CER-ETH - Center of Economic Research at ETH Zurich

Optimal control of pollutants with delayed stock accumulation

Ralph Winkler

Working Paper 08/91
July 2008

Economics Working Paper Series

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Optimal control of pollutants with delayed stock accumulation

Ralph Winkler*

CER-ETH – Center of Economic Research at ETH Zurich, Switzerland

July 2008

Abstract: We study the optimal control of a pollutant that accumulates with a delay. We find that optimal paths are, in general, non-monotonic and oscillatory, but monotonic if the objective function is additively separable. Hence, using additively separable objective functions as an approximation to a general objective function may be a misspecification. With a numerical example we illustrate that an additively separable approximation performs considerably worse in delayed compared to instantaneous stock accumulation.

Keywords: additively separable objective, approximated objective, delayed optimal control, optimal pollution control

JEL-Classification: Q50, C61

Correspondence:
Ralph Winkler
CER-ETH – Center of Economic Research at ETH Zurich
Zürichbergstrasse 18, 8092 Zurich, Switzerland
phone: +41 44 632 8614, fax: +41 44 632 1867, email: mail@ralph-winkler.de

* I am grateful to Jürgen Eichberger, Malte Faber, Hans Gersbach, Hans Haller, Natali Hritonenko, Ulf Moslener, John Proops, Till Requate, João Rodrigues, Yuri Yatsenko and seminar participants at the University of Kiel in February 2004 and at the University of Girona in May 2008 for comments on an earlier draft, and to the Simulation and Optimization Group of the Interdisciplinary Center for Scientific Computing, University of Heidelberg for a free license of the MUSCOD-II software package. Financial support by the Deutsche Forschungsgemeinschaft (German Research Foundation) under the graduate programme "Environmental and Resource Economics" and by the European Commission under the Marie Curie Intra-European Fellowship scheme, No. MEIF-CT-2003-501536, is gratefully acknowledged.

1 Introduction

Numerous environmental problems we face today are caused by pollutants which accumulate stocks with a delay to their emissions. Often such delays are due to a time consuming transportation processes, in which the pollutants travel from the emitting source to the place where they accumulate. As a prime example think of chlorofluorocarbons (CFCs), which cause the depletion of the stratospheric ozone layer that shields the earth's surface from ultraviolet radiation. Once released, the CFCs need 5–10 years to reach a height of about 30 km, where the depletion of the ozone layer starts. Other examples for pollutants with delayed stock accumulation include nitrate and pesticide run-off from agricultural cultivation, which seep away and accumulate in the groundwater and decrease its quality as drinking water (UNEP 2002).

In this paper we analyze delayed pollution stock accumulation in a continuous time optimal control framework, thereby merging two distinct strands of economic literature. A first strand deals with the optimal control of instantaneously accumulating stock pollutants. While some contributions (e.g., Gradus and Smulders 1993, Keeler et al. 1972, Van der Ploeg and Withagen 1991, Smith 1977, Goulder and Mathai 2000) analyze steady-state growth in highly aggregated Ramsey-type optimal growth models with environmental pollution, others focus on the complex system dynamics of environmental problems caused by stock pollutants. For example, Falk and Mendelsohn (1993) analyze the optimal emissions of greenhouse gases, Baumgärtner et al. (2007) and Moslener and Requate (2007) analyze the dynamic interaction of different stock pollutants, and Goeschl and Perino (2007) investigate optimal R&D expenditures if technologies give rise to stock pollutants. A second strand deals with the delayed accumulation of capital. While Asea and Zak (1999), Bambi (2008) and Rustichini (1989) analyze growth models with one capital good, Benhabib and Rustichini (1991), Boucekkine et al. (1997) and Boucekkine et al. (2005) investigate cyclical behavior in vintage capital models. Heinzel and Winkler (2007) and Winkler (2008) analyze optimal structural change in the energy sector, assuming that power plants exhibit a time-to-build feature.

A shared result across the second literature strand is that capital accumulation models with delays exhibit oscillatory optimal paths. Therefore, one would expect a similar result for delayed stock accumulation models. If we, however, introduce a delay into a standard pollution control model (e.g., Falk and Mendelsohn 1993), the optimal paths show monotonic behavior, as in the case of instantaneous stock accumulation. The reason for this difference is that the standard pollution control model exhibits an objective function which is additively separable in the stock and the control, while this is not the case in capital accumulation models. We show in section 2 that for delayed stock accumulation the functional form of the objective function plays a crucial role for the qualitative system dynamics. While delayed stock accumulation problems exhibit oscillatory paths in general (no matter if we accumulate capital or pollutants), the optimal paths are monotonic if the objective function is additively separable in the control and the stock variables.

The importance of this finding crucially hinges upon the reason why objective functions are mostly assumed to be additively separable in pollution control problems. In section 3 we show that the additively separable form can be thought of as an approximation to a more complex general objective function. If this is the case, one has to be careful with imposing such an approximation in delayed accumulation problems, as this may be a misspecification of the original problem. Moreover, we show that this *qualitative* difference may also be *quantitatively* relevant. Therefore, we set up a simple delayed pollution control model in section 4, which exhibits an objective function which is not additively separable in the stock and the control. We approximate an additively separable objective function and solve both the general and the approximated control problem numerically for instantaneous and delayed stock accumulation. We find that the approximation performs considerably worse in case of delayed compared to instantaneous stock accumulation. Our result challenges the use of additively separable welfare functions as “good” approximations to more general objective functions, at least in the case of delayed stock accumulation.

2 A generic pollution control model with delayed stock accumulation

We introduce a generic optimal control problem with one pollutant that accumulates with a delay and show that the system dynamics crucially depends on whether the objective function is additively separable in the emissions and the stock. Throughout the paper partial derivatives are denoted by subscripts, derivatives with respect to the sole argument are denoted by primes and derivatives with respect to time are denoted by dots.

Suppose a social planner, who solves the following optimization problem:

$$\max_{e(t)} \int_0^\infty F\big(s(t), e(t)\big) \exp[-\rho t] \, dt \ , \quad \rho > 0 \ , \tag{1}$$

subject to:

$$\begin{aligned} \dot{s}(t) &= e(t-\tau) - \gamma s(t) \ , \quad \tau, \gamma > 0 \ , && \text{(2a)} \\ e(t) &\geq 0 \ , && \text{(2b)} \\ s(0) &= s_0 \geq 0 \ , && \text{(2c)} \\ e(t) &= \xi(t) \geq 0 \ , \quad t \in [-\tau, 0) \ , && \text{(2d)} \end{aligned}$$

where ρ is the constant and positive discount rate, τ denotes the delay between the emission and the accumulation of the pollutant and γ is the constant and positive deterioration rate of the stock s. F is a twice continuously differentiable generic felicity function, which is a function of emissions e and the pollution stock s. We assume that F is increasing in e and decreasing in s, strictly and jointly concave in both arguments, and satisfies Inada conditions. These assumptions ensure that there exists a unique interior solution. In addition, we impose that emissions are non-negative and specify initial conditions for the pollution stock, s_0, and an initial emission path ξ for the time interval $[-\tau, 0)$.

The dynamics of the stock s is governed by the delayed differential equation (2a), which represents the standard form (except for the delay) used in stock accumulation problems in economics. At time t the stock increases by the emissions at time $t-\tau$, and deteriorates at the constant rate γ. Thus, the model exhibits inertia, as the stock s reacts with a delay to a variation of the control e. As a consequence, the path of the stock s in the time interval $t \in [0, \tau]$, which we denote by $\sigma(t)$, is completely determined by the initial stock s_0 and the initial path ξ:

$$\sigma(t) = s_0 \exp[-\gamma t] + \int_0^\tau \xi(t' - \tau) \exp[-\gamma(t - t')] \, dt' \ , \qquad t \in [0, \tau] \ . \tag{3}$$

To solve problem (1), we apply the generalized Maximum principle derived in El-Hodiri et al. (1972) for delayed optimal control problems and derive the following present-value Hamiltonian $\mathcal{H}$:[1]

$$\mathcal{H} = F\big(s(t), e(t)\big) \exp[-\rho t] + \lambda(t+\tau) e(t) - \gamma \lambda(t) s(t) \ , \tag{4}$$

where λ denotes the costate variable or shadow price of the stock s.

Assuming that the Hamiltonian $\mathcal{H}$ is continuously differentiable with respect to emis-

[1] As the assumed properties of F ensure an interior solution, we do not need to explicitly check for the non-negativity of emissions e.

sions e, the following necessary conditions hold for an optimal solution:

$$F_e\big(s(t), e(t)\big) \exp[-\rho t] = -\lambda(t+\tau) \ , \tag{5a}$$

$$\dot{\lambda}(t) = \gamma\lambda(t) - F_s\big(s(t), e(t)\big) \exp[-\rho t] \ . \tag{5b}$$

These necessary conditions are also sufficient for the existence of a unique interior solution if, in addition to the assumed curvature properties of F, the following transversality condition holds:

$$\lim_{t\to\infty} [\lambda(t)s(t)] = 0 \ . \tag{5c}$$

Solving equation (5b) by using the transversality condition (5c) yields:

$$\lambda(t) = \int_t^\infty F_s\big(s(t'), e(t')\big) \exp[-\rho t'] \exp[-\gamma(t'-t)] \, dt' \ . \tag{6}$$

Now, the economic interpretation of the necessary and sufficient conditions is straightforward. The shadow price $\lambda(t)$ equals the net present value of all future disutility stemming from a marginal increase of stock s at time t. Equation (5a) says that along the optimal path the net present value of the utility gain of a marginal increase in emissions has to equal the net present value of the utility loss induced by the resulting increase in the pollution stock. As the pollution stock accumulates with a timelag τ, the net present value of the utility loss induced by a marginal unit of emissions at time t is equal to $\lambda(t+\tau)$.

We eliminate the shadow price $\lambda(t)$ by differentiating equation (5a) with respect to time and inserting into equation (5a). We derive, together with the equation of motion (2a), the following system of differential equations, the solution of which determines the optimal emission path $e(t)$ $(t \geq 0)$ and the optimal stock dynamics $s(t)$ $(t \geq \tau)$:

$$\begin{aligned} \dot{e}(t) &= \frac{1}{F_{ee}\big(s(t), e(t)\big)} \Big\{ (\gamma+\rho) F_e\big(s(t), e(t)\big) + F_s\big(s(t+\tau), e(t+\tau)\big) \exp[-\rho\tau] \\ &\quad + F_{es}\big(s(t), e(t)\big) \big[\gamma s(t) - e(t-\tau)\big] \Big\} \ , & (7a) \\ \dot{s}(t) &= e(t-\tau) - \gamma s(t) \ , & (7b) \end{aligned}$$

Note that $\dot{e}$ and $\dot{s}$ also depend on *advanced* (i.e., at a later time) and on *retarded* (i.e., at an earlier time) variables. Hence, (7) forms a system of *neutral type functional differential equations* which is difficult to solve analytically, as even the linear approximation around the stationary state exhibits, in general, no closed form analytical solution. Nevertheless, it is possible to derive some qualitative properties of the system dynamics, which are summarized in the following propositions.

Proposition 1 (Stationary state)
The system of functional differential equations (7) exhibits a unique stationary state $(s^\star, e^\star)$*, which is determined by the following system of implicit equations:*

$$-\frac{F_s(s^\star, e^\star)}{F_e(s^\star, e^\star)} = (\gamma + \rho) \exp[\rho\tau] \ , \tag{8a}$$

$$e^\star = \gamma s^\star \ . \tag{8b}$$

The proof is given in the appendix.

To derive qualitative properties of the system dynamics in a neighborhood around the unique stationary state, we linearize the system of functional differential equations (7) around the stationary state $(s^\star, e^\star)$. The following proposition states the properties of the roots of the characteristic equation of the linearized system.

Proposition 2 (Roots of the characteristic equation)
The characteristic equation of the linear approximation of the system of functional differential equations (7) exhibits

- *in general an infinite number of complex roots with unbounded positive real parts and an infinite number of complex roots with unbounded negative real parts, and*
- *one positive and one negative real solution, in the special case that* $F_{es}(s^\star, e^\star) = 0$.

The proof is given in the appendix.

Proposition 2 implies that the stationary state $(s^\star, e^\star)$ is, in either case, a saddle point. As a consequence, there exists a unique optimal path which converges towards the stationary state.[2] The qualitative system dynamics crucially depends on the functional form of the felicity function F. In general, the convergence towards the stationary state is oscillatory, as it is governed by the superposition of an infinite number of exponentially damped oscillations. In the special case that $F_{es}(s^\star, e^\star) = 0$ the convergence towards the stationary state is exponentially and, therefore, monotonic.

[2] If the characteristic equation exhibits purely imaginary roots, the system dynamics may exhibit *limit-cycles*. That is, the optimal paths oscillate around the stationary state without converging towards or diverging from it. As purely imaginary roots can only occur "accidentally" (by well chosen endogenous parameters), we do not explicitly consider limit cycles in the following. Limit-cycles in the case of delayed optimal control problems have been discussed, among others, by Rustichini (1989) and Asea and Zak (1999).

To better understand what happens with the system dynamics in case of $F_{es}(s^\star, e^\star) = 0$, we consider the special case that the felicity function F is additively separable in both arguments (which guarantees that $F_{es} = 0$). Then, the system of functional differential equations (7) reduces to a system of ordinary first order differential equations, where the initial stock s_0 and the initial path ξ translate into the new initial condition $\tilde{s}(0) = \tilde{s}_0 = \sigma(\tau)$, which is given by equation (3) for $t = \tau$. To see this, we set $F\big(e(t), s(t)\big) = E\big(e(t)\big) + S\big(s(t)\big)$ and $\tilde{s}(t) = s(t+\tau)$ and derive:

$$\dot{e}(t) = \frac{1}{E''\big(e(t)\big)} \left\{(\gamma+\rho)E'\big(e(t)\big) + S\big(\tilde{s}(t)\big)\exp[-\rho\tau]\right\} , \tag{9a}$$

$$\dot{\tilde{s}}(t) = e(t) - \gamma\tilde{s}(t) . \tag{9b}$$

We see that the crucial difference between equation (9a) and the corresponding equation (7a) is that equation (9a) only depends on $e(t)$ and $s(t+\tau)$, and not simultaneously on $e(t)$, $e(t+\tau)$, $e(t-\tau)$, $s(t)$ and $s(t+\tau)$. That is, the same variable is not evaluated at different times but only at one point in time. In addition, the time structure of equation (9a) and (9b) is identical (e is evaluated at a time which lies τ earlier than the time at which s is evaluated) and, therefore, can be reduced to an ordinary differential equation by variable transformation. Thus, the result that for F additively separable the system dynamics is monotonic hinges on two crucial features. On the one hand, with F being additively separable, the first and second derivatives depend only either on s or e and the cross derivative F_{es} vanishes, which leads to the fact that in equation (9a) the same variable is only evaluated at one point in time. On the other hand, the equation of motion of the stock (9b) is of the same structure, as it depends on variables which are only evaluated at one point in time and, in addition, the delay structure is the same as the resulting delay structure in equation (9a).[3]

In summary, if the felicity function is additively separable, the optimal system dynamics shows a monotonic convergence towards the stationary state, as is the case for instantaneous stock accumulation. However, if the felicity function is not additively separable, the system dynamics exhibits, in general, oscillatory behavior. As the importance of this finding crucially depends on when do models exhibit additively separable felicity functions and why, we shall discuss felicity functions in environmental economics in the next section.

[3] This implies that the monotonicity result is not valid for arbitrary equations of motion. In fact, the most general form of the equation of motion preserving this characteristic is $\dot{s} = G\big(e(t-\tau)\big) - H\big(s(t)\big)$, where G and H are two monotonically increasing functions.

3 Felicity functions in environmental economics

In most economic models the felicity function F is represented by an instantaneous welfare function V, which measures the utility of the optimizing agent at time t. In environmental economics, the instantaneous welfare function can be of two types:

1. V solely depends on consumption. In this case "nature" does not influence the agents' utility directly, but only indirectly as source of consumption or input factors of production. In general, it is assumed that emissions increase and the pollutant stock decreases consumption:

$$F = V(c) \ , \qquad c = c(e, s, \cdot) \ , \quad c_e > 0 \ , \quad c_s < 0 \ . \tag{10}$$

2. V depends on consumption and "environmental quality" q. The idea is that environmental quality is a direct source of utility. In general, it is assumed that emissions increase consumption, while the pollutant stock decreases environmental quality:

$$F = V(c, q) \ , \qquad c = c(e, \cdot) \ , \quad c_e > 0 \ , \qquad q = q(s, \cdot) \ , \quad q_s < 0 \ . \tag{11}$$

As V is, in general, an increasing and concave function, the felicity function is not per se additively separable in either case.

However, most environmental economic models assume felicity functions, which are additively separable. In fact, in partial equilibrium models the objective is often to find the net present minimum of the sum of abatement costs, depending on the amount of pollutant emissions, and costs due to environmental damage, depending on the pollutant stock (e.g., Falk and Mendelsohn 1993, Goulder and Mathai 2000, and Moslener and Requate 2001). Thus, the felicity function F reads

$$F = -\big[C(e) + D(s)\big] \ , \tag{12}$$

where C are the abatement costs, depending on emissions e, and D are the costs caused by environmental damage, which hinge upon the pollution stock s. Obviously, this is an approximation either to the first case (neglecting diminishing marginal utility of consumption and consumption additively separable in the emissions and the stock), or the second case (neglecting diminishing marginal utility of consumption and utility additively separable in consumption and environmental quality). In general equilibrium models, instantaneous utility V is often of the second type, but assumed to be additively separable in consumption and environmental quality (e.g., Baumgärtner et al. 2007, Goeschl and Perino 2007, and Van der Ploeg and Withagen 1991):[4]

$$F = U\big(c(e)\big) - D\big(q(s)\big) \ , \tag{13}$$

where U denotes utility derived from consumption and D denotes utility loss caused by environmental damage. The main motivation for the additively separable form is analytical tractability. Therefore, we also consider this form to be rather an approximation than the real underlying relationship.

From the discussion in the previous section it is clear that, in the case of delayed stock accumulation, the real felicity functions (10) and (11) give rise to oscillatory paths, while the approximations (12) and (13) exhibit monotonic paths. In the following, we illustrate that in case of delayed stock accumulation the solution of the approximated problem falls not only *qualitatively* short of the solution of the real problem, but also *quantitatively*.

4 General versus additively separable felicity functions: an illustration

In this section, we present a simple delayed stock accumulation model, which is of the generic class discussed in section 2. Although it is inspired by the environmental problem of the emission of chlorofluorocarbons (CFCs), it is applicable to various stock pollutants. CFCs are a prime example of stock pollutants accumulating with a delay. They have been widely used as cooling agents in refrigeration and air conditioning, as propellants in aerosols sprays and foamed plastics, and as solvents for organic matters and compounds. The CFCs have been valued due to their favorable chemical and biological characteristics. They are chemically inert, not inflammable and non-toxic. Unfortunately, in the stratosphere the CFCs cause the depletion of the ozone layer which shields the earth's surface from ultraviolet radiation. Once released, the CFCs need 5–10 years to reach a height of about 30 km, where the depletion of the ozone layer starts. Hence, the stock of stratospheric CFCs reacts to the emissions of CFCs with a lag time of 5–10 years.

Consider an economy with a constant population normalized to 1. We assume that labor is the sole input to the two available production processes in the economy. The first production process produces a consumption good c with constant returns to labor.

[4] Exceptions which do not assume an additively separable felicity function include Gradus and Smulders (1993), and Keeler et al. (1972).

Without loss of generality, we assume that one unit of labor produces one unit of the consumption good. Thus, the amount of consumption equals the amount of labor l_1 applied to consumption good production:

$$c(t) = l_1(t) \ . \tag{14}$$

In addition, the production of each unit of consumption good gives rise to one unit of gross emissions. The second production process is an abatement process with decreasing returns to scale. Denoting the amount of labor employed to the abatement process by l_2, the amount of abated emissions a is given by:

$$a(t) = \sqrt{\alpha l_2(t)} \ , \quad \alpha > 0 \ . \tag{15}$$

Then, net emissions e equal to

$$e(t) = c(t) - a(t) = l_1(t) - \sqrt{\alpha l_2(t)} \ . \tag{16}$$

The emissions accumulate with a delay τ to a pollutant stock s and decay at the constant rate γ, as given by equation (2a). The pollution stock s imposes a negative externality with increasing marginal damage on the economy, as it reduces the effective labor force l:[5]

$$l(t) = 1 - \beta s(t)^2 \ , \quad \beta > 0 \ . \tag{17}$$

We consider a social planner who maximizes the net present value of all future consumption by distributing the effective labor force among the two production processes.[6] As an additional input of labor in both processes increases consumption, the labor constraint holds with equality along the optimal path: $l_1(t) + l_2(t) = 1$. Hence, we derive for consumption c:[7]

$$c(t) = c\big(e(t), s(t)\big) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\big(1 - \beta s(t)^2 - e(t)\big) + \alpha^2}\right] \ . \tag{18}$$

---

[5] In the case of CFCs, one might think of an increase in the rate of skin cancer with increasing stock of the pollutant, which prevents increasingly more people from working.

[6] For the sake of simplicity we abstract from decreasing marginal returns of consumption. However, considering a concave instantaneous utility function would not alter our results qualitatively.

[7] In addition, we assume that the pollution stock s in the time interval $t \in [0, \tau)$, which is completely determined by the initial stock s_0 and the initial path ξ, is always smaller than $\sqrt{1/\beta}$. Otherwise, the total labor force would be annihilated before emission control becomes effective.

α	β	γ	ρ	τ	s_0	ξ
3	0.002	0.1	0.03	0/10	10	1

Table 1: Exogenous parameters and initial values used for the numerical optimization.

Thus, we derive an optimal control problem with delayed stock accumulation of the class discussed in section 2 with a felicity function $F = c(e, s)$ as given by equation (10).

Obviously, $F = c(e, s)$ is not additively separable, and thus the optimal paths will, in general, be non-monotonic. However, if we suppose that the labor costs of abatement are small (i.e., $\alpha >> 1$) and the consumption loss due to the stock externality is small compared to overall consumption (i.e., $\beta << 1$), we can construct an additively separable *approximation* of F. In a first step, we approximate the felicity function (18) by its first order Taylor series around $\beta = 0$:

$$c\big(e(t), s(t)\big) \approx \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\big(1 - e(t)\big) + \alpha^2}\right] - \underbrace{\frac{\alpha}{\sqrt{\alpha^2 + 4\alpha\big(1 - e(t)\big)}}}_{A}\beta s(t)^2 \ . \quad (19)$$

Second, as α is large, α^2 outweighs $4\alpha\big(1 - e(t)\big)$ and A is approximately equal to 1. Thus, we obtain for the approximated felicity function $\hat{F}$

$$\hat{F}\big(e(t), s(t)\big) = \frac{1}{2}\left[2e(t) - \alpha + \sqrt{4\alpha\big(1 - e(t)\big) + \alpha^2}\right] - \beta s(t)^2 \ . \quad (20)$$

Note that $\hat{F}$ is now additively separable in e and s, and of the form of equation (13). According to the discussion in section 2, we expect that the approximated felicity function $\hat{F}$ gives rise to monotonic optimal paths.

Now, we analyze the quality of the approximated felicity function (20) compared to the original felicity function (18). From the derivation of the approximation it is obvious that the approximation is better the larger is α and the smaller is β. However, in the following we show that, for given parameter values α and β, the additively separable felicity function (20) is a much better approximation to the original felicity function (18) in case of instantaneous stock accumulation, compared to the case of delayed stock accumulation.

As the optimal control problems with both the original and the approximated felicity function cannot be solved analytically, we numerically solve the optimal paths of emissions and the pollution stock with the advanced optimal control software package MUSCOD-II (Diehl et al. 2001, Leineweber et al. 2003), developed by the Simulation and Optimization Group of the Interdisciplinary Center for Scientific Computing, University of Heidelberg. Details of the numerical optimization method are discussed in Brandt-Pollmann et al. (2008). The time horizon for the numerical optimization has been set to 200 years, and all parameters have been chosen so that the system at time $t = 200$ is very close to the stationary state (for a more convenient exposition, the figures just show times up to $t = 100$). The exogenous parameters and the initial values used for the numerical optimization are given in Table 1. The parameter values have primarily been chosen so as to illustrate clearly the different effects, and do not necessarily reflect the characteristics of real environmental pollution problems.

To illustrate the qualitative difference between instantaneous and delayed stock accumulation, we computed two different scenarios, one with instantaneous stock accumulation ($\tau = 0$), and one with delayed stock accumulation ($\tau = 10$). For both scenarios we derive the optimal paths for the original and the approximated felicity function. From the implicit functions (8), we compute the stationary state values for the original and the approximated additively separable felicity function, given the parameter values stated in Table 1. For $\tau = 0$, one derives $(s^\star, e^\star) \approx (6.14, 0.61)$ for the original and $(s^\star, e^\star) \approx (6.12, 0.61)$ for the approximated case, and for $\tau = 10$, one derives $(s^\star, e^\star) \approx (6.66, 0.67)$ for the original and $(s^\star, e^\star) \approx (6.93, 0.69)$ for the approximated felicity function. We see that the stationary state values for the original and the approximated felicity function are closer together in case of instantaneous stock accumulation. Thus, by just comparing long-run stationary states the additively separable felicity function seems a better approximation to the original one for instantaneous than for delayed stock accumulation.

As Figure 1 (upper half) shows, this first impression is reinforced by the system dynamics. For instantaneous stock accumulation (left side), the optimal paths for original and approximated felicity functions show the same *qualitative* behavior. The emission paths start substantially below their stationary state levels of about 0.61 and converge monotonically towards them. Starting from an initial value of 10, the pollution stocks decrease monotonically towards their stationary state values of about 6.14 for the original and 6.12 for the approximated felicity function. Moreover, both the original and the approximated optimal paths are also *quantitatively* very close together.

The right side of Figure 1 shows the optimal paths for delayed stock accumulation. Since there is a delay of $\tau = 101$, the path for the pollution stock in the time interval

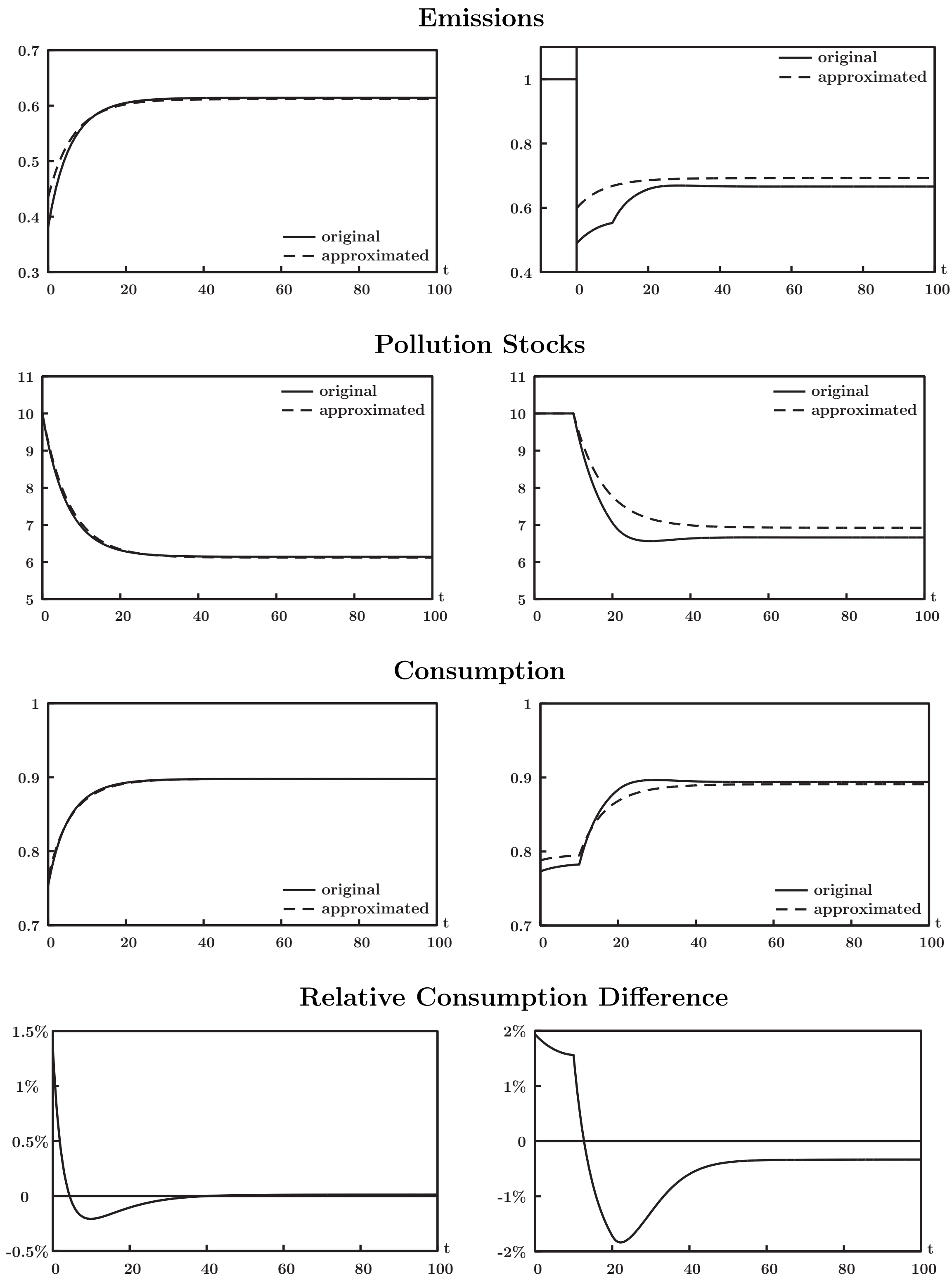

Figure 1: Optimal system dynamics in case of instantaneous (left) and delayed (right) stock accumulation for the original and the approximated felicity functions.

τ	ΔW	$\Delta(s^\star, e^\star)$	$\max|\Delta c|$	$\int_0^{100}(\Delta c)^2\,dt$
0	-0.001%	-0.03%	1.35%	$1.48 \cdot 10^{-4}$
10	-5.55%	3.95%	1.94%	$6.15 \cdot 10^{-2}$

Table 2: Four different indices for the quality of approximation in case of instantaneous and delayed stock accumulation

$t \in [0, 10]$ is completely determined by the initial value s_0 and the initial path ξ for both the original and the approximated problem. We set the initial path to $\xi = 1$, which is shown as the emission path in the time interval $t \in [-10, 0]$ in the graph. This implies that the stock stays constant at $s_0 = 10$ for $t \in [0, \tau]$. The emission paths reflect the results of section 2: the additively separable felicity function displays monotonic convergence towards its stationary state value, whereas the original felicity function displays non-monotonic and oscillatory. Moreover, the differences between approximated and original felicity functions are over the whole time horizon substantially larger than in the case of instantaneous stock accumulation. The differences are highest in the short run. As a consequence, also the optimal path for the pollution stock differs markedly from the optimal path derived from the additively separable approximation. The path exhibits a pronounced dip between $t = 10$ and $t = 30$, which corresponds to the emissions between $t = 0$ and $t = 20$, because of the delay τ.

In addition, we compute the real consumption, as given by equation (18), given the optimal paths of the emissions and the pollution stock derived by the original and the approximated felicity function. Figure 1 (lower half) shows the consumption paths and the relative difference in the consumption paths.[8] Again, we see that in the case of delayed stock accumulation the difference in consumption is considerably higher than in the case of instantaneous stock accumulation. To compare the quality of approximation of the additively separable approximation more quantitatively, we compute four indices summarized in Table 2: ΔW denotes the relative difference in overall welfare, $\Delta(s^\star, e^\star)$ denotes the relative difference in the long-run stationary state values, $\max|\Delta c|$ denotes the maximal (in absolute terms) consumption difference in instantaneous utility, and $\int_0^{100}(\Delta c)^2\,dt$ is the integral of the squared consumption differences over the first 100 year. While the first measures captures approximation quality in terms of welfare (and, thus, is the most important), the other measures capture differences in approximation quality with respect to the optimal paths. For all indices the approximation is better the smaller

[8] Note that consumption is identical to instantaneous welfare in this model.

is the corresponding value. We see that according to all four indices the approximation quality is much better for $\tau = 0$ than for $\tau = 10$. In particular, the difference in welfare loss is substantial. While the overall welfare loss due to the approximation is just 0.001% for $\tau = 0$, it amounts to considerable 5.55% for $\tau = 10$.

5 Conclusion

We have studied the optimal control of stock pollutants which accumulate with a delay. For a generic problem with one stock and one control, we have shown that the system dynamics crucially depend on the functional form of the felicity function. In general, the system dynamics is characterized by non-monotonic and oscillatory paths, but exhibits monotonic optimal paths if the felicity function is additively separable in the control and the stock.

This result is important for the design of delayed optimal control models in a twofold manner. First, one has to be aware that the functional form of the objective influences the qualitative behavior of the optimal paths. Second, as illustrated by a numerical example, approximations of the objective, which result in good quantitative approximations of the original problem in case of instantaneous stock accumulation, may result in considerably worse approximations in case of delayed stock accumulation.

Although the model discussed in this papers is highly abstract, there are some general conclusions which can be drawn for the optimal control of delayed stock pollutants. First, there is an additional moment of inertia, because of the delay between the control and the accumulation of stock, which demands increased caution and alertness in the handling of delayed pollutants such as CFCs. Second, the application of an easy to handle additively separable objective as a good approximation to the original problem might be a misspecification because of the change of the qualitative behavior of the optimal paths. Third, non-monotonic and oscillatory optimal paths may be difficult to implement by environmental policy. Thus, even if the optimal (non-monotonic) emission path is known, it might be not applicable because of institutional constraints.

However, this paper only scratches the tip of the iceberg of delayed optimal control problems and leaves many open questions for future research. For example, it is not obvious how the transition from $\tau = 0$ to $\tau > 0$ takes place *quantitatively*, although the results clearly show that there is a *qualitative* difference in the optimal control of instantaneous and delayed accumulation problems. Another interesting question, is the analysis of 'second best' *monotonic* optimal paths, if the *non-monotonic* first best optimal path is not applicable due to institutional constraints.

Appendix

A.1 Proof of proposition 1

To derive the stationary state of the system of functional differential equations, we set $\dot{s} = \dot{e} = 0$, which yields the implicit equations (8). The assumed curvature properties for F assure the existence of a unique solution.

A.2 Proof of proposition 2

Linearizing the system of functional differential equations (7) around the stationary state $(s^\star, e^\star)$ yields the following system of differential-difference equations:[9]

$$\begin{aligned} \dot{e}(t) &\approx (\gamma+\rho)(e(t)-e^\star) + \frac{F^\star_{es}}{F^\star_{ee}} \exp[-\rho\tau](e(t+\tau)-e^\star) - \frac{F^\star_{es}}{F^\star_{ee}}(e(t-\tau)-e^\star) \\ &\quad + \frac{F^\star_{es}}{F^\star_{ee}}(2\gamma+\rho)(s(t)-s^\star) + \frac{F^\star_{ss}}{F^\star_{ee}} \exp[-\rho\tau](s(t+\tau)-s^\star) \ , \qquad \text{(A.1a)} \\ \dot{s}(t) &\approx (e(t-\tau)-e^\star) - \gamma(s(t)-s^\star) \ . \qquad \text{(A.1b)} \end{aligned}$$

The general solution of the system of differential-difference equations can be written as an infinite sum of exponential functions (Bellman and Cooke 1963, Boucekkine et al. 2005). Introducing the following abbreviations:

$$X = \frac{F^\star_{es}}{F^\star_{ee}} \exp[-\rho\tau] \ , \quad Y = \frac{F^\star_{es}}{F^\star_{ee}} \ , \quad Z = \frac{F^\star_{ss}}{F^\star_{ee}} \exp[-\rho\tau] + \gamma(\gamma+\rho) > 0 \ , \qquad \text{(A.2)}$$

we derive for the characteristic equation

$$Q(x) = x^2 - \rho x - X \exp[\tau x](x+\gamma) + Y \exp[-\tau x](x-\rho-\gamma) - Z = 0 \ . \qquad \text{(A.3)}$$

$Q(x)$ is a *quasi-polynomial*, which exhibits, in general, an infinite number of roots with negative real part and an infinite number of roots with positive real part. To see this, note first that the characteristic roots are symmetric around $\rho/2$, i.e., if x^0 is a characteristic root, then $\rho - x^0$ is also a characteristic root (one can easily verify that $Q(x^0) = Q(\rho - x^0)$). Second, we introduce the new variable $y = \tau x$ and multiply Q with $\tau^2 \exp[y]$

$$Q(y) = \exp[y]\left[y^2 - y\rho\tau - \tau^2 Z\right] - \exp[2y]\tau X(y+\gamma\tau) + \tau Y(y-\gamma\tau-\rho\tau) \ , \qquad \text{(A.4)}$$

[9] Functions evaluated at the stationary state $(s^\star, e^\star)$ are denoted by a star.

in order to apply Theorem 13.1 of Bellman and Cooke (1963: 441). As $Q(y)$ has no *principal term*, i.e., a term where the highest power of y and the highest exponential term appear jointly,[10] $Q(y)$ has "an unbounded number of zeros with arbitrarily large positive real part" (ibid). But as the characteristic roots are symmetric around $\rho/2$, this also implies an unbounded number of roots with arbitrarily large negative real part.

For $F^{\star}_{es} = 0$, the characteristic equation (A.3) reduces to a quadratic equation:

$$Q(x) = x^2 - \rho x - Z \ , \tag{A.5}$$

which exhibits the two real solutions

$$x_1 = \frac{\rho}{2} - \sqrt{\left(\frac{\rho}{2}\right)^2 + Z} < 0 \ , \qquad x_2 = \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 + Z} > \rho \ . \tag{A.6}$$

[10] In this case, the principal term would be a term with $y^2 \exp[2y]$.

References

Asea, P. K. and P. J. Zak (1999): 'Time-to-build and cycles', *Journal of Economic Dynamics and Control*, **23:** 1155–75.

Bambi, M. (2008): 'Endogenous growth and time-to-build: the AK case', *Journal of Economic Dynamics and Control*, **32:** 1015–40.

Baumgärtner, S., Jöst, F. and R. Winkler (2007): 'Optimal dynamic scale and structure of a multi-pollution economy'. *Working Paper Series in Economics No. 50*, University of Lüneburg.

Bellman, R. and K. L. Cooke (1963): *Differential-Difference Equations.* Academic Press, New York.

Benhabib, J. and A. Rustichini (1991): 'Vintage capital, investment, and growth', *Journal of Economic Theory*, **55:** 323–39.

Boucekkine, R., Germain, M. and O. Licandro (1997): 'Replacement echoes in the vintage capital growth model', *Journal of Economic Theory*, **74:** 333–48.

Boucekkine, R., Licandro, O., Puch, L. A. and F. del Rio (2005): 'Vintage capital and the dynamics of the AK model', *Journal of Economic Theory*, **120:** 39–72.

Brandt-Pollmann, U., Winkler, R., Moslener, U. and J. Schlöder (2008): 'Numerical solution of optimal control problems with constant control delays', *Computational Economics*, **31:** 181–206.

Diehl, M., Leineweber, D. B. and A. A. S. Schäfer (2001): 'Muscod-II users' manual'. *Preprint 2001-25*, Interdisciplinary Center for Scientific Computing, University of Heidelberg.

El-Hodiri, M. A., Loehman, E. and A. Whinston (1972): 'An optimal growth model with time lags', *Econometrica*, **40:** 1137–46.

Falk, I. and R. Mendelsohn (1993): 'The economics of controlling stock pollutants: An efficient strategy for greenhouse gases', *Journal of Environmental Economics and Management*, **25:** 76–88.

Goeschl, T. and G. Perino (2007): 'Innovation without magic bullets: Stock pollution and R&D sequences', *Journal of Environmental Economics and Management*, **54:** 146–61.

Goulder, L. H. and K. Mathai (2000): 'Optimal CO_2 abatement in the presence of induced technological change', *Journal of Environmental Economics and Management*, **39:** 1–38.

Gradus, R. and S. Smulders (1993): 'The trade-off between environmental care and long-term growth', *Journal of Economics*, **58:** 25–51.

Heinzel, C. and R. Winkler (2007): 'The role of environmental and technology policies in the transition to a low-carbon energy industry'. *Working Paper No. 07/71*, CER-ETH – Center of Economic Research at ETH Zurich.

Keeler, E. M., Spence, E. M. and R. Zeckhauser (1972): 'The optimal control of pollution', *Journal of Economic Theory*, **4:** 19–34.

Leineweber, D. B., Bauer, I., Bock, H. G. and J. P. Schlöder (2003): 'An efficient multiple shooting based reduced SQP strategy for large-scale dynamic process optimization – part I: Theoretical aspects', *Computers and Chemical Engineering*, **27:** 157–66.

Moslener, U. and T. Requate (2001): 'Optimal abatement strategies for various interacting greenhouse gases – is the global warming potential a useful indicator'. *Discussion-Paper No. 360*, Alfred Weber-Institute of Economics, University of Heidelberg.

Moslener, U. and T. Requate (2007): 'Optimal abatement in dynamic multipollutant problems when pollutants can be complements or substitutes', *Journal of Economic Dynamics and Control*, **31:** 2293–316.

Van der Ploeg, F. and C. Withagen (1991): 'Pollution control and the Ramsey problem', *Environmental and Resource Economics*, **1:** 215–30.

Rustichini, A. (1989): 'Hopf bifurcation for functional differential equations of mixed type', *Journal of Dynamics and Differential Equations*, **1:** 145–77.

Smith, V. L. (1977): 'Control theory applied to natural and environmental resources – an exposition', *Journal of Environmental Economics and Management*, **4:** 1–24.

Winkler, R. (2008): 'Optimal compliance with emission constraints. Dynamic characteristics and the choice of technique', *Environmental and Resource Economics*, **39:** 411–32.

Working Papers of the Center of Economic Research at ETH Zurich

(PDF-files of the Working Papers can be downloaded at www.cer.ethz.ch/research).

08/91 R. Winkler
Optimal control of pollutants with delayed stock accumulation

08/90 S. Rausch and T. F. Rutherford
Computation of Equilibria in OLG Models with Many Heterogeneous Households

08/89 E. J. Balistreri, R. H. Hillberry and T. F. Rutherford
Structural Estimation and Solution of International TradeModels with Heterogeneous Firms

08/88 E. Mayer and O. Grimm
Countercyclical Taxation and Price Dispersion

08/87 L. Bretschger
Population growth and natural resource scarcity: long-run development under seemingly unfavourable conditions

08/86 M. J. Baker, C. N. Brunnschweiler, and E. H. Bulte
Did History Breed Inequality? Colonial Factor Endowments and Modern Income Distribution

08/85 U. von Arx and A. Ziegler
The Effect of CSR on Stock Performance: New Evidence for the USA and Europe

08/84 H. Gersbach and V. Hahn
Forward Guidance for Monetary Policy: Is It Desirable?

08/83 I. A. MacKenzie
On the Sequential Choice of Tradable Permit Allocations

08/82 I. A. MacKenzie, N. Hanley and T. Kornienko
A Permit Allocation Contest for a Tradable Pollution Permit Market

08/81 D. Schiess and R. Wehrli
The Calm Before the Storm? - Anticipating the Arrival of General Purpose Technologies

08/80 D. S. Damianov and J. G. Becker
Auctions with Variable Supply: Uniform Price versus Discriminatory

08/79 H. Gersbach, M. T. Schneider and O. Schneller
On the Design of Basic-Research Policy

08/78 C. N. Brunnschweiler and E. H. Bulte
Natural Resources and Violent Conflict: Resource Abundance, Dependence and the Onset of Civil Wars

07/77 A. Schäfer, S. Valente
Habit Formation, Dynastic Altruism, and Population Dynamics

07/76 R. Winkler
Why do ICDPs fail? The relationship between subsistence farming, poaching and ecotourism in wildlife and habitat conservation

07/75 S. Valente
International Status Seeking, Trade, and Growth Leadership

07/74 J. Durieu, H. Haller, N. Querou and P. Solal
Ordinal Games

07/73 V. Hahn
Information Acquisition by Price-Setters and Monetary Policy

07/72 H. Gersbach and H. Haller
Hierarchical Trade and Endogenous Price Distortions

07/71 C. Heinzel and R. Winkler
The Role of Environmental and Technology Policies in the Transition to a Low-carbon Energy Industry

07/70 T. Fahrenberger and H. Gersbach
Minority Voting and Long-term Decisions

07/69 H. Gersbach and R. Winkler
On the Design of Global Refunding and Climate Change

07/68 S. Valente
Human Capital, Resource Constraints and Intergenerational Fairness

07/67 O. Grimm and S. Ried
Macroeconomic Policy in a Heterogeneous Monetary Union

07/66 O. Grimm
Fiscal Discipline and Stability under Currency Board Systems

07/65 M. T. Schneider
Knowledge Codification and Endogenous Growth

07/64 T. Fahrenberger and H. Gersbach
Legislative Process with Open Rules

07/63 U. von Arx and A. Schäfer
The Influence of Pension Funds on Corporate Governance

07/62 H. Gersbach
The Global Refunding System and Climate Change

06/61 C. N. Brunnschweiler and E. H. Bulte
The Resource Curse Revisited and Revised: A Tale of Paradoxes and Red Herrings

06/60 R. Winkler
Now or Never: Environmental Protection under Hyperbolic Discounting

06/59 U. Brandt-Pollmann, R. Winkler, S. Sager, U. Moslener and J.P. Schlöder
Numerical Solution of Optimal Control Problems with Constant Control Delays

06/58 F. Mühe
Vote Buying and the Education of a Society

06/57 C. Bell and H. Gersbach
Growth and Enduring Epidemic Diseases

06/56 H. Gersbach and M. Müller
Elections, Contracts and Markets

06/55 S. Valente
Intergenerational Transfers, Lifetime Welfare and Resource Preservation

06/54 H. Fehr-Duda, M. Schürer and R. Schubert
What Determines the Shape of the Probability Weighting Function?

06/53 S. Valente
Trade, Envy and Growth: International Status Seeking in a Two-Country World

06/52 K. Pittel
A Kuznets Curve for Recycling

06/51 C. N. Brunnschweiler
Cursing the blessings? Natural resource abundance, institutions, and economic growth

06/50 C. Di Maria and S. Valente
The Direction of Technical Change in Capital-Resource Economics

06/49 C. N. Brunnschweiler
Financing the alternative: renewable energy in developing and transition countries

06/48 S. Valente
Notes on Habit Formation and Socially Optimal Growth

06/47 L. Bretschger
Energy Prices, Growth, and the Channels in Between: Theory and Evidence

06/46 M. Schularick and T.M. Steger
Does Financial Integration Spur Economic Growth? New Evidence from the First Era of Financial Globalization

05/45 U. von Arx
Principle guided investing: The use of negative screens and its implications for green investors

05/44 Ch. Bjørnskov, A. Dreher and J.A.V. Fischer
The bigger the better? Evidence of the effect of government size on life satisfaction around the world