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# Default Risk in Stochastic Volatility Models

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## Abstract

We consider a stochastic volatility model of the mean-reverting type to describe the evolution of a firm's values instead of the classical approach by Merton with geometric Brownian motions. We develop an analytical expression for the default probability. Our simulation results indicate that the stochastic volatility model tends to predict higher default probabilities than the corresponding Merton model if a firm's credit quality is not too low. Otherwise the stochastic volatility model predicts lower probabilities of default. The results may have implications for various financial applications.

Keywords: stochastic volatility, Merton model, default probabilities, rate of mean reversion

JEL Classification: G13, G21, G32

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# 1 Introduction

In credit risk modeling the KMV/CreditMetrics setting is one of the most popular approaches. It is based on the classical Merton setting, which describes the non-observable firm's asset values as a geometric Brownian motion. A firm defaults if, at the time of servicing the debt, its assets are lower than its outstanding debt. The Merton model is attractive because of its formal elegance, based on the option pricing theory developed by Black and Scholes (1973), and its ease of implementation. But it is well known that the underlying assumptions are quite unrealistic. Three different research strategies have been pursued to model credit risk since Merton developed his approach: extensions of the Merton model, the First Passage Model developed by Black and Cox (1976), and reduced-form models.<sup>1</sup>

We follow the first line of research for two reasons. First, many researchers and practitioners are applying the model. Second, the Basel Committee on Banking Supervision has used the Merton approach to calculate capital regulatory requirements. One crucial weakness of this approach is that the empirical evidence from time series of equity prices contradicts the Merton model. The observation goes back at least to Black (1976), who discussed the fat tail characteristics of return distributions. Moreover, the Merton model also predicts that the implied volatility of options is constant through time, which is false.

Since the late 1980s, stochastic volatility models have been developed (see the classical papers by Wiggins (1987), Hull and White (1987), Scott (1982), Stein and Stein (1991), and Heston (1993)) to explain some of the empirical features of the joint time series of stock and option prices, as the volatility process is not perfectly correlated with the Brownian motion and thus has an independent random component. We use flexible stochastic volatility models that have been developed in the seminal contributions of Fouque, Papanicolaou and Sircar (2001) and of Fouque, Sircar and Sølna (2006).

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<sup>1</sup>Structural and reduced-form models can be reconciled. In particular, assuming incomplete information of investors regarding the process dynamics that trigger the default in structural models produces default characteristics consistent with reduced-form models (see Elizalde (2005) for a survey).

Then, we address some key issues in credit risk management and banking regulation:

- How can default probabilities be determined in stochastic volatility models?
- How much do they differ from estimations based on the simple Merton model?

Concerning the latter, this difference is too small if trying to translate the recent results in Fouqué, Sircar, Sølna (2006) to the context of approximating the default probabilities (in the nonleverage case there is actually no difference). This is due to the fact that in this case one has to work under the subjective probability measure and the order of their asymptotic expansion is too small.

Our main contribution in this paper is a new analytical expression for the default probability for a class of fast mean reverting SV models, which offers a higher degree of accuracy compared to the results of Fouqué et al. (2006). The price to be paid is that the usual calibration procedure can no more be applied, since our asymptotic expansion explicitly depends on the solution of a Poisson equation and not just on the numerical properties of the latter. The extension to a larger class of models and a solution to the calibration problem is ongoing work.

Our further results are as follows: An analytical result and simulations indicate that stochastic volatility models tend to predict higher default probabilities than the corresponding Merton model with constant volatility if a firm's default risk is not too high. Hence, the Merton model underestimates the default probability of firms with high credit-worthiness. Otherwise the stochastic volatility model predicts lower probabilities of default.

Our paper contributes to four strands of the literature. First, a number of different specifications of stochastic volatility models have been suggested. We use flexible stochastic volatility of the mean-reverting type (Fouque, Papanicolaou, and Sircar 2001 and Fouque, Sircar, Sølna 2006).<sup>2</sup>

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<sup>2</sup>Recently, Bansal and Yaron (2004) and Tauchen (2004) have provided interesting general equilibrium foundations for stochastic volatility.

Second, the accuracy and contribution of the default forecasting model based on Merton's (1974) bond pricing model and developed by the KMV corporation has attracted substantial literature. Duffie and Wang (2004), for instance, show that KMV-Merton probabilities have significant predictive power in a model of default probabilities over time and can generate a term structure of default probabilities. Campbell, Hilscher, and Szilagyi (2004) estimate hazard models, finding that KMV-Merton-type default probabilities seem to have relatively little forecasting power after conditioning on other variables. Bharath and Shumway (2004) conclude that KMV-Merton-type default probabilities have some predictive power for default but they are not sufficient statistically. In this paper we explore the robustness of the Merton model with respect to stochastic volatility. Our analytical formula may help to increase the power of the Merton model in predicting default probabilities.

Third, the Merton model has been extended in many ways. Black and Cox (1976) allow for safety covenants and subordination arrangements, while Turnbull (1979) includes corporate taxes and bankruptcy costs. Kim, Ramaswamy, and Sunderesan (1993) allow the riskless interest rate to follow a square root process correlated with firm value. They show that default risk is not sensitive to the volatility of interest rates but is sensitive to interest rate expectations. Longstaff and Schwartz (1995) model stochastic interest rates correlated with the firm process, an exogenous early default, and an exogenous recovery rate. Finally, Leland (1994) and Leland and Toft (1996) endogenize the bankruptcy decision while accounting for taxes and bankruptcy costs. We suggest that the introduction of the stochastic volatility feature will predict higher default probabilities if firms are not highly leveraged and lower default probabilities for less healthy firms. As discussed in the section on implications, this result might be useful to improve the empirical accuracy of structural models.

The paper is organized as follows: In the next section we introduce the class of stochastic volatility models we are working with. In the third section we develop an analytical formula for the probability of default in this setting. In the fourth section we perform simulation exercises. In section five, we derive implications. Section six concludes.

## 2 A Stochastic Volatility Model to Describe a Firm's Value

We assume that the value  $S(t)$  of a firm has the following dynamics:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_t dW(t), \quad \text{with } \sigma_t = f(Y(t)), \quad (1)$$

$$dY(t) = \alpha(m - Y(t)) dt + \beta dZ(t), \quad t \geq 0, \quad (2)$$

where  $(W(t))_{t \geq 0}$  and  $(Z(t))_{t \geq 0}$  are standard Brownian motions that are stochastically independent<sup>3</sup>,  $m \in \mathbb{R}$  and  $\mu, \alpha, \beta$  are some positive constants. The first equation has the standard interpretation. The infinitesimal return  $\frac{dS(t)}{S(t)}$  has mean  $\mu dt$  with a constant rate of return  $\mu$  and random fluctuations governed by the volatility process  $\sigma_t = f(Y(t))$ , with  $f$  being some positive function of a process  $(Y(t))_{t \geq 0}$  driving the volatility. We note that  $(Y(t))_{t \geq 0}$  and  $(S(t))_{t \geq 0}$  are Feller processes.<sup>4</sup>

The stochastic differential equation for  $(Y(t))_{t \geq 0}$  is of the mean-reverting type. Here  $\alpha$  is the rate of mean reversion and  $m$  is the long-term level of  $(Y(t))_{t \geq 0}$ . The drift-term pulls  $Y(t)$  toward  $m$  and hence  $\sigma_t$  approaches the mean value of  $f(Y(t))$  with respect to the long-term distribution of  $(Y(t))_{t \geq 0}$ . The variable  $\beta$  is a positive real number that describes the volatility of the volatility.

The central assumption is that the volatility process is fast mean-reverting, i.e. the volatility level fluctuates randomly around its mean level and the episodes of high/low volatility are rather short. This volatility regime can be described by assuming that the parameter  $\epsilon = 1/\alpha$  is small when compared with other time scales. In this sense, there are several possible interpretations for  $(Y(t))_{t \geq 0}$ . For instance,  $Y(t)$  may refer to the level of ambiguity in financial markets, i.e. to the average level of confidence of market participants in their own statistical forecasts. Changing levels of ambiguity translate into different levels of volatility (see e.g. Chen et al. 2009 and Faria et al. 2009).

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<sup>3</sup>The second Brownian motion  $(Z(t))_{t \geq 0}$  can be correlated with the Brownian motion  $(W(t))_{t \geq 0}$ . In this paper we assume as a benchmark case that the instantaneous correlation is zero.

<sup>4</sup>We refer the reader to Karatzas and Shreve (1988) or Øksendal (1998) for details concerning stochastic differential equations and the related stochastic calculus.

Let  $\Phi(y)$  be the density function of the invariant distribution<sup>5</sup> of  $(Y(t))_{t \geq 0}$  and denote

$$L_{\Phi}^1(\mathbb{R}) := \left\{ g : \int_{\mathbb{R}} \Phi(y) |g(y)| dy < \infty \right\}. \quad (3)$$

Throughout the paper we use as a shortcut the following notation for the average with respect to the invariant distribution:

$$\langle g \rangle := \int_{\mathbb{R}} \Phi(y) g(y) dy, \quad (4)$$

for all  $g \in L_{\Phi}^1(\mathbb{R})$ . Furthermore we assume that  $f \in \mathcal{C}^2$ ,  $f^{2k} \in L_{\Phi}^1(\mathbb{R})$ ,  $k \leq 2$ . Then the *effective volatility*  $\bar{\sigma}$  is defined by  $\bar{\sigma}^2 = \langle f^2 \rangle$ , i.e. it is the average volatility with respect to invariant distribution of  $(Y(t))_{t \geq 0}$ .

The above stochastic volatility settings extend the classical Merton model in a natural way. For  $f(x) = e^x$  we obtain the Scott univariate SV model, and for  $f(x) = |x|$  we get the Stein-Stein model.

Following the Merton model the capital structure of the firm comprises equity and a zero-coupon bond with maturity  $T$  and face value  $B$ . Under these assumptions, equity represents a call option on the firm's assets with maturity  $T$  and strike price  $B$ . Then the firm will default at time  $T$  if  $S_T < B$ . The variable  $B$  is the default barrier. We use  $PD(T, B|t, x, y)$  to denote the default probability of the firm at current time  $t$ , when its value of debt is  $B$  at time  $T > t$  and  $S(t) = x$  and  $Y(t) = y$ . Following Fouqué et al. (2001), under a fast mean-reverting (FMR) volatility regime, the best approximation of the above SV model in the class of models with constant volatility is the following Merton model:

$$\frac{dS(t)}{S(t)} = \mu dt + \bar{\sigma} dW(t), \quad t \geq 0. \quad (5)$$

Denoting  $PD_0(T, B|t, x)$  the corresponding default probability in this Merton model, we have

$$PD_0(T, B|t, x) = \Phi_{NS} \left( \frac{\log B - \log x - (T - t) \cdot (\mu - \frac{1}{2} \bar{\sigma}^2)}{\sqrt{T - t} \cdot \bar{\sigma}} \right), \quad (6)$$

for all  $(t, x) \in [0, T) \times [0, \infty)$ , where  $\Phi_{NS}$  is the cdf of a standard normal random variable.

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<sup>5</sup>Note that for this Ornstein-Uhlenbeck process  $(Y(t))_{t \geq 0}$  the invariant distribution is  $N(m, \nu^2)$ , where  $\nu = \frac{\beta}{\sqrt{2\alpha}}$ .

Such a simple description for  $PD(T, B|t, x, y)$  is of course not possible, however, we obtained a new analytical approximation in the context of the asymptotic theory of Fouqué et al. (2001). In the following we restrict our attention to the case  $x > B, t < T$ , since this is the most interesting one for practical purposes.

### 3 Analytical Development of Default Probability

#### 3.1 The Main Result

In this section we derive an analytical approximation for the default probability in the above described SV framework. In the following we assume that the usual conditions for the asymptotic theory of Fouqué et al. (2001) are satisfied. This is a general technique to construct approximations for the solutions of a large class of PDEs including those used in option pricing theory. In particular, for the above SV scenario the corresponding approximations are obtained by performing an expansion of the PDE solution in powers of  $\sqrt{\epsilon}$ , where  $\epsilon = \frac{1}{\alpha}$ . In general it is possible to obtain an analytical description only for the first two terms of this expansion. Then the corresponding approximation is the leading order plus the first correction.

For the special case where the quantity of interest is the default probability in the above SV model, then this first correction is zero, however an analytical description of the second correction is still possible and this leads to the default probability approximation formula given below. In this context, performing an expansion of the probability of default in powers of  $\sqrt{\epsilon}$  reduces actually to an expansion in powers of  $\epsilon$ , which means far less terms as usually are needed for a good approximation. This is a strong motivation for trying to derive analytically as many terms as possible from the corresponding asymptotic expansion of the default probability.

As  $(Y_t)_{t \geq 0}$  is a Feller process the infinitesimal generator exists which is denoted by  $\mathcal{L}_0$ . Let



$\phi \in L^1_{\Phi}(\mathbb{R})$  be a solution of the Poisson equation

$$\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle.$$

Then we have the following

**Theorem 1**

A corrected default probability formula under the above stochastic volatility setting can be explicitly given by

$$\min\left\{1, \frac{\tilde{P}D(T, B|t, x, y) + |\tilde{P}D(T, B|t, x, y)|}{2}\right\}, \quad (7)$$

$$\tilde{P}D(T, B|t, x, y) := PD_0(T, B|t, x) + \frac{1}{\alpha}PD_1(T, B|t, x, y), \quad (8)$$

$$PD_1(T, B|t, x, y) = -\frac{1}{2}(\phi(y) - \langle \phi \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} - \frac{1}{4}(T - t)(\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle)$$

$$\cdot \left(2x^2 \frac{\partial^2 PD_0}{\partial x^2} + 4x^3 \frac{\partial^3 PD_0}{\partial x^3} + x^4 \frac{\partial^4 PD_0}{\partial x^4}\right), \quad (9)$$

for all  $(t, x, y) \in [0, T) \times (B, \infty) \times \mathbb{R}$ , where  $PD_0(T, B|t, x)$  is the default probability in the Merton setting with volatility  $\bar{\sigma}$ .

The proof and some elementary facts about the above Poisson equation and its solutions can be found in the Appendix.

### 3.2 Discussion

Several remarks are useful to put the result in perspective.

(i) As already mentioned in Fouqué et al. (2000), this type of approximations performs poorly close to  $T$  or to the other frontiers of the corresponding domain for  $x$ . However, our approximation is more accurate than the corresponding one in Fouqué et al. (2006), since we performed the expansion in powers of  $\epsilon$ , instead of  $\sqrt{\epsilon}$ , while preserving the same number of terms. Moreover, unlike Fouqué et al. (2006), our approximation also depends on  $y$ , which gives the chance to capture with this analytical formula a larger amount of the relevant market informations.

(ii) A weaker,  $y$ -independent version of the previous corrected default probability formula can be obtained by taking expectations with regard to  $y$ :

$$\min\left\{1, \frac{\overline{PD}(T, B|t, x) + |\overline{PD}(T, B|t, x)|}{2}\right\}, \quad (10)$$

where

$$\begin{aligned} \overline{PD}(T, B|t, x) &:= PD_0(T, B|t, x) + \frac{1}{\alpha} \overline{PD}_1(T, B|t, x), \\ \overline{PD}_1(T, B|t, x) &= -\frac{1}{4}(T-t)(\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) \cdot \left( 2x^2 \frac{\partial^2 PD_0}{\partial x^2} + 4x^3 \frac{\partial^3 PD_0}{\partial x^3} + x^4 \frac{\partial^4 PD_0}{\partial x^4} \right), \end{aligned} \quad (11)$$

for all  $(t, x) \in [0, T) \times (B, \infty)$ .

(iii) In practice, all the parameters of the above presented formulae can be estimated from equity prices data using nonlinear filtering methods, see e.g. Jazwinski (1970) or Tanizaki (1996). In this way, additional asymptotic expansions for the parameter calibration can be alleviated.

### 3.3 Relationship to the Merton Model

With the above notations we obtain in the limit for  $\alpha \rightarrow \infty$  (or  $\epsilon \rightarrow 0$ ),

$$\lim_{\alpha \rightarrow \infty} \tilde{PD}(T, B|t, x, y) = PD_0(T, B|t, x).$$

When the rate of mean reversion becomes very large, the stochastic volatility model converges to the Merton model with a constant volatility. Hence the Merton model corresponding to the effective volatility  $\bar{\sigma}$  is closest to our SV model. Thus the difference between the SV and Merton settings can be studied by examining  $\tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x)$ , as a function of  $\alpha$ , which will be done in the following.

For fixed  $T, B, t, x, y$  we introduce

$$\delta(\alpha) = \tilde{PD}(T, B|t, x, y) - PD_0(T, B|t, x) = \frac{1}{\alpha} PD_1(T, B|t, x, y), \quad \forall \alpha > 0. \quad (12)$$

Then, by observing that

$$\frac{d^n \delta}{d\alpha^n} = (-1)^n \cdot n! \frac{1}{\alpha^{n+1}} PD_1(T, B|t, x, y), \quad \forall n \in \mathbf{N}, \alpha > 0 \quad (13)$$

we obtain the following

**Corollary**

A.) If  $PD_1(T, B|t, x, y) \neq 0$ , then

$$\text{sgn}(PD_1(T, B|t, x, y)) \cdot (\tilde{P}D(T, B|t, x, y) - PD_0(T, B|t, x)) > 0, \quad \forall \alpha > 0 \quad (14)$$

and the function

$$\alpha \rightarrow |\tilde{P}D(T, B|t, x, y) - PD_0(T, B|t, x)| \quad (15)$$

is strictly monotonically decreasing.

B.) If  $PD_1(T, B|t, x, y) = 0$ , then

$$\tilde{P}D(T, B|t, x, y) = PD_0(T, B|t, x), \quad \forall \alpha > 0. \quad (16)$$

The above result regarding the difference between the default probability in our SV model and the default probability in the corresponding Merton setting will be graphically illustrated and discussed in the next section.

## 4 Examples

In this section we perform simulations using the derived analytical formula for default probability in order to illustrate how default probabilities in the stochastic volatility model relate to the default risk estimated by the Merton model.

As a basic case, we adopt the Scott SV model and we use the following set of parameters (see Duan, Gauthier, and Zaanoun, 2004):  $t = 0, T = 1, \mu = 0.1, \nu = 0.26, m = -0.5358, Y_0 = -0.5$ . The current value of the first firm was taken to be  $S_0 = 100$  monetary units. For the corresponding Merton model we took the diffusion coefficients equal to the respective components of  $\bar{\sigma}$ . We now vary the face value of debt  $B$  and the rate of mean reversion  $\alpha$  and compare the resulting default probabilities under the stochastic volatility model and under the Merton model.

From these simulation exercises we report the following figures (Figure 1 to 4 in the appendix). In each figure, the default barrier  $B$  is fixed. The  $x$ -axis represents the values for  $\alpha$ , which we vary between 0 and 1200. The  $y$ -axis stands for the default probability. The straight line in each figure is the default probability under the Merton setting, as the default risk does not depend on  $\alpha$ . The other curve shows how the default probability depends on the rate of mean reversion under the stochastic volatility model. We report the figures for  $B = 40, 55, 60, 65$  in the appendix.

The figures illustrate some characteristics of the stochastic volatility model.<sup>6</sup> First, the default probability under SV converges monotonically to the PD under the Merton setting when the rate of mean reversion becomes large. Intuitively, when the rate of mean reversion becomes very large, the SV model converges to the Merton model.

Second, for lower rates of mean reversion, the default prediction of the SV model differs from the PD of the Merton model by a wide margin which is not present with an approximation of a lower order. Third, if the default barrier is not too high (and thus the default risk is not too high, either), the probability of default is higher under stochastic volatility. By performing simulations with varying  $B$ , we can observe that this pattern occurs for default probabilities in the Merton setting below 6%. The opposite pattern occurs for higher default risk when stochastic volatility models predict lower default rates.

The result requires subtle interpretation. Consider a firm with low default risk as measured by the Merton approach. With a low rate of mean reversion under SV, volatility fluctuates. If volatility is high, the default probability will increase. The opposite will occur if volatility shrinks. Since the default risk is not too high, the first effect dominates, as higher volatility raises the default probability disproportionately. The decline of the PD when volatility declines in comparison to effective volatility is comparatively small. For high default risk, the relative importance of both effects is reversed.

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<sup>6</sup>We performed a host of simulation exercises by varying  $B$  from zero to 80 for different sets of parameters. All simulations confirmed the qualitative results which we report here. Details are available upon request.

## 5 Implications

Our results have several implications. First, taking into account the stochastic feature can yield quite different predictions for probabilities of default. In particular, firms that appear to be healthy and are assessed to be of good credit quality may turn out to have much higher default risk under SV. Moreover, the rate of mean reversion has a major impact on the probability of default. Such properties have direct implications for risk management, credit analysis, and the pricing of credit derivatives.

Second, it is well known that structural models of default have difficulties in generating credit spread that are comparable to the ones observed in practice. The comprehensive study by Eom, Helwege and Huang (2004) shows that the Merton spreads are low compared to empirical values and that more recent conceptual innovations by Longstaff and Schwartz (1995), and Collin-Dufresne and Goldstein (2001) tend to over-predict credit risk of bonds that are highly leveraged and under-predict spreads of safer bonds. As our model predicts higher default probabilities for healthy firms and lower defaults otherwise in comparison to the Merton model, our model might be useful in further applications to solve the empirical puzzle concerning the default spreads of corporate bonds.

## 6 Conclusion

In this paper we propose a simple formula for default probability in SV models that extends the Merton model. The SV model is more flexible than the Merton setting. The latter can be obtained by taking the limit  $\alpha \rightarrow \infty$  which produces a Brownian motion with constant volatility. Therefore, assuming that a firm dynamics can be described with the SV model under the fast mean reverting volatility regime less restrictive than the Merton framework. Moreover, as argued in the introduction the SV model provides a better description of the observed distributional properties of financial time series.

Our simulation exercises suggest that the stochastic volatility feature is important in assess-

ing the default risk of firms. Hence our formula can enrich the tools of credit risk management. Numerous issues deserve further research. The method of higher order approximation could be applied to more general correlation structures. Moreover, calibration of solutions with higher-order approximations requires the calibration of solutions of the Poisson equation. These topics constitute an entire research project.

## 7 Appendix

### 7.1 Proof of the Theorem

Considering the pricing problem in Fouqué et al. (2001), the default probability function  $PD(T, B|t, x, y)$  satisfies

$$\frac{\partial PD}{\partial t} + \frac{1}{2}f^2(y)x^2\frac{\partial^2 PD}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2 PD}{\partial y^2} + \alpha(m-y)\frac{\partial PD}{\partial y} + \mu x\frac{\partial PD}{\partial x} = 0, \quad (17)$$

on  $(t, x, y) \in [0, T) \times (B, \infty) \times \mathbb{R}$  with terminal condition  $PD(T, B|T, x, y) = h_{PD}(x) = \frac{1}{2}(1 + \text{sgn}(B - x))$ , for all  $x \geq B, y \in \mathbb{R}$ .

With the usual rescaling method for modeling fast mean reversion in the volatilities

$$\alpha = \frac{1}{\epsilon}, \quad \beta = \nu \frac{\sqrt{2}}{\sqrt{\epsilon}}, \quad \epsilon > 0,$$

( $m$  and  $\nu$  fixed constants,  $\nu > 0$ ) we can rewrite the above problem as

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \mathcal{L}_1\right)PD^\epsilon = 0, \quad (18)$$

where  $PD^\epsilon$  is the rescaled default probability,

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \quad (19)$$

$$\mathcal{L}_1 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y) \cdot x^2 \cdot \frac{\partial^2}{\partial x^2} + \mu x \cdot \frac{\partial}{\partial x}. \quad (20)$$

The idea is to expand  $PD^\epsilon$  in powers of  $\epsilon$ :

$$PD^\epsilon = PD_0 + \epsilon PD_1 + \epsilon^2 PD_2 + \dots, \quad (21)$$

where  $PD_k, k = 0, 1, \dots$ , are function of  $(t, x, y)$  to be determined.

As usual in the asymptotic theory of Fouqué et al. (2000), we are primarily interested in the first two terms  $PD_0 + \epsilon PD_1$ .

Substituting (21) in (18), we get that  $PD_0$  is the default probability in the Merton setting with constant volatility  $\bar{\sigma}$  and  $PD_1$  has to verify

$$\mathcal{L}_0 PD_1 = -\mathcal{L}_1 PD_0, \quad (22)$$

on  $[0, T) \times (B, \infty) \times \mathbb{R}$ , with the terminal condition  $PD_1(T, B|T, x, y) = 0$  and with the centering condition  $\langle \mathcal{L}_1 PD_1 \rangle = 0$ .

Hence, to prove our assertion it is sufficient to show that the function  $PD_1$  described in the Theorem satisfies the above conditions.

Firstly observe that

$$\langle \mathcal{L}_1(x^n \cdot \frac{\partial^n PD_0}{\partial x^n}) \rangle = 0, \quad \forall n \in \mathbb{N}. \quad (23)$$

Using property (23) and the fact that  $\phi$  is a solution of the above Poisson equation, we obtain

$$\mathcal{L}_1 PD_0 = \frac{1}{2}(f^2(y) - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2},$$

and

$$\mathcal{L}_0 PD_1 = -\frac{1}{2}(f^2(y) - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2}.$$

By summing up both equations, we obtain the PDE (22) and it only remains to verify the centering condition, which results as follows:

$$\begin{aligned} \langle \mathcal{L}_1 PD_1 \rangle &= \frac{1}{2} \langle (f^2 - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_1}{\partial x^2} \rangle - \frac{1}{2} \langle \phi \rangle \langle \mathcal{L}_1 \left( x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} \right) \rangle \\ &\quad + \langle \mathcal{L}_1 \left( PD_1 + \frac{1}{2} \phi x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} \right) \rangle. \end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} &\langle \mathcal{L}_1 \left( PD_1 + \frac{1}{2} \phi x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} \right) \rangle \\ &= \frac{1}{4} (\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) \left( 2x^2 \frac{\partial^2 PD_0}{\partial x^2} + 4x^3 \frac{\partial^3 PD_0}{\partial x^3} + x^4 \frac{\partial^4 PD_0}{\partial x^4} \right). \end{aligned}$$

From property (23) we have  $\langle \mathcal{L}_1 \left( x^2 \cdot \frac{\partial^2 PD_0}{\partial x^2} \right) \rangle = 0$ . Hence, it remains to show that

$$\begin{aligned} &\langle (f^2 - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_1}{\partial x^2} \rangle \\ &= -\frac{1}{2} (\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) \left( 2x^2 \frac{\partial^2 PD_0}{\partial x^2} + 4x^3 \frac{\partial^3 PD_0}{\partial x^3} + x^4 \frac{\partial^4 PD_0}{\partial x^4} \right). \quad (24) \end{aligned}$$



For this purpose observe first that

$$\begin{aligned}
& \langle (f^2 - \langle f^2 \rangle) PD_1 \rangle \\
&= -\frac{1}{2} (\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) x^2 \frac{\partial^2 PD_0}{\partial x^2} - \frac{1}{4} (T - t) (\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) \langle (f^2 - \langle f^2 \rangle) \rangle \\
&\quad \cdot \left( 2x^2 \frac{\partial^2 PD_0}{\partial x^2} + 4x^3 \frac{\partial^3 PD_0}{\partial x^3} + x^4 \frac{\partial^4 PD_0}{\partial x^4} \right) \\
&= -\frac{1}{2} (\langle f^2 \phi \rangle - \langle f^2 \rangle \langle \phi \rangle) x^2 \frac{\partial^2 PD_0}{\partial x^2}.
\end{aligned}$$

Finally, using the fact that

$$\frac{1}{2} \langle (f^2 - \langle f^2 \rangle) \cdot x^2 \cdot \frac{\partial^2 PD_1}{\partial x^2} \rangle = \frac{1}{2} x^2 \cdot \frac{\partial^2}{\partial x^2} \left( \langle (f^2 - \langle f^2 \rangle) PD_1 \rangle \right)$$

we obtain (24), which completes the proof. □

## 7.2 Some facts about the Poisson Equation

Following Fouqué, Papanicolaou, and Sircar (2001), we report here some elementary facts about the Poisson equation

$$\mathcal{L}_0 \phi = g, \tag{25}$$

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \quad m \in \mathbb{R}, \nu > 0.$$

where  $g \in L^1_{\Phi}(\mathbb{R})$  (see (3)) The centering condition  $\langle g \rangle = 0$  (see (4)) is necessary for (25) to admit a solution  $\phi \in L^1_{\Phi}(\mathbb{R})$  and then it can be shown that the derivative of  $\phi$  can be explicitly given by

$$\phi' = \frac{1}{\nu^2 \Phi} \int_{-\infty}^{\cdot} g \cdot \phi. \tag{26}$$

The solutions of the Poisson equation above satisfy the following growth condition: if  $|g(y)| \leq C_1(1 + |y|^n)$ ,  $n \in \mathbb{N}$ , then  $|\phi(y)| \leq C_2(1 + |y|^n)$  where  $C_1$  and  $C_2$  are two constants.

This property ensures that all terms involving  $\phi$  in the Theorem are well defined.

### 7.3 Figures

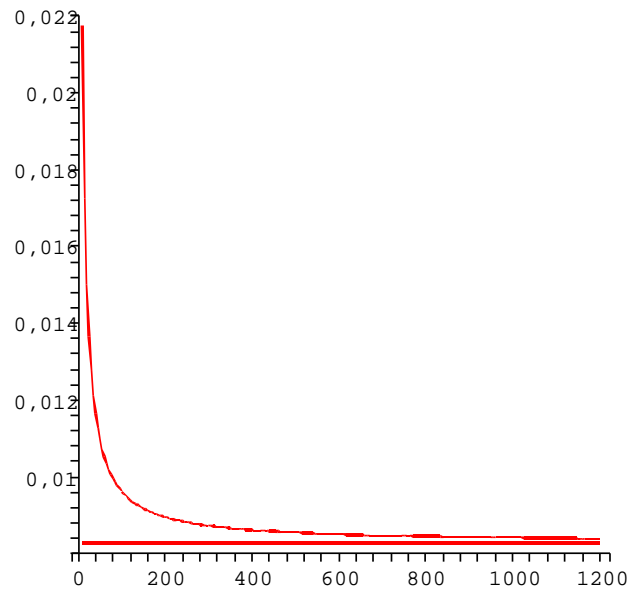


Figure 1: Default barrier:  $B = 40$ .

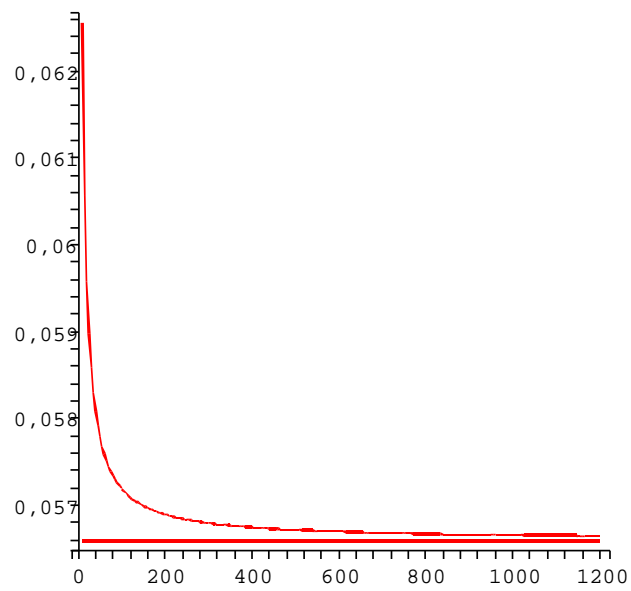


Figure 2: Default barrier:  $B = 55$ .

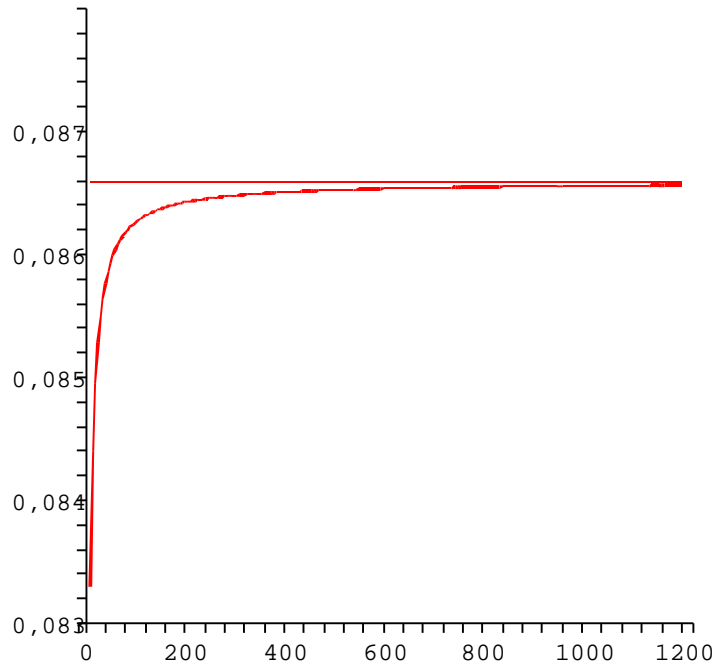


Figure 3: Default barrier:  $B = 60$ .

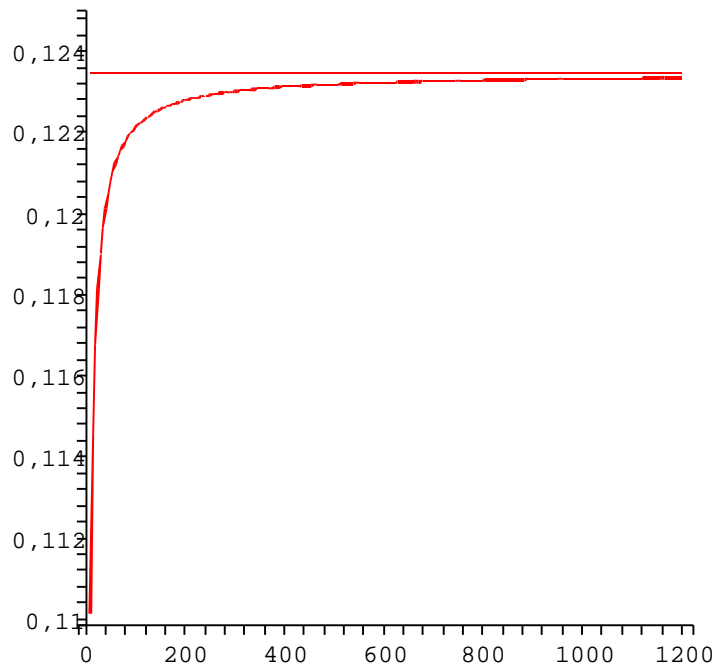


Figure 4: Default barrier:  $B = 65$ .

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