$$\Phi_{1,v}=\frac{(\tilde{z}-x_v)(1+J\Gamma e^{-x_v J}\cos y_v J)-J\Gamma y_v e^{-x_v J}\sin y_v J}{(\tilde{z}^2-2\tilde{z}x_v+x_v^2+y_v^2)(1+J^2\Gamma^2 e^{-2Jx_v}+2J\Gamma e^{-x_v J}\cos y_v J)} \quad (68)$$

Now let's study the second piece of the solution of capital, namely

$$\sum_{v\neq\tilde{v}}(n_v+P_v)e^{\lambda_v t} = \sum_{v\neq\tilde{v}}\left(n_v e^{\lambda_v t}+\bar{n}_v e^{\bar{\lambda}_v t}\right)+\sum_{v\neq\tilde{v}}\left(P_v e^{\lambda_v t}+\bar{P}_v e^{\bar{\lambda}_v t}\right) \quad (69)$$

$$= 2\sum_{v\neq\tilde{v}}\left[\varsigma_v\cos y_v t+\omega_v\sin y_v t\right]e^{-x_v t}+\sum_{v\neq\tilde{v}}\left(P_v e^{\lambda_v t}+\bar{P}_v e^{\bar{\lambda}_v t}\right) \quad (70)$$

which after some algebra and taking into account the trigonometric relations $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\sin(a+b) = \sin a \cos b + \cos a \sin b$ becomes

$$\sum_{v\neq\tilde{v}}(n_v + P_v)e^{\lambda_v t} = 2\sum_{v\neq\tilde{v}}\left[(\varsigma_v + \hat{a}\Phi_{1,v})\cos y_v t - (\omega_v + \hat{a}\Phi_{2,v})\sin y_v t\right]e^{x_v t} \tag{71}$$

where

$$\Phi_{2,v} = \frac{y_v + J\Gamma e^{-x_v J}\left[(\tilde{z} - x_v)\sin y_v J + y_v \cos y_v J\right]}{(\tilde{z}^2 - 2\tilde{z}x_v + x_v^2 + y_v^2)(1 + J^2\Gamma^2 e^{-2Jx_v} + 2J\Gamma e^{-x_v J}\cos y_v J)} \tag{72}$$

Optimal path of capital and consumption when divisible labor supply

The linearized system of mixed functional equations in capital and consumption is

$$\dot{k}(t) = \eta_{11}k(t-\tau) + \eta_{12}c(t) \tag{73}$$

$$\dot{c}(t) = \rho c(t) - \eta_{22}c(t+\tau) + \eta_{21}k(t). \tag{74}$$

After some algebra, it is possible to rewrite the equations (73) and (74) as a second order mixed functional differential equation in capital

$$\ddot{k}(t-\tau) - \eta_{11}\dot{k}(t-2\tau) + \eta_{22}\dot{k}(t) - \rho\dot{k}(t-\tau) + \rho\eta_{11}k(t-2\tau) - (\eta_{11}\eta_{22} + \eta_{12}\eta_{21})k(t-\tau) = 0 \tag{75}$$

with boundary conditions $k(t) = \xi(t)$ and $\dot{k}(t) = \eta_{11}\xi(t) - \eta_{12}\zeta(t)$ in $t \in [-2\tau, 0]$.[5]

In order to find a series expansion of solution of the equation (75), the finite Laplace transformation technique has been introduced as suggested by Bellman and Cooke (Chapter 6, pages 197-205). After fixing the finite interval of time where we want to obtain the series expansion, in our case $t \in [0, T]$, an $a < 0 - 2\tau$ and a $b > T + 2\tau$ has been chosen. The finite Laplace transformation of $k(t)$ in the interval (a, b) is

$$\mathtt{L}(k(t)) = \int_a^b k(t)e^{-zt}dt. \tag{76}$$

Applying it on equation (75), and taking the linear properties of the Laplace transformation into account, we obtain

$$\mathtt{L}(\ddot{k}(t-\tau)) - \eta_{11}\mathtt{L}(\dot{k}(t-2\tau)) + \eta_{22}\mathtt{L}(\dot{k}(t)) - \rho\mathtt{L}(\dot{k}(t-\tau)) + \rho\eta_{11}\mathtt{L}(k(t-2\tau)) - (\eta_{11}\eta_{22} + \eta_{12}\eta_{21})\mathtt{L}(k(t-\tau)) = 0. \tag{77}$$

After the calculation of the values of $\mathtt{L}(\dot{k}(t))$, $\mathtt{L}(k(t-\tau))$, $\mathtt{L}(\dot{k}(t-2\tau))$, $\mathtt{L}(\dot{k}(t-\tau))$, $\mathtt{L}(k(t-2\tau))$, and $\mathtt{L}(\ddot{k}(t-\tau))$, we substitute them into equation (77), and rewrite it as

$$\mathtt{L}(k(t)) = \frac{\phi(z)}{h(z)} = g(z) \tag{78}$$

with $\phi(z)$ and $h(z)$ analytic functions. The solution of capital can be obtained through the inverse finite Laplace transformation on the set of circle contours C_ℓ with $\ell = 1, 2, ...$, centre in the origin of the complex plane, and radius y_ℓ. Then the solution of capital is

$$k(t) = \oint g(z)e^{zt}dz. \tag{79}$$

[5]Observe also that the characteristic equation of (75) is exactly the same of the linearized system (**??**).

Taking into account that $g(z)$ is not complex differentiable in all of this domain due to the singularities represented by the roots of $h(z)$, by using the Residue theorem the solution of (79) is:

$$\begin{aligned} k(t) &= \sum_{z_r \in C_\ell} \texttt{Res}\left(\frac{\phi(z)}{h(z)}, e^{zt}\right) \\ &= \sum_{z_r \in C_\ell} n_{z_r} e^{z_r t} \end{aligned} \tag{80}$$

where the residues $n_{z_r} = \frac{\phi(z_r)}{h'(z_r)}$ are defined in the complex field $\texttt{C}$. On the other hand, the solution of consumption can be obtained through equation (73), observing that $\dot{k}(t) = \sum_{z_r \in C_\ell} z_r n_{z_r} e^{z_r t}$.

Spectrum of Roots - Discrete Time

In this Appendix section the table of the moduli of the roots in the discrete time case is reported.

Table 1: **Modulus of roots when indivisible labor supply** ($\chi = 0.0$)

	Time to Build parameter (J)							
$h_c(z)$	1	2	3	4	5	6	7	8
$\lvert z_1 \rvert$	–	11.03	3.31^c	2.43	1.95^c	1.74	1.59^c	1.5
$\lvert z_2 \rvert$		0.94	3.31^c	2.12^c	1.95^c	1.69^c	1.59^c	1.48^c
$\lvert z_3 \rvert$			0.95	2.12^c	1.69^c	1.69^c	1.53^c	1.48^c
$\lvert z_4 \rvert$				0.95	1.69^c	1.48^c	1.53^c	1.42^c
$\lvert z_5 \rvert$					0.95	1.48^c	1.35^c	1.42^c
$\lvert z_6 \rvert$						0.96	1.35^c	1.27^c
$\lvert z_7 \rvert$							0.961	1.27^c
$\lvert z_8 \rvert$								0.963
$h_k(\lambda)$								
$\lvert \lambda_1 \rvert$	–	1.06	1.06	1.06	1.055	1.052	1.05	1.05
$\lvert \lambda_2 \rvert$		0.09	0.30^c	0.47^c	0.59^c	0.68^c	0.74^c	0.79^c
$\lvert \lambda_3 \rvert$			0.30^c	0.47^c	0.59^c	0.68^c	0.74^c	0.79^c
$\lvert \lambda_4 \rvert$				0.41	0.51^c	0.58	0.66^c	0.71^c
$\lvert \lambda_5 \rvert$					0.51^c	0.596^c	0.66^c	0.71^c
$\lvert \lambda_6 \rvert$						0.596^c	0.63^c	0.67
$\lvert \lambda_7 \rvert$							0.63^c	0.68^c
$\lvert \lambda_8 \rvert$								0.68^c

[c] Same modulus indicates conjugate complex roots and not repeated real roots

Table 2: **Modulus of roots when divisible labor supply** ($\chi = -0.15$)

	Time to Build parameter (J)							
$H(z)$	1	2	3	4	5	6	7	8
$\lvert z_1 \rvert$	–	13.47	3.75^c	2.53	2.02^c	1.77	1.61^c	1.51
$\lvert z_2 \rvert$	–	1.89	3.75^c	2.41^c	2.02^c	1.76^c	1.61^c	1.50^c
$\lvert z_3 \rvert$		0.21	1.81	2.41^c	1.94^c	1.76^c	1.60^c	1.50^c
$\lvert z_4 \rvert$		0.18	0.35^c	1.72	1.94^c	1.71^c	1.60^c	1.505^c
$\lvert z_5 \rvert$			0.35^c	0.47	1.63	1.71^c	1.57^c	1.505^c
$\lvert z_6 \rvert$			0.32	0.45^c	0.54^c	1.57	1.57^c	1.49^c
$\lvert z_7 \rvert$				0.45^c	0.54^c	0.61	1.51	1.49^c
$\lvert z_8 \rvert$				0.41	0.52^c	0.60^c	0.65^c	1.46
$\lvert z_9 \rvert$					0.52^c	0.60^c	0.65^c	0.693
$\lvert z_{10} \rvert$					0.49	0.58^c	0.64^c	0.69^c
$\lvert z_{11} \rvert$						0.54	0.64^c	0.69^c
$\lvert z_{12} \rvert$							0.62^c	0.68^c
$\lvert z_{13} \rvert$							0.62^c	0.68^c
$\lvert z_{14} \rvert$							0.59	0.66^c
$\lvert z_{15} \rvert$								0.66^c
$\lvert z_{16} \rvert$								0.63

^{c} Same modulus indicates conjugate complex roots and not repeated real roots

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