A test for weakly separable preferences

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Abstract: We identify necessary and sufficient conditions under which a finite data set of price vectors and consumption bundles can be rationalized by a weakly separable utility function. Our result could be understood as a generalization of Afriat's Theorem.

Keywords: Afriat's Theorem, utility function, revealed preference, generalized axiom of revealed preference, consistent family of preferences

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1. Introduction

Suppose we observe a consumer making purchases from ℓ goods, with a typical observation t consisting of the bundle $x^t \in R^\ell_+$ chosen by the consumer and the price vector $p^t \in R^\ell_{++}$ that the consumer faces. A finite data set $\mathcal{O} = \{(p^t, x^t)\}_{t\in\mathcal{T}}$ is said to be *rationalized* by the function $U : R^\ell_+ \to R$ if, for all $t \in \mathcal{T}$, the observed bundle x^t maximizes $U(x)$ in the set

$$B^t = \{x \in R^\ell_+ : p^t \cdot x \leqslant p^t \cdot x^t\}. \tag{1}$$

Afriat's Theorem gives us the precise condition under which a data set can be rationalized by a *well-behaved*, i.e., strongly monotone[1] and continuous, utility function U. It says that this is possible if and only if the data set obeys the *generalized axiom of revealed preference* or GARP, for short (see Afriat (1967) and Varian (1982)). This property requires that there be no strict revealed preference cycles on the set of observed consumption bundles $\mathcal{X} = \{x^t\}_{t\in\mathcal{T}}$, where a bundle x^t is said to be revealed preferred (revealed strictly preferred) to another bundle x^s if $p^t \cdot x^t \geqslant (>)\, p^t \cdot x^s$. GARP is an easy to check property, either directly or

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[1] This means that $U(x') > U(x)$ whenever $x' > x$.

via a linear program, so Afriat's Theorem has become the cornerstone of a large empirical literature on consumer demand.

It is common in empirical and theoretical work to impose more conditions on the utility function, in addition to requiring it to be well-behaved. A particularly common and convenient property is *weak separability* (which we shall often refer to in this paper simply as 'separability'). For this reason, it is useful to develop a characterization of data sets that could be rationalized by utility functions with this added feature. The objective of this paper is to solve this problem.

We assume that the set of goods can be divided into non-overlapping subsets X_1, X_2,..., $X_{\bar{J}}$, with X_j consisting of $\ell_j \geqslant 1$ goods. We could interpret the set X_j in different ways. For example, it could represent a particular category of goods, say clothing items or food items, it could represent goods consumed in period j in an inter-temporal model, or goods consumed in state j in a model with uncertainty. In these cases and in many others, one may wish to check whether the agent has a preference over bundles in X_j that is independent of her consumption of goods outside that set. We denote the agent's consumption bundle by $x = (x_1, x_2, ..., x_{\bar{J}})$, where the subvector $x_j \in R_+^{\ell_j}$ gives his consumption of X_j-goods. The data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in \mathcal{T}}$ is said to be rationalized by a weakly separable utility function if there are well-behaved functions $U_j : R_+^{\ell_j} \to R_+$ and $F : R_+^{\bar{J}} \to R$ such that $G : \Pi_{j=1}^{\bar{J}} R_+^{\ell_j} \to R$, given by

$$G(x) = F(U_1(x_1), U_2(x_2), ..., U_M(x_M)), \tag{2}$$

rationalizes $\mathcal{O}$. It is not hard to see that if $\mathcal{O}$ can be rationalized by G, then the segmented data set $\mathcal{O}_j = \{(p_j^t, x_j^t)\}_{t \in \mathcal{T}}$ can be rationalized by U_j (where p_j^t refers to the subvector of p^t corresponding to the prices of X_j-goods). Consequently, $\mathcal{O}_j$ must obey GARP. Thus it is clear that for $\mathcal{O}$ to have a weakly separable rationalization, it is necessary that $\mathcal{O}$ and $\mathcal{O}_j$ (for all j) obey GARP. It may be reasonable to imagine that these conditions are also sufficient for such a rationalization, but that turns out to be false (see Section 4).

To formulate a correct set of necessary and sufficient conditions, it is helpful to examine Afriat's Theorem more closely. Given a data set $\mathcal{O}$, the revealed preference relations impose an incomplete order on $\mathcal{X} = \{x^t\}_{t \in \mathcal{T}}$. It is easy to show that so long as there are no strict revealed preference cycles – which is precisely what GARP guarantees – there will be a preference $\succeq$ (i.e., a reflexive, transitive, *and complete* relation) defined on $\mathcal{X}$ that is *consistent with those revealed relationships*, i.e. $x^t \succeq (\succ)\, x^s$ if x^t is revealed preferred (revealed strictly preferred) to x^s. The non-trivial part of Afriat's Theorem constructs a well-behaved utility function, defined on the consumption space R^ℓ_+, that rationalizes the data and agrees with $\succeq$ on $\mathcal{X}$. In other words, we could understand the role of GARP as ensuring the existence of a consistent preference $\succeq$ on $\mathcal{X}$; rationalization of the data set then follows from the existence of $\succeq$. It is this concept of a consistent preference that will play a key role in our characterization of data sets that are rationalizable by weakly separable utility functions.

Let $\left\{\{\succeq_j\}_{j=1}^{\bar{J}}, \succeq\right\}$ be a collection of preferences, where $\succeq$ is a preference on $\mathcal{X}$ and $\succeq_j$ a preference on $\mathcal{X}_j$. First, we require that the preference $\succeq_j$ be a consistent preference on $\mathcal{X}_j$. Given this, $x^t \in \mathcal{X}$ is said to be conditionally revealed preferred to $x^s \in \mathcal{X}$ if there is a bundle $x' \in \Pi_{j=1}^{\bar{J}}\mathcal{X}_j$ such that $p^t \cdot x^t \geqslant p^t \cdot x'$ and $x'_j \succeq_j x^s_j$ for all j; the conditional revealed preference is said to be strict if either $p^t \cdot x^t > p^t \cdot x'$ or $x'_j \succ_j x^s_j$ for some j. The collection $\left\{\{\succeq_j\}_{j=1}^{\bar{J}}, \succeq\right\}$ is said to be a *consistent family of preferences* if (i) for all j, $\succeq_j$ is a consistent preference and (ii) the preference $\succeq$ is consistent with the conditional revealed relations on $\mathcal{X}$. The main result of our paper says that

> *the data set $\mathcal{O} = \{p^t, x^t\}_{t \in \mathcal{T}}$ can be rationalized by a weakly separable utility function if and only if it admits a consistent family of preferences.*

Furthermore, if a consistent family of preferences exists, then we can choose U_j such that it extends $\succeq_j$ and G such that it extends $\succeq$. This theorem provides a test of whether a data set has a weakly separable rationalization since whether or not a consistent family of preferences exists on a finite set of bundles is clearly a finite problem.

Where is the difficulty in proving our main result? It is easy to see that the existence of a consistent family is a necessary condition: we need only choose $\succeq_j$ to be the preference generated on U_j on $\mathcal{X}_j$ and $\succeq$ to be the preference generated by G on $\mathcal{X}$. To prove sufficiency, we first notice that Afriat's Theorem guarantees that there is a well-behaved sub-utility function U_j that rationalizes $\mathcal{O}_j$ and extends $\succeq_j$. This collection of sub-utility functions $\{U_j\}_{j=1}^{\bar{J}}$, transforms the set $\mathcal{O}$ into $\widehat{\mathcal{O}} = \{(K^t, u^t\}_{t\in\mathcal{T}}$, where $u^t = (U_1(x_1^t), U_2(x_2^t), ..., U_{\bar{J}}(x_{\bar{J}}^t)$ and

$$K^t = \{(U_1(x'_1), U_2(x'_2), ..., U_{\bar{J}}(x'_{\bar{J}}) : x' \in B^t\},$$

with B^t given by (1). The next and final step of the proof consists of finding a function F that rationalizes $\widehat{\mathcal{O}}$ in the sense that, for all $t \in \mathcal{T}$, $F(u^t) \geqslant F(u)$ for all $u \in K^t$; with this we can guarantee that G, as defined by (2) rationalizes the data. By a recent result of

Forges and Minelli (2009), which generalizes Afriat's Theorem to non-linear constraint sets, we know that F exists so long as $\widehat{\mathcal{O}}$ obeys a generalized version of GARP. However, there are many functions that rationalize $\mathcal{O}_j$ and extend $\succeq_j$, so there are many possible versions of $\widehat{\mathcal{O}}$. The principal difficulty in the proof lies in carefully constructing the correct sub-utility functions U_j so as to ensure that $\widehat{\mathcal{O}}$ obeys GARP.

It is well-known that the rationalizing utility function provided by Afriat's Theorem is not just well-behaved but also concave. In our main theorem, the rationalizing function we construct is such that the U_js can always be chosen to be concave functions but F need not be a concave function. Indeed, we give (in Section 4) an example of a data set that admits a weakly separable rationalization, but where the weakly separable utility function can *never* be a quasi-concave function.[2] This example highlights the difference between our main theorem and the revealed preference test of weak separability developed by Varian (1983).[3] Varian provides necessary and sufficient conditions under which a data set is rationalized by a utility function G such that U_js and F are well-behaved and concave. The concavity assumption that he imposes on both the sub-utility and aggregator functions makes for a relatively straightforward proof of the validity of his conditions. The example we provide shows that rationalization in the sense of Varian is substantively different from the one considered in this paper.

Our main result can be extended to test for models of utility maximization involving nested layers of weak separability. For example, we may be interested in testing whether a data set can be rationalized by a utility function of the form

$$G(x_0, x_1, x_2, x_3) = U_0(x_0, U_1(x_1, U_2(x_2, U_3(x_3)))),$$

where x_i is a bundle of X_i-goods (or $i = 0, 1, 2, 3$). For example, $x_i \in R_+$ could be the agent's consumption in period i and the agent has a forward looking utility function defined over consumption in three periods. We show that a data set is rationalizable by a utility function with nested weak separability if and only if it admits a consistent family of preferences on the sets of observed consumption bundles, where the notion of consistency is now modified to take into account the nested structure.

The rest of the paper is organized as follows. Section 2 gives a quick survey of basic concepts and Afriat's Theorem. The Forges-Minelli Theorem plays a crucial role in the

[2] This is surprising since we know, from Afriat's Theorem, that the data set can be rationalized by a well-behaved and concave utility function; such a function cannot, however, be weakly separable as well.

[3] For implementations of Varian's test of weak separability, see Cherchye et al. (2011) and its references.

proof of our main result and Section 3 is devoted to re-presenting and extending that result in a way that suits our purposes. Section 4 presents the main result and discusses various issues related to it. Section 5 shows how the test developed in the previous section can be extended to test for nested weak separability.

2. Basic Concepts and Afriat's Theorem

Let $\mathcal{O} = \{(p^t, x^t)\}_{t \in \mathcal{T}}$ be a finite set of T elements, where $p^t \in R^\ell_{++}$ and $x^t \in R^\ell_+$. We interpret $\mathcal{O}$ as a set of observations, where x^t is the observed bundle of ℓ goods chosen by the agent (the *demand bundle*) at the price vector p^t. Given a price vector $p \in R^\ell_{++}$ and income $w > 0$, the agent's *budget set* is defined as the set $B(p, w) = \{x \in R^\ell_+ : p \cdot x \leqslant w\}$. A function $U : R^\ell_+ \to R$ is said to *rationalize* the set $\mathcal{O}$ if, at all $t \in \mathcal{T}$, $U(x^t) \geqslant U(x)$ for all $x > 0$ such that $p^t \cdot x \leqslant p^t \cdot x^t$. In other words, x^t is the bundle that maximizes the agent's utility function U within the budget set $B^t \equiv B(p^t, w^t)$, where $w^t = p^t \cdot x^t$.

We are interested in finding conditions under which $\mathcal{O}$ is rationalizable by a *well-behaved* utility function U; by this we mean that U is continuous and strongly monotone (i.e. $U(x') > U(x)$ whenever $x' > x$). For this purpose, it is useful to introduce a number of concepts. Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x^t\}_{t \in \mathcal{T}}$. ($\mathcal{X}$ includes, if necessary, multiple copies of the same vector.) For x^t, $x^s \in \mathcal{X}$, we say that x^t is *directly revealed preferred to* x^s if $p_t \cdot x_s \leqslant p_t \cdot x_t$; when this inequality is strict, we say that x^t is *directly revealed strictly preferred to* x^s. We denote these relations by $x^t \succeq^* x^s$ and $x^t \succ\!\!\succ^* x^s$ respectively. We say that x^t is *revealed preferred* to x^s (and denote it by $x^t \succeq^{**} x^s$) if there are observations t_1, t_2,..., t_n such that $x^t \succeq^* x^{t_1}$, $x^{t_1} \succeq^* x^{t_2}$, ..., $x^{t_{n-1}} \succeq^* x^{t_n}$, and $x^{t_n} \succeq^{**} x^s$; x^t is said to be *revealed strictly preferred* to x^s (denoted by $x^t \succ\!\!\succ^{**} x^s$) if any of the direct preferences in this sequence is strict. If $x^t \succeq^{**} x^s$ and $x^s \succeq^{**} x^t$, then we say that x^t and x^s are *revealed indifferent* and denote it by $x^t \sim^{**} x^s$. We refer to $\succeq^{**}$ and $\succ\!\!\succ^{**}$, as the *revealed relations* (or *revealed preference relations*) *of* $\mathcal{O}$.

The set $\mathcal{O}$ is said to obey the *generalized axiom of revealed preference* (GARP) if whenever there are observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfying

$$p^{t_1} \cdot x^{t_2} \leqslant p^{t_1} \cdot x^{t_1};\ p^{t_2} \cdot x^{t_3} \leqslant p^{t_2} \cdot x^{t_2}; \ldots ; p^{t_{n-1}} \cdot x^{t_n} \leqslant p^{t_{n-1}} \cdot x^{t_{n-1}};\ \text{and}\ p^{t_n} \cdot x^{t_1} \leqslant p^{t_n} \cdot x^{t_n}$$

then all the inequalities have to be equalities. One could re-formulate GARP in terms of the revealed relations: the data set $\mathcal{O}$ obeys GARP if whenever there are observations (p^{t_i}, x^{t_i})

(for $i = 1, 2, ..., n$) satisfying

$$x^{t_1} \succeq^* x^{t_2},\ x^{t_2} \succeq^* x^{t_3},\ ..., x^{t_{n-1}} \succeq^* x^{t_n}, \text{ and } x^{t_n} \succeq^* x^{t_1}, \tag{3}$$

then none of direct revealed preferences in this sequence can be replaced with the strict revealed preference relation $\succ\!\succ^*$.[4, 5]

Afriat's Theorem says that *the data set $\mathcal{O}$ can be rationalized by a well-behaved utility function if and only if it obeys GARP.* Standard proofs of this result (see, for example, Fostel et al. (2004)) proceed by showing that GARP holds if and only if there are numbers $\lambda^t > 0$ and ϕ^t (for every $t \in T$) that obey the *Afriat inequalities*, i.e.,

$$\phi^t \leqslant \phi^k + \lambda^k p^k \cdot (x^t - x^k) \text{ for all } k \neq t. \tag{4}$$

The utility function $U : R^l_+ \to R$ given by

$$U(x) = \min_{(p^t, x^t) \in \mathcal{O}} \left\{ \phi^t + \lambda^t p^t \cdot (x - x^t) \right\} \tag{5}$$

can then be shown to rationalize $\mathcal{O}$, with $U(x^t) = \phi^t$ for all $x^t \in \mathcal{X}$. This function is clearly well-behaved and it is also concave.

Everything we have said so far is completely standard in the literature. To understand Afriat's Theorem better and to extend it in a way that is crucial for our purpose, we now present this result again more carefully, taking an approach that is less familiar.

We call a binary relation on a set a *preference relation* (or simply a preference) if it reflexive, transitive, and complete. Let $\succeq$ be a preference on $\mathcal{X}$. We write $x \succ y$ if $x \succeq y$ but $y \not\succeq x$ and $x \sim y$ if $x \succeq y$ and $y \succeq x$. The preference $\succeq$ is said to be *consistent with the revealed relations of $\mathcal{O}$* (or simply, a *consistent preference*) if it has the following two properties: (i) $x \succeq y$ whenever $x \succeq^{**} y$ and (ii) $x \succ y$ whenever $x \succ\!\succ^{**} y$.[6] Note that (i) also implies that $x \sim y$ if $x \sim^{**} y$. It is clear that to check that (i) and (ii) holds, we need only check these properties for the direct relations $\succeq^*$ and $\succ\!\succ^*$. The following result (whose simple proof we shall skip) says that the existence of a consistent preference $\succeq$ on $\mathcal{X}$ is *necessary* for rationalizability by a well-behaved utility function.

[4] To say the obvious, even if $x^t \succeq^* x^s$ and $x^s \not\succeq^* x^t$ we do not obtain $x^t \succ\!\succ^* x^s$.

[5] GARP was first introduced by Varian (1983), who defined a data set $\mathcal{O}$ as obeying GARP if the following holds: whenever there are two observed bundles x^t and x^r with $x^t \succeq^{**} x^r$, then $x^r \not\succ\!\succ^* x^s$. This definition is plainly equivalent to the one we have given, which is sometimes also referred to as *cyclical consistency* (Varian (1983)). In defining GARP the way we did, we follow Fostel et al (2004), amongst others.

[6] Note it does not say that $x \sim y$ implies that $x \sim^{**} y$ nor does it say that $x \succ y$ implies that $x \succ\!\succ^{**} y$.

Proposition 1 *Suppose that $\mathcal{O}$ is drawn from an agent who maximizes a locally non-satiated utility function U.*[7] *Then the preference $\succeq_U$ on $\mathcal{X}$ induced by U (i.e., $x^t \succeq_U x^s$ if $U(x^t) \geqslant U(x^s)$) is consistent with the revealed relations of $\mathcal{O}$.*

When does a data set $\mathcal{O}$ admit a preference on $\mathcal{X}$ that agrees with its revealed relations? It is intuitive that GARP, which precludes the existence of strict revealed preference cycles on $\mathcal{X}$, should be both a necessary and sufficient condition for the existence of a consistent preference. The following result is proved in the Appendix.

Proposition 2 *The set $\mathcal{O}$ admits a preference on $\mathcal{X}$ that is consistent with its revealed relations if and only if it obeys GARP.*

Note that Proposition 2 can be thought of as an elementary version of Afriat's Theorem, where GARP guarantees the existence of a preference $\succeq$ on $\mathcal{X}$ that rationalizes the data in the following sense: $x^t \succeq (\succ)\, x$ for all $x \in \mathcal{X}$ such that $p^t \cdot x \leqslant (<)\, p^t \cdot x^t$. The difference between this result and the actual Afriat's Theorem is that in the latter the agent's consumption space (and thus the domain of his utility function) is taken to be R^ℓ_+ rather than $\mathcal{X}$. The following theorem is the main result of this section and the converse of Proposition 1; it says that any consistent preference on $\mathcal{X}$ can be extended to a utility function that is defined on the whole consumption space R^ℓ_+ and that rationalizes the data.

Theorem 1 *Suppose $\mathcal{O}$ admits a preference $\succeq$ (on $\mathcal{X}$) that is consistent with its revealed relations. Then there exists a well-behaved and concave function $U : R^\ell_+ \to R$ with the following properties:* (i) *it rationalizes $\mathcal{O}$ and* (ii) *the preference on $\mathcal{X}$ induced by U coincides with $\succeq$, i.e., $U(x^t) > (=)\, U(x^s)$ if and only if $x^t \succ (\sim)\, x^s$ for x^t, x^s in $\mathcal{X}$.*

Remark: The function can be chosen to have the form given by (5).

With Theorem 1 we can now understand Afriat's Theorem as the concatenation of two results: first, GARP guarantees the existence of a consistent preference on $\mathcal{X}$ that rationalizes the data (by Proposition 2); second, the consistent preference on $\mathcal{X}$ can be extended to a well-behaved and concave utility function defined on R^ℓ_+ that rationalizes the data. Property (ii) in Theorem 1 highlights a feature that hitherto has gone unnoticed or, at least, has not been given much prominence: *any* consistent preference on $\mathcal{X}$ can be extended to a rationalizing

[7] Local non-satiation means that, at any bundle $x \in R^\ell_+$ and any open neighborhood of x, there is x' in that neighborhood such that $U(x') > U(x)$. Note that any well-behaved utility function is strongly monotone and hence locally non-satiated.

and well-behaved utility function. In other words, no consistent preference on $\mathcal{X}$ can be eliminated by rationality. This strengthening of Afriat's Theorem allows us to control more carefully the utility function used to rationalize a data set and turns out to be crucial in helping us develop tests for weakly separable preferences (in Section 4 and 5). It may also be of useful in other ways. For example, a modeler may have some information on the agent's preference over $\mathcal{X}$, in addition to that revealed by the agent's demand at different price vectors; Theorem 1 says that this information can always be incorporated into the rationalizing utility function, so long as it is consistent with the revealed relations.

Instead of proving Theorem 1 we will prove a more general result in the next section (Theorem 2), where the agent's budget sets are allowed to be non-linear. The proof gives an explicit procedure for constructing the rationalizing utility function U. Indeed, it proceeds by first showing that the existence of $\succeq$ guarantees the existence of Afriat inequalities (4) that also satisfy $\phi^t \geqslant (>)\, \phi^r$ if $x^t \succeq (\succ)\, x^r$. The utility function given by (5) can then be shown to rationalize the data and to obey property (ii).

3. A revealed preference test for non-linear budget sets

There are a number of results that extend Afriat's Theorem to account for budget (more generally, constraint) sets that are nonlinear, including Matzkin (1991), Chavas and Cox (1993) and Forges and Minelli (2009). The last of these is most relevant for our purposes. Forges and Minelli consider a scenario where an observer has access to a set of observations, with each observation consisting of a (possibly) nonlinear constraint set and a choice from that set. There is a natural and obvious generalization of the GARP property for such a set of observations; Forges and Minelli pointed out that this generalized GARP property is necessary and sufficient for the observations to be rationalizable. The utility function they construct for the rationalization has a form similar to the classic Afriat-form (see (5)); in particular, it is the minimum of a finite family of functions, though the functions in that family are no longer linear in x. Therefore, this utility function need not be concave and indeed one could construct data sets where any rationalizing utility function *must not* be concave. In other words, Afriat's Theorem has a general extension to nonlinear constraint sets, so long as we do not require the concavity of the utility function rationalizing the data.

In this section we re-present and generalize the Forges-Minelli Theorem in a form that makes it convenient for our application of the result in Sections 4 and 5. This generalization is analogous to our generalization of Afriat's Theorem in Theorem 1. Our version of the result emphasizes the flexibility of the utility function rationalizing the data set; in particular, it is possible to construct a utility function that agrees with *any* consistent preference on the observed choices. Our result also differs from Forges and Minelli's in that we obtain a slightly stronger property on the rationalizing utility function – it is well-behaved (i.e., strongly monotone and continuous) – rather than monotone[8] and continuous, but this is because we impose a slightly stronger restriction on the constraint sets. In our applications of the Forges-Minelli Theorem, this modification turns out to be convenient.

A set $K \subset R^\ell_+$ is said to be a *regular* if it has the following properties: (i) there is $x \gg 0$ such that $x \in K$; (ii) K is *monotone*, i.e., if $x \in K$ then any $x' \in R^\ell_+$ such that $x' \leqslant x$ is also in K; and (iii) K is compact (iv) if x is on the *upper boundary of* K, (i.e., if, for all $y \gg x$, $y \notin K$) then λx is not on the upper boundary of K for all $\lambda \in [0, 1)$; and (v) if x is on the upper boundary of K, then $y \notin K$ for all $y > x$. Clearly, the classical budget set $B(p, w)$ (for $p \gg 0$ and $w > 0$) is a regular set. In our formulation of the Forges-Minelli Theorem, we shall be requiring the constraint sets to be regular. Forges and Minelli imposed conditions (i) to (iv) on their constraint sets, but not (v).[9]

Denoting the upper boundary of K by ∂K, it is straightforward to check that, when K is regular, for every nonzero $x \in R^\ell_+$, there is a unique $\mu > 0$ such that $\mu x \in \partial K$. Define $g : R^\ell_+ \to R$ by $g(x) = 1/\mu$ for $x > 0$ and $g(0) = 0$; we shall refer to g as K's *gauge function*. This function is continuous, 1-homogeneous and—because of (iv)—it is strongly monotone.[10] Lastly, the set K can be characterized by the gauge function, i.e., $K = \{x \in R^\ell_+ : g(x) \leqslant 1\}$.

Let $\mathcal{O} = \{(K^t, x^t)\}_{t \in \mathcal{T}}$ be a finite set of T elements, where $K^t \subset R^\ell_{++}$ is a regular set and $x^t \in \partial K^t$. We interpret $\mathcal{O}$ as a set of observations, where x^t is the observed bundle of ℓ goods chosen by the agent from the constraint set K^t. The function $U : R^\ell_+ \to R$ *rationalizes the set* $\mathcal{O}$ if $x^t \in \arg\max\{U(x) : x \in K^t\}$ for all $t \in \mathcal{T}$.

We are interested in finding conditions under which $\mathcal{O}$ is rationalizable. Fortunately, all the concepts and results we introduced in the previous section carry over naturally to this

[8] The utility function U is monotone if $U(x') > U(x)$ whenever $x' \gg x$.

[9] For example, conditions (i) to (iv) will permit a constraint set like $[0, 1] \times [0, 1]$ but $[0, 1] \times [0, 1] \cup \{(0, r) : r \in [1, 2]\}$ is excluded by (iv). Neither is a regular set, with the former excluded by (v). Adding assumption (v) leads to a stronger conclusion: the rationalizing utility function in Theorem 2 is continuous and strongly monotone, while the rationalizing function in the original Forges-Minelli Theorem (Proposition 3 in their paper) is continuous and monotone. If we drop (v), then the same proof we give for Theorem 2 will still go through, except that it leads to a monotone (rather than strongly monotone) utility function.

[10] Suppose $x' > x$ but $g(x') = g(x) = 1/m$. This implies that mx' and mx are both on the upper boundary of K, which is excluded by (v) since $mx' > mx$.

more general setting. Denote (as before) the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x^t\}_{t\in\mathcal{T}}$. For x^t, $x^s \in \mathcal{X}$, we say that x^t is *directly revealed preferred to* x^s if $x^s \in K^t$; if there is $\lambda > 1$ such that $\lambda x^s \in K^t$ (so $x^s \notin \partial K^t$), we say that x^t is *directly revealed strictly preferred to* x^s. We denote these relations by $x^t \succeq^* x^s$ and $x^t \succ\!\!\succ^* x^s$ respectively. From these we may construct the revealed preferred ($\succeq^{**}$), revealed strictly preferred ($\succ\!\!\succ^{**}$), and revealed indifference ($\sim^{**}$) relations, in exactly the same way as in Section 2.

It is clear that Proposition 1 remains true in this setting, i.e., if some locally non-satiated utility function U rationalizes $\mathcal{O}$ then it induces a preference on $\mathcal{X}$ that is consistent with the revealed relations of $\mathcal{O}$. There is also an analog to Proposition 2: the existence of a consistent preference on $\mathcal{X}$ is equivalent to GARP. In this context, $\mathcal{O}$ is said to obey GARP if the following holds: *whenever there are observations (K^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfying $x^{t_2} \in K^{t_1}$, $x^{t_3} \in K^{t_2}$,..., $x^{t_n} \in K^{t_{n-1}}$, and $x^{t_1} \in K^{t_n}$, then $x^{t_2} \in \partial K^{t_1}$, $x^{t_3} \in \partial K^{t_2}$,..., and $x^{t_1} \in \partial K^{t_n}$.* More succinctly, the property could be described as such: if $x^{t_1} \succeq^* x^{t_2}$, $x^{t_2} \succeq^* x^{t_3}$,..., $x^{t_{n-1}} \succeq^* x^{t_n}$, and $x^{t_n} \succeq^* x^{t_1}$, then none of the direct revealed preferences can be replaced with the strict preference $\succ\!\!\succ^*$. Last but not least, Theorem 1 generalizes in the following way, which is our version of the Forges-Minelli Theorem.

Theorem 2 *Suppose $\mathcal{O} = \{(K^t, x^t)\}_{t\in\mathcal{T}}$ admits a preference $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. Then there exists a well-behaved utility function $U : R^\ell_+ \to R$ with the following properties: (i) it rationalizes $\mathcal{O}$ and (ii) the preference on $\mathcal{X}$ induced by U coincides with $\succeq$, i.e., $U(x^t) > (=) U(x^s)$ if and only if $x^t \succ (\sim) x^s$ for x^t, x^s in $\mathcal{X}$. The utility function U can be chosen to take the form*

$$U(x) = \min_{t\in\mathcal{T}} \left\{\phi^t + \lambda^t(g^t(x) - 1)\right\} \tag{6}$$

where $g^t : R^\ell_+ \to R_+$ is the gauge function of K^t, ϕ^t and λ^t are scalars with $\lambda^t > 0$, and

$$\phi^t \leqslant \phi^k + \lambda^k(g^k(x^t) - 1) \ \ \textit{for all } k \neq t \tag{7}$$

so that $U(x^t) = \phi^t$.

Remark 1: This result includes Theorem 1 as a special case, where $K^t = B(p^t, p^t \cdot x^t)$. With no loss of generality, we may normalize p^t so that $p^t \cdot x^t = 1$ for all $t \in \mathcal{T}$; the gauge function of K^t is then simply $g^t(x) = p^t \cdot x$, so (6) has precisely the same form as (5) and (7) reduces to the Afriat inequalities (4).

Remark 2: Since GARP guarantees the existence of a consistent preference on $\mathcal{X}$, it follows from Theorem 2 that if $\mathcal{O}$ obeys GARP then $\mathcal{O}$ can be rationalized by a well-behaved utility function.

The proof of Theorem 2 requires the following lemma. In one guise or another, this lemma is well-known but we include it here for completeness.

LEMMA 1 *Given the data set $\mathcal{O}$, suppose there are numbers ϕ^t and $\lambda^t > 0$ (for every $t \in \mathcal{T}$) that obey the inequalities (7). Then the function $U : R^l_+ \to R$ given by (6) rationalizes $\mathcal{O}$, satisfies $U(x^t) = \phi^t$, and is well-behaved.*

Proof: The fact that $U(x^t) = \phi^t$ follows immediately from the definition of U and (7). Note that U is a continuous utility function since the gauge functions g^t are all continuous and it is strongly monotone since $\lambda_t > 0$ for all t and g^t is strongly monotone. To see that it generates the observations in $\mathcal{O}$, let $x \in K^s$ and thus $g^k(x) \leqslant 1$. It follows from the definition of U that $U(x) \leqslant \phi_s$ and so $U(x) \leqslant U(x^s)$. Therefore, $x^s \in \arg\max_{x \in K^s} U(x)$. **QED**

Proof of Theorem 2: With no loss of generality, write $\mathcal{X} = \{x^1, x^2, ..., x^T\}$, where either $x^{n+1} \succ x^n$ or $x^{n+1} \sim x^n$, for $n = 1, 2, ...T-1$. We need to find numbers ϕ^s and $\lambda^s > 0$ (for $s = 1, 2, ..., T$) that (a) obey the inequalities (7) and (b) satisfy $\phi^{n+1} > (=) \phi^n$ if $x^{n+1} \succ (\sim)\, x^n$. Then Lemma 1 guarantees that U (as defined by (6)) rationalizes the data set and satisfies $U(x^n) = \phi^n$. Note that the latter property, together with (b), guarantee that (ii) holds, i.e., the restriction of U to $\mathcal{X}$ coincides with $\succeq$. We shall find ϕ^n and λ^n with a step-by-step approach, explicitly constructing the numbers ϕ^n and λ^n one at a time.

Denote $g^i(x^j) - 1$ by a^{ij}. Choose ϕ^1 to be any number and λ^1 to be any positive number. Since $x^j \succeq x^1$ for all $j > 1$, we have $a^{1j} \geqslant 0$ (because if not, $x^1 \ggcurly^{**} x^j$ and $x^1 \succ x^j$ by the consistency of $\succeq$). Suppose $x^2 \succ x^1$; then $\min_{j>1} a_{1j} > 0$. This is because if $a^{1J} = 0$ for some $J > 1$, then $x^1 \succeq x^J$ (again, by the consistency of $\succeq$), which is impossible since $x^J \succ x^1$. So there is ϕ^2 such that

$$\phi^1 < \phi^2 < \min_{j>1} \{\phi^1 + \lambda^1 a^{1j}\}. \tag{8}$$

On the other hand, if $x^2 \sim x^1$, then we can choose ϕ^2 such that

$$\phi^1 = \phi^2 \leqslant \min_{j>1} \{\phi^1 + \lambda^1 a^{1j}\}. \tag{9}$$

Now choose $\lambda^2 > 0$ sufficiently small so that

$$\phi^1 \leqslant \phi^2 + \lambda^2 a^{21}.$$

Clearly this is possible if $a^{21} \geqslant 0$. If $a^{21} < 0$, then $x^2 \succ x^1$ (by consistency of $\succeq$), in which case $\phi^2 > \phi^1$, and the inequality is still possible for a λ^2 sufficiently small.

We now go on to choose ϕ^3 and λ^3. It follows from $x^j \succeq x^i$ for all $j > 2$ and $i = 1, 2$ that $a^{ij} = p^i \cdot (x^j - x^i) \geqslant 0$ for $i = 1, 2$. Once again we consider two cases: when $x^3 \succ x^2 \succeq x^1$ and when $x^3 \sim x^2 \succeq x^1$. In the case of the former, we know that $\min_{j>2} a^{2j} > 0$ since, if $a^{2J} = 0$ for some $J > 2$, then $x^2 \succeq x^J$ which contradicts $x^J \succ x^2$. Therefore,

$$\phi^2 < \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}.$$

Similarly, $\min_{j>2} a^{1j} > 0$; if $a^{1J} = 0$ for some $J > 2$, then $x^1 \succeq x^J$ which contradicts $x^J \succ x^1$. Therefore,

$$\phi^2 < \min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\};$$

this is the case because either ϕ^2 was chosen to satisfy (8) or $\phi^2 = \phi^1$. We conclude that there is ϕ^3 such that

$$\phi^2 < \phi^3 < \min\left\{\min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}, \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}\right\}.$$

We turn to the case where $x^3 \sim x^2 \succeq x^1$. It follows from (8) and (9) that $\phi^2 \leqslant \min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}$. We also know that $a^{2j} \geqslant 0$ for all $j > 2$. Therefore, we can choose ϕ^3 such that

$$\phi^2 = \phi^3 \leqslant \min\left\{\min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}, \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}\right\}.$$

Now choose $\lambda^3 > 0$ sufficiently small so that

$$\phi^i \leqslant \phi^3 + \lambda^3 a^{3i} \text{ for } i = 1, 2.$$

Clearly this is possible if $a^{3i} \geqslant 0$. If $a^{3i} < 0$, then $x^3 \succ x^i$, in which case $\phi^3 > \phi^i$, and the inequality is still possible for a λ^3 sufficiently small.

Repeating this argument, we choose ϕ^k (for $k = 2, 3, ..., T$) such that if $x^k \succ x^{k-1}$ then

$$\phi^{k-1} < \phi^k < \min_{s \leqslant k-1}\left\{\min_{j>k-1}\{\phi^s + \lambda^s a_{sj}\}\right\} \tag{10}$$

and if $x^k \sim x^{k-1}$ then

$$\phi^{k-1} = \phi^k \leqslant \min_{s \leqslant k-1}\left\{\min_{j>k-1}\{\phi^s + \lambda^s a_{sj}\}\right\}; \tag{11}$$

and $\lambda_k > 0$ (for $k = 2, 3, ..., N$) such that

$$\phi^i \leqslant \phi^k + \lambda^k a^{ki} \text{ for } i \leqslant k - 1. \tag{12}$$

For any fixed m, (10) and (11) guarantee that $\phi^m \leqslant \phi^s + \lambda^s a^{sm}$ for $s < m$ (setting $k = m$ and letting $j = m$), while (12) guarantees that this inequality holds for $s > m$ (with $k = s$ and $i = m$). In other words, we have found λ^s and ϕ^s to obey the (7). **QED**

Our final objective in this section is to develop a sharper version of Theorem 2 that allows us to control utility levels achieved at constraint sets *outside* the set of observations. While this result may be independently interesting, our reason for proving it is to use it later in Sections 4 and 5 to establish the validity of our revealed preference test for separability. The basic message of Theorem 3 is easy to explain. Given the utility function U, the *indirect utility* at some regular set K is the highest utility achievable in K, i.e., $I_U(K) = \sup\{U(x) : x \in K\}$. Suppose U rationalizes $\mathcal{O}$ and agrees with some consistent preference $\succeq$. Clearly, $I_U(K) \geqslant U(x^t)$ for all $x^t \in K \cap \mathcal{X}$. The corollary goes further by saying that we could always choose U such that, if x^s is ranked (by $\succeq$) strictly above all bundles in $K \cap \mathcal{X}$, then $I_U(K) < U(x^s)$. In other words, $I_U(K)$ could be chosen so that it will *not* be higher than the utility of any bundle which it is not 'required' (by $\succeq$) to be higher.

Example 1. Figure 1 depicts a situation with four observations, where the bundle x^3 is revealed strictly preferred to x^2. A consistent preference on $\mathcal{X} = \{x^1, x^2, x^3, x^4\}$ is

$$x^4 \succ x^3 \succ x^2 \succ x^1. \tag{13}$$

We know from Theorem 1 that there is a utility function that rationalizes the data and gives the ordering (13). Now consider the budget set K with the bold budget line; that budget set contains x^3, so its indirect utility must be higher than the utility of x^3. The claim in Theorem 3 is that we can find a concave and well-behaved utility function U rationalizing $\{(p^t, x^t)\}_{t=1}^4$ such that

$$U(x^4) > I_U(K) > U(x^3) > U(x^2) > U(x^1); \tag{14}$$

crucially the indirect utility of K is strictly lower than that of x^4. The reader will have no difficulty verifying the claim in this simple case by choosing a point on the bold budget line to represent the agent's demand and then drawing in a family of convex indifference curves compatible with (14).

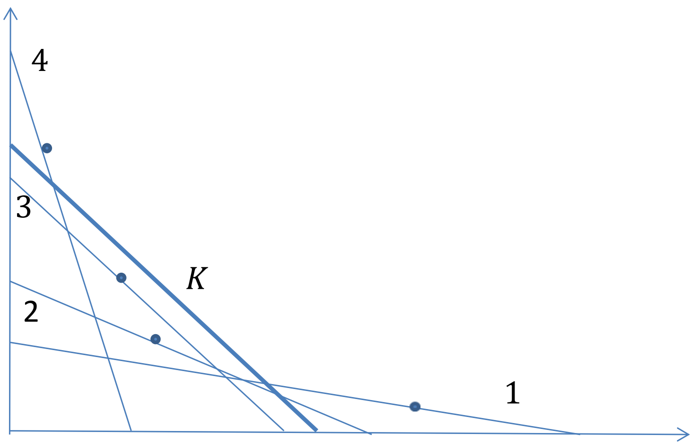

Figure 1: Data set with four observations

It is necessary to introduce a number of concepts formally before we state the result. For any regular set K such that $K \cap \mathcal{X}$ is nonempty, we define

$$\beta(K) = \{x' \in K \cap \mathcal{X} : x' \succeq x \,\forall\, x \in K \cap \mathcal{X}\}. \tag{15}$$

This is the set of elements that are ranked by $\succeq$ (at least weakly) ahead of the other elements in $K \cap \mathcal{X}$; clearly it is nonempty so long as $K \cap \mathcal{X}$ is nonempty. If $K \cap \mathcal{X}$ is empty, we let $\beta(K) = \{0\}$. Let $\alpha(K)$ be the set of elements in $\mathcal{X}$ ranked strictly above $\beta(K)$, i,e, $\alpha(K) = \{x \in \mathcal{X} : x \succ y \,\forall\, y \in \beta(K)\}$ and define

$$\bar{\beta}(K) = \{x'' \in \mathcal{X} : y \succeq x'' \,\forall\, y \in \alpha(K)\}. \tag{16}$$

In other words, $\bar{\beta}(K)$ is the set of elements in $\mathcal{X}$ ranked *just above* $\beta(K)$. Note that this set is empty if K contains a highest ranked element of $\mathcal{X}$ (according to $\succeq$).

THEOREM 3 *Suppose $\mathcal{O}$ admits a preference $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. For any finite collection of regular sets $\mathfrak{C} = \{K^m\}_{m \in M}$, there is a well-behaved utility function $U : R^\ell_+ \to R$ satisfying (i) and (ii) in Theorem 2 and the following properties:*

$$U(\bar{\beta}(K^m)) > I_U(K^m) \geqslant U(\beta(K^m)) \;\; \textit{for all } m \in M \textit{ and} \tag{17}$$

$$U(\beta(K^m)) = I_U(K^m) \textit{ if } \beta(K^m) \subseteq \partial K^m. \tag{18}$$

REMARK: Since *any* rationalization of $\mathcal{O}$ that extends $\succeq$ must satisfy $I_U(K^m) \geqslant U(\beta(K^m))$, the substantive part of (17) lies in the claim that U can be chosen such that $I_U(K^m)$ is bounded above *strictly* by $U(\bar{\beta}(K^m))$.

We prove Theorem 3 in the Appendix. The broad strategy of the proof is easy to explain: we explicitly choose bundles $x^m \in K^m$ such that the augmented data set $\widetilde{\mathcal{O}} = \mathcal{O} \cup \{(K^m, x^m)\}_{m\in M}$ admits a consistent preference $\succeq'$ on $\mathcal{X} \cup \{x^m\}_{m\in M}$ that (i) extends the given preference $\succeq$ and (ii) obeys the preference ordering implied by (17) and (18). If M consists of just one element, this extension is quite straightforward (see Lemma 4 in the Appendix); when M has multiple elements, the proof is more delicate. In any case, once $\widetilde{\mathcal{O}}$ has been constructed, an application of Theorem 2 then guarantees the existence of a well-behaved function U with the required properties. Indeed U can be explicitly constructed, with the form given by (6). Furthermore, when all the constraint sets (including $\{K^m\}_{m\in M}$) are classical budget sets, then U is also a concave function.

4. Weakly separable preferences

We assume that the agent chooses from a finite set of goods which can be divided into non-overlapping subsets X_1, X_2,..., $X_{\bar{J}}$, with X_j consisting of $\ell_j \geqslant 1$ goods. We denote the set $\{1, 2, ..., \bar{J}\}$ by $\mathcal{J}$ and agent's consumption bundle by $x = (x_1, x_2, ..., x_{\bar{J}})$, where the subvector $x_j \in R^{\ell_j}_+$ gives his consumption of goods in X_j. The prices of the goods in X_j is denoted by the vector $p_j \in R^{\ell_j}_{++}$, so $p = (p_1, p_2, ..., p_{\bar{J}})$. Given a finite data set $\mathcal{O} = \{(p^t, x^t)\}_{t\in\mathcal{T}}$, the set of observed consumption bundles of X_j-goods (alone) is denoted by $\mathcal{X}_j$, i.e., $\mathcal{X}_j = \{x^t_j\}_{t\in\mathcal{T}}$. As in the previous section, $\mathcal{X} = \{x^t\}_{t\in\mathcal{T}}$.

We would like to know when $\mathcal{O}$ can be *rationalized by a weakly separable utility function* (or, briefly, by a *separable utility function*). By this we mean that there are well-behaved functions $U_j : R^{\ell_j}_+ \to R_+$ and $F : R^{\bar{J}}_+ \to R$ such that $G : \Pi^{\bar{J}}_{j=1} R^{\ell_j}_+ \to R$, given by

$$G(x) = F(U_1(x_1), U_2(x_2), ..., U_M(x_M)), \tag{19}$$

rationalizes $\mathcal{O} = \{(p^t, x^t)\}_{t\in\mathcal{T}}$. There is no loss of generality in assuming that $U_j(0) = 0$ (so $U_j(x_j) > 0$ for all $x_j > 0$) and we shall be imposing this convenient condition throughout this section. Note that G is well-behaved provided both U_j and F are well-behaved.

Included in our formulation is the important case where there is just a single non-trivial separable commodity group. In this case, the goods can be partitioned into two sets, X_0 and X_1 (consisting of $\ell_0 > 0$ and $\ell_1 > 0$ goods respectively), and the agent's preference over bundles of X_1-goods is independent of her consumption of X_0-goods but her preference over bundles of X_0-goods is dependent on her sub-utility from X_1-goods. This feature is captured by a utility function $G : R^\ell_+ \to R$ such that, at $x = (x_0, x_1)$,

$$G(x) = U_0(x_0, U_1(x_1)), \tag{20}$$

where U_0 and U_1 are well-behaved functions. The general formulation in (19) reduces to this case if we set $X_0 = \bigcup_{j=2}^{M} X_j$ and, for all $j \geqslant 2$, X_j consists of just a single good with U_j chosen (with no loss of generality) to be the identity function.

The separable structure of G in (19) makes it possible for the agent's decision to be interpreted as a two-stage decision. At price $p \gg 0$ and income $w > 0$, we define

$$K(p, w) = \left\{ u \in R^{\bar{J}}_+ : \; u = (u_1, ..., u_{\bar{J}}),\, u_j = U_j(x_j),\, x \in B(p, w) \right\}. \tag{21}$$

$K(p, w)$ is the image of the budget set $B(p, w)$ under the collection of subutility functions $\mathcal{U} = \{U_j\}_{j \in \mathcal{J}}$; it gives the possibilities open to the agent at (p, w) in terms of the subutilities derived from X_j-goods. The next result is well-known and has a straightforward proof which we shall omit.

Proposition 3 *Suppose that G has the form given by (19) where U_j is locally nonsatiated (for all j), and F is strongly monotone. Then the following are equivalent:*
[1] $\bar{x} \in \arg\max\{G(q) : q \in B(p, w)\}$
[2] *for each j, $\bar{x}_j \in \arg\max\{U_j(x_j) : x_j \in B(p_j, p_j \cdot \bar{x}_j)\}$ and $\bar{u} \in \arg\max\{F(u) : u \in K(p, w)\}$, where $\bar{u} = (u(\bar{x})j)_{j=1}^{\bar{J}}$;*
[3] *for each j, $\bar{x}_j \in \arg\max\{U_j(x_j) : x_j \in B(p_j, p_j \cdot \bar{x}_j)\}$ and, denoting $p_j \cdot \bar{x}_j$ by $\bar{w}_j$,*

$$(\bar{w}_1, \bar{w}_2, ..., \bar{w}_{\bar{J}}) \in \arg\max \left\{ F\left(\{I_{U_j}(p_j, w_j)\}_{j=1}^{\bar{J}} \right) : \sum_{j=1}^{\bar{J}} w_j = w \right\}.$$ [11]

Statement [3] in Proposition 3 says that the agent's decision can be thought of as consisting of two parts. First, the agent's choice of X_j-goods maximizes U_j at the level of expenditure assigned to X_j-goods. Second, the expenditure devoted to each category of goods is determined by maximizing F, after taking into account the indirect utilities $I_{U_j}(p_j, w_j)$. The formulation [2] is closely related to [3], but imagines the agent as choosing from the constraint set $K(p, w)$, having first worked out the sub-utilities derived from the X_j-goods; this formulation of the agent's choice problem turns out to be useful for our purposes.

To help us rationalize $\mathcal{O}$ with a separable utility function, we introduce a new concept called *conditional GARP*. First, observe that if $\mathcal{O}$ can be rationalized by a utility function

[11] Recall that $I_{U_j}(p_j, w_j) = \sup\{U_j(x_j) : p_j \cdot x_j \leqslant w_j\}$.

of the form (19) then each *segmented data set* $\mathcal{O}_j = \{(p^t_j, x^t_j)\}_{t\in\mathcal{T}}$ must be rationalized by U_j (by Proposition 3) and hence $\mathcal{O}_j$ will obey GARP. Now suppose $\mathcal{O}$ has the property that $\mathcal{O}_j$ obeys GARP (for all j) and let $\succeq_j$ be a preference on $\mathcal{X}_j$ that is consistent with its revealed relations. For $x \in \mathcal{X}$ and $x' \in \Pi_{j=1}^{\bar{J}}\mathcal{X}_j$, we say that x is directly revealed preferred (directly strictly revealed preferred) to x' if there is $t \in \mathcal{T}$ such that $(p^t, x) \in \mathcal{O}$, and $p^t \cdot x' \leqslant (<)\, p^t \cdot x$. We denote this relation by $x \succeq^* (\succ\!\!\succ^*)\, x'$. For bundles x and $\bar{x}$ in $\mathcal{X}$, we say that x *is directly revealed preferred to* $\bar{x}$ *conditional on* $\{\succeq_j\}_{j\in\mathcal{J}}$ and denote this by $x \succeq^\star \bar{x}$ if there is a bundle x' such that

(a) $x' \in \Pi_{j=1}^{\bar{J}}\mathcal{X}_j$ with $x \succeq^* x'$, and

(b) $x'_j \succeq_j \bar{x}_j$ for all j.

If, in addition, either $x \succ\!\!\succ^* x'$ or $x'_j \succ_j \bar{x}_j$ for some j then we say that x *is directly strictly revealed preferred to* $\bar{x}$ *conditional on* $\{\succeq_j\}_{j\in\mathcal{J}}$ and denote this by $x \succ\!\!\succ^\star x'$. From these we may, by taking the transitive closure, construct the revealed preferred ($\succeq^{\star\star}$) and revealed strictly preferred ($\succ\!\!\succ^{\star\star}$) relations conditional on $\{\succeq_j\}_{j=1}^{\bar{J}}$. When there is no danger of confusion, we shall refer to $\succeq^{\star\star}$ and $\succ\!\!\succ^{\star\star}$ collectively as the conditional revealed relations, omitting the reference to $\{\succeq_j\}_{j\in\mathcal{J}}$.

The set $\mathcal{O}$ is said to obey *GARP conditional on* $\{\succeq_j\}_{j\in\mathcal{J}}$ (or simply *conditional GARP*) if, whenever there are observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfying

$$x^{t_1} \succeq^\star x^{t_2},\; x^{t_2} \succeq^\star x^{t_3},\; ...,\; x^{t_{n-1}} \succeq^\star x^{t_n}, \text{ and } x^{t_n} \succeq^\star x^{t_1}, \tag{22}$$

we cannot replace $\succeq^\star$ with $\succ\!\!\succ^\star$. Notice that conditional GARP reduces to GARP (in the standard sense) if each X_j consists of just one good. Put differently, we could think of the role of conditional GARP in the following way: if X_j consists of just one good, there can only be one consistent preference $\succeq_j$ on X_j, namely, $x'_j \succeq_j x_j$ if $x'_j \geqslant x_j$; in contrast, when X_j has more than one good, the preference $\succeq_j$ on $\mathcal{X}_j$ that is consistent with the revealed relations of $\mathcal{O}_j$ is typically non-unique and so it has to be explicitly specified when defining the revealed relations. It is possible for a data set to satisfy GARP conditional on one set of consistent preferences and to fail GARP when conditioned on another; we illustrate this phenomenon in Example 2 later in this section.

The next result shows that conditional GARP is necessary for rationalization with a separable utility function.

PROPOSITION 4 *Suppose $\mathcal{O}$ is collected from an agent who maximizes a function G of the form (19), where U_j are locally non-satiated and F is strongly monotone. Then $\mathcal{O}$ obeys GARP conditional on $\{\succeq_j\}_{j\in\mathcal{J}}$, where $\succeq_j$ is the preference on $\mathcal{X}_j$ induced by U_j.*

Proof: By Proposition 3, $x_j^t \in \arg\max_{x_j\in B_j^t} U_j(x_j)$, where $B_j^t = \{x_j \in R_+^{\ell_j} : p_j^t \cdot x_j \leqslant p_j^t \cdot x_j^t\}$, for all $t \in \mathcal{T}$. Therefore, by Proposition 1, $\succeq_j$ is consistent with the revealed relations of $\mathcal{O}_j = \{(p_j^t, x_j^t)\}_{t\in\mathcal{T}}$. Let x and x' be two bundles in $\mathcal{X}$. If x is conditionally directly revealed preferred to x' then there is $x'' \in \Pi_{j=1}^{\bar{J}} \mathcal{X}_j$ such that $x \succeq^* x''$ and $x''_j \succeq_j x'_j$ for all j. Since $x \succeq^* x''$, we have $G(x) \geqslant G(x'')$ because otherwise the agent is better off choosing x. Since F is increasing in u_j for all $j \in \mathcal{J}$,

$$G(x'') = F(U_1(x''_1), ...U_{\bar{J}}(x''_{\bar{J}})) \geqslant G(x') = F(U_1(x'_1), ..., U_{\bar{J}}(x'_{\bar{J}})).$$

Therefore, $x \succeq^\star x'$ implies that $G(x) \geqslant G(x')$. Now suppose $x \succ\!\!\succ^\star x'$; then $x \succeq^* x''$, $x''_j \succeq_j x'_j$ for all $j \in \mathcal{J}$, *and* either $q \succ\!\!\succ^* q''$ or $x''_j \succ_j x'_j$ for some j. If $x''_j \succ_j x'_j$ for some j, then $U_j(x''_j) > U_j(x'_j)$ and thus $G(x) \geqslant G(x'') > G(x')$ (because F is strongly monotone). By definition, if $x \succ\!\!\succ^* x''$, then there is price vector p^t at which $(p^t, x) \in \mathcal{O}$ and $p^t \cdot x > p^t \cdot x''$. Since U_1 is locally non-satiated, there is bundle $x'''_1 \in R_+^{\ell_1}$ such that $U_1(x'''_1) > U_1(x''_1)$ and the bundle $x''' = (x'''_1, x''_2, ...x''_{\bar{J}})$ obeys $p^t \cdot x \geqslant p^t \cdot x'''$. Since F is strongly monotone, $G(x''') > G(x'')$ and since x''' is affordable to the agent at observation t, $G(x) \geqslant G(x''')$. So we obtain $G(x) > G(x'') \geqslant G(x')$.

To recap, we have shown that $G(x) \geqslant G(x')$ if $x \succeq^\star x'$ and $G(x) > G(x')$ if $x \succ\!\!\succ^\star x'$. Suppose the observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, ..., n$) satisfy (22). Then we obtain

$$G(x^{t_1}) \geqslant G(x^{t_2}) \geqslant G(x^{t_3}) ... \geqslant G(x^{t_n}) \geqslant G(x^{t_1}),$$

which holds only if $G(x^{t_i}) = G(x^{t_k})$ for all i and k. Thus none of conditional revealed preferences in (22) can be replaced with $\succ\!\!\succ^\star$. **QED**

Suppose $\mathcal{O}$ obeys GARP conditional on the consistent relations $\{\succeq_j\}_{j\in\mathcal{J}}$. Then we know (by Proposition 2) that there is a preference $\succeq$ on $\mathcal{X}$ that is consistent with its conditional revealed preference relations. In this case, we say that $\succeq$ is a *consistent preference on $\mathcal{X}$, conditional on $\{\succeq_j\}_{j\in\mathcal{J}}$* and refer to these preferences collectively as a *consistent family of preferences*. The next result spells out the conditions under which a data set can be rationalized by a weakly separable utility function; it is the main result of this paper and the converse of Proposition 4.

THEOREM 4 *Suppose that the data set $\mathcal{O}$ admits a consistent family of preferences: $\succeq_j$ on $\mathcal{X}_j$ for all $j \in \mathcal{J}$ and $\succeq$ on $\mathcal{X}$. Then $\mathcal{O}$ can be rationalized by a utility function G of the form (19), where* (i) *U_j is a well-behaved and concave function that rationalizes $\mathcal{O}_j$, with the preference induced by U_j on $\mathcal{X}_j$ coinciding with $\succeq_j$;* (ii) *F is a well-behaved function and the preference induced by G on $\mathcal{X}$ coincides with $\succeq$.*

REMARK: In the case where X_j consists of just one good (for all j), the only consistent preference on $\mathcal{X}_j$ says that $x'_j \succeq_j x_j$ if $x'_j \geqslant x_j$; we can choose U_j to be the identity function and this theorem reduces to saying that $\mathcal{O}$ can be rationalized by some well-behaved utility function that agrees with $\succeq$ on $\mathcal{X}$. In other words, we recover Afriat's Theorem.

It is instructive to consider how this theorem characterizes data sets that are rationalizable by a utility function of the form (20). Given a data set $\mathcal{O}$ and a partition of the goods into sets X_0 and X_1, suppose that the segment $\mathcal{O}_1 = \{p_1^t, x_1^t\}_{t\in\mathcal{T}}$ obeys GARP. Then $\mathcal{O}_1$ will admit a consistent preference $\succeq_1$ on $\mathcal{X}_1 = \{x_1^t\}_{t\in\mathcal{T}}$. Given $\succeq_1$, we can construct the revealed preference relations on $\mathcal{X} = \{x^t\}_{t\in\mathcal{T}}$, which reduces to the following form in this context: for $x = (x_0, x_1)$ and $\bar{x} = (\bar{x}_0, \bar{x}_1)$ in $\mathcal{X}$, $x \succeq^\star \bar{x}$ if there is a bundle $x'_1 \in \mathcal{X}_1$ such that $x \succeq^* (\bar{x}_0, x'_1)$ and $x'_1 \succeq_1 \bar{x}_1$; if, in addition, either $x \ggcurly^* (\bar{x}_0, x'_1)$ or $x'_1 \succ_1 \bar{x}_1$, then $x \ggcurly^\star \bar{x}$. We can check whether these conditional revealed relations obey GARP; if they do, there will be a preference $\succeq$ on $\mathcal{X}$ that extends those relations. Theorem 4 says the following in this context: *if the data set $\mathcal{O}$ admits a consistent family of preferences comprising $\succeq_1$ on $\mathcal{X}_1$ and $\succeq$ on $\mathcal{X}$, then there is a utility function G of the form (20) that rationalizes $\mathcal{O}$, such that G agrees with $\succeq$ on $\mathcal{X}$ and U_1 agrees with $\succeq_1$ on $\mathcal{X}_1$.*[12]

Example 2. Consider a data set with the following two observations, drawn from an agent who is choosing a consumption bundle out of two X_0-goods and two X_1-goods.
$p^1_{X_1} = (2,1)$, $p^1_{X_0} = (1,3/2)$, $x_1^1 = (0,1)$, $x_0^1 = (1,2)$, $w^1 = 5$
$p^2_{X_1} = (1,2)$, $p^2_{X_0} = (3/2,1)$, $x_1^2 = (1,0)$, $x_0^2 = (2,1)$, $w^2 = 5$.
In this case, $\mathcal{X} = \{x^1, x^2\}$ and $\mathcal{X}_1 = \{x_1^1, x_1^2\}$. It is easy to check that $(x_0^1, x_1^1) \ggcurly^* (x_0^2, x_1^1)$ and $(x_0^2, x_1^2) \ggcurly^* (x_0^1, x_1^2)$ but $(x_0^1, x_1^1) \not\succeq^* (x_0^2, x_1^2)$ and $(x_0^2, x_1^2) \not\succeq^* (x_0^1, x_1^1)$. Furthermore, there is no revealed preference relation between x_1^1 and x_1^2, so any preference between those two bundles is consistent with the revealed relations.

Suppose we choose $x_1^1 \sim_1 x_1^2$ as the consistent preference on $\mathcal{X}_1$. Then $(x_0^1, x_1^1) \ggcurly^\star (x_0^2, x_1^2)$ because $(x_0^1, x_1^1) \ggcurly^* (x_0^2, x_1^1)$ and $x_1^1 \sim_1 x_1^2$. On the other hand, $(x_0^2, x_1^2) \ggcurly^\star (x_0^1, x_1^1)$ because

[12]Of course, the existence of such a consistent family is also necessary for rationalizability by a utility function G of the form (20).

$(x_0^2, x_1^2) \gg^\star (x_0^1, x_1^2)$ and $x_1^1 \sim_1 x_1^2$. So the data set clearly violates GARP conditional on $x^1 \sim_1 x^2$. On the other had, suppose we choose $x_1^1 \succ_1 x_1^2$ as the consistent preference on $\mathcal{X}_1$. Then we obtain $(x_0^1, x_1^1) \gg^\star (x_0^2, x_1^2)$ because $(x_0^1, x_1^1) \gg^* (x_0^2, x_1^1)$ and $x^1 \succ_1 x^2$. This is the only revealed preference relation on $\mathcal{X}$, so this data set obeys GARP conditional on $x^1 \succ_1 x^2$. In short, $x_1^1 \succ_1 x_1^2$ and $x^1 \succ x^2$ form a consistent family of preferences. By Theorem 4, the data set is rationalizable by a utility function of the form (20), with $U_1(x_1^1) > U_2(x_1^2)$ and $G(x^1) > G(x^2)$.

The proof of Theorem 4

To have some sense of where the difficulty lies in proving Theorem 4, first note that the existence of well-behaved and concave utility functions U_j that obey property (i) is guaranteed by Theorem 1. These functions map the data set $\mathcal{O}$ into the set

$$\widehat{\mathcal{O}} = \{(K^t, u^t)\}_{t \in \mathcal{T}}, \tag{23}$$

where $u^t = \left(U_1(x_1^t, U_2(x_2^t), ..., U_{\bar{J}}(x_{\bar{J}}^t)\right)$ and $K^t = K(p^t, p^t \cdot x^t)$ is the image of B^t (in the sense defined by (21)). We denote the direct revealed preference relations of $\widehat{\mathcal{O}}$ (in the sense defined in Section 3) by $\succeq^*$ and $\gg^*$. The U_j functions are not unique and thus neither is the image $\widehat{\mathcal{O}}$, so the issue is whether we can construct those functions in such a way that $\widehat{\mathcal{O}}$ obeys GARP. If we can, Theorem 2 guarantees the existence of a well-behaved function F that rationalizes $\widehat{\mathcal{O}}$ and, by Proposition 3, the resulting G (as defined by (19)) will rationalize $\mathcal{O}$.

$\widehat{\mathcal{O}}$ obeys GARP if and only if there exists a preference $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}} = \{u^t\}_{t \in T}$ that is consistent with the revealed relations of $\widehat{\mathcal{O}}$. An obvious candidate for $\widehat{\gtrsim}$ is the following:

$$u^t \mathrel{\widehat{\gtrsim}} u^s \text{ if } x^t \succeq x^s. \tag{24}$$

The relation $\widehat{\gtrsim}$ is a preference because $\succeq$ is a preference. What needs to be shown is that, for some choice of U_js, $\widehat{\gtrsim}$ is also consistent with the revealed relations if $\widehat{\mathcal{O}}$. As we shall show in the proof of Theorem 4, this consistency is guaranteed so long as the U_j functions are chosen to satisfy the following property:

$$u^t \succeq^* (\gg^*)\, u^s \Longrightarrow x^t \succeq^\star (\gg^\star)\, x^s. \tag{25}$$

Proof of Theorem 4: Lemma 2 (stated after this proof) provides a way of constructing the functions U_j such that the property stated in (25) holds, in addition to property (i). We claim that the preference $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}}$ (as defined by (24)) is consistent with $\succeq^*$ and $\gg^*$. This means showing that (a) $u^t \widehat{\gtrsim} u^s$ if $u^t \succeq^* u^s$ and (b) $u^t \widehat{>} u^s$ if $u^t \gg^* u^s$.

To show (a), observe that if $u^t \widehat{\gtrsim}^* u^s$, then $x^t \geq^\star x^s$ (by (25)). By definition, $\geq$ is consistent with $\geq^\star$, and so we obtain $x^t \geq x^s$ which, in turn, implies (by (24)) that $u^t \widehat{\gtrsim} u^s$. To show (b), first note that

$$u^t \widehat{>} u^s \text{ if } x^t > x^s. \tag{26}$$

Given the definition of $\widehat{\gtrsim}$ in (24), for (26) *not* to be true, there must exist $\bar{t}$ and $\bar{s}$ such that $u^{\bar{t}} = u^t$ and $u^{\bar{s}} = u^s$ with $x^{\bar{t}} \leq x^{\bar{s}}$. But this cannot occur: if $u^t = u^{\bar{t}}$, then $x^t \geq^\star x^{\bar{t}}$ and $x^{\bar{t}} \geq^\star x^t$ (by (25)) and the consistency of $\geq$ guarantees that $x^t \sim x^{\bar{t}}$; similarly, $x^s \sim x^{\bar{s}}$ and thus $x^{\bar{t}} > x^{\bar{s}}$ if $x^t > x^s$. By (25), if $u^t \gg^* u^s$, then $x^t \gg^\star x^s$. This implies that $x^t > x^s$ (by the consistency of $>$ with $\gg^\star$) and thus $u^t \widehat{>} u^s$ (by (26)).

Theorem 2 guarantees that there is a well-behaved function F that rationalizes $\widehat{\mathcal{O}}$ and induces $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}}$. That G (defined by (19)) rationalizes $\mathcal{O}$ follows from Proposition 3 and it follows from the definition of $\widehat{\gtrsim}$ that G agrees with $\geq$ on $\mathcal{X}$. **QED**

Our claim that U_j can be chosen to obey (25) is stated formally below.

LEMMA 2 *Suppose that the data set $\mathcal{O}$ admits consistent preferences $\geq_j$ on $\mathcal{X}_j$ for all j. Then there are well-behaved and concave utility functions U_j that satisfy condition (i) in Theorem 4 and the property (25).*

The proof of Lemma 2 is in the Appendix. By definition, $u^t \geq^* u^s$ if $u^s \in K^t$, which means that there is w_j such that $\sum_{j=1}^{\bar{J}} w_j \leqslant p^t \cdot x^t$, with $I_{U_j}(p_j^t, w_j) \geqslant u_j^s$ for all j. In other words, the agent can divide his total expenditure in a way that allows him (at price p^t) to achieve sub-utilities greater than u_j^s (for each j). If $x^t \geq^\star x^s$, there will be bundles $x_j'' \in \mathcal{X}_j$ such that $x_j'' \geq_j x_j^s$, and $\sum_{j=1}^{\bar{J}} p_j^t \cdot x_j'' \leqslant p^t \cdot x^t$. Thus we obtain $u^t \geq^* u^s$, so long as U_j agrees with $\geq_j$. The principal claim in Lemma 2 (property (25)) is that, by a suitable choice of U_js we can ensure that this is the *only* way of achieving $u^t \geq^* u^s$. The following example gives a sense of what is involved in the proof of Lemma 2.

Example 3. Suppose we would like to rationalize a data set $\mathcal{O}$ with a utility function of the form (20) and we know that $\mathcal{O}$ admits a consistent family of preferences consisting of $\geq_1$ on $\mathcal{X}_1$ and $\geq$ on $\mathcal{X}$. Theorem 1 guarantees that there is some well-behaved utility function U_1 that rationalizes $\mathcal{O}_1$ and agrees with $\geq_1$. This induces $\widehat{\mathcal{O}} = \{(K^t, u^t)\}_{t \in \mathcal{T}}$, where $u^t = (x_0^t, U_1(x_1^t))$ and

$$K^t = \left\{ \left(x_0^t, U_1(x_1^t)\right) : x^t \in B(p^t, p^t \cdot x^t) \right\}.$$

To guarantee that $\mathcal{O}$ can be rationalized by a utility function G of the form (20) that extends

$\succeq$, it suffices that $\widehat{\succeq}$, as defined by (24), is consistent with the revealed relations of $\widehat{\mathcal{O}}$. In turn, this holds if U_1 can be chosen to obey property (25), which in this context can be written in the following way:

$$\big(x_0^t, U_1(x_1^t)\big) \succeq^* (\succ\!\!\succ^*)\ (x_0^s, U_1(x_1^s)) \Longrightarrow x^t \succeq^\star (\succ\!\!\succ^\star)\ x^s. \tag{27}$$

The claim in Lemma 2 is that this is possible.

To have a sense of what this claim involves, suppose that the data set consists of four observations and the agent's demand for X_1 goods is depicted in Figure 1. Clearly, $\mathcal{O}_1$ obeys GARP and the preference $\succeq_1$ defined by $x_1^4 \succ_1 x_1^3 \succ_1 x_1^2 \succ_1 x_1^1$ is a consistent preference on $\mathcal{X}_1$. Furthermore, suppose that the set K, demarcated by the bold line, is the budget set

$$\{x \in R_+^2 : p_{X_1}^3 \cdot x_1 \leqslant p^3 \cdot x^3 - p_{X_0}^3 \cdot x_0^2\};$$

i.e., it is the budget left for spending on X_1 goods should the agent buy x_0^2 at $t = 3$. It follows from Figure 1 that $(x_0^3, x_1^3) \succ\!\!\succ^\star (x_0^2, x_1^2)$ and thus $(x_0^3, U_1(x_1^3))\widehat{\succ\!\!\succ}^*(x_0^2, U_1(x_1^2))$. Suppose that $x_0^4 = x_0^2$, so that the budget left for spending on X_1 goods should the agent buy x_0^4 at $t = 3$ is again the set K. Notice that in this case $(x_0^3, x_1^3) \not\succeq^\star (x_0^4, x_1^4)$. For (27) to hold, we require U_1 such that $(x_0^3, U_1(x_1^3)) \not\succeq^\star (x_0^4, U_1(x_1^4))$; equivalently, $U_1(x_1^4) > I_{U_1}(K)$. Clearly, this could be guaranteed. Beyond this simple example, arguments of this type involve the application of Theorem 3, and this theorem plays a crucial role in the proof of Lemma 2.

Is there a simpler characterization?

Theorem 4 says that a finite data set is rationalizable by a weakly separable utility function if and only if it admits a consistent family of preferences. To check whether a consistent preference family exists is plainly a finite problem but it is also computationally intensive, so it would be convenient if there is an easily verifiable condition on the data set that guarantees its existence. There is at least one natural and computationally straightforward condition that springs to mind and that we should investigate.

To fix ideas, consider the simpler problem of rationalizing a data set with a utility function of the form (20) and define the relations $\succeq^\sharp$ and $\succ\!\!\succ^\sharp$ on $\mathcal{X}$ in the following way: for $x = (x_0, x_1)$ and $x' = (x_0', x_1')$ in $\mathcal{X}$, we say that $x \succeq^\sharp x'$ if there is $x_1'' \in \mathcal{X}_1$ such that (a) $x \succeq^* (x_0', x_1'')$ and (b) $x_1'' \succeq_1^* x_1'$. If, in addition, either $x \succ\!\!\succ^* (x_0', x_1'')$ or $x_1'' \succ\!\!\succ_1^* x_1'$, then $x \succ\!\!\succ^\sharp x'$. Notice that $\succeq^\sharp \subseteq \succeq^\star$ and $\succ\!\!\succ^\sharp \subseteq \succ\!\!\succ^\star$. It is clear that if the data set is rationalizable by (20), then the relations $\succeq^\sharp$ and $\succ\!\!\succ^\sharp$ will not have any strict cycles, but is this condition also sufficient for rationalizability? Unfortunately, the answer is negative.

Example 4: Consider a data set with the following four observations, drawn from an agent who is choosing a consumption bundle out of two X_1-goods and four X_0-goods.
$p^1_{X_1} = (1, 2)$, $p^1_{X_0} = (2.5, 1, 100, 100)$, $x^1_1 = (1, 0)$, $x^1_0 = (2, 1, 0, 0)$, $w^1 = 7$
$p^2_{X_1} = (2, 1)$, $p^2_{X_0} = (1, 1.5, 100, 100)$, $x^2_1 = (0, 1)$, $x^2_0 = (1, 2, 0, 0)$, $w^2 = 5$
$p^3_{X_1} = (2, 1)$, $p^3_{X_0} = (100, 100, 2.5, 1)$, $x^3_1 = (0, 1)$, $x^3_0 = (0, 0, 2, 1)$, $w^3 = 7$
$p^4_{X_1} = (1, 2)$, $p^4_{X_0} = (100, 100, 1, 1.5)$, $x^4_1 = (1, 0)$, $x^4_0 = (0, 0, 1, 2)$, $w^4 = 5$.

One could check that the following holds: (i) $(x^1_0, x^1_1) \ggcurly^* (x^2_0, x^1_1)$ and $(x^1_0, x^1_1) \ggcurly^* (x^2_0, x^2_1)$; (ii) $(x^2_0, x^2_1) \ggcurly^* (x^1_0, x^2_1)$; (iii) $(x^3_0, x^3_1) \ggcurly^* (x^4_0, x^3_1)$ and $(x^3_0, x^3_1) \ggcurly^* (x^4_0, x^4_1)$; and (iv) $(x^4_0, x^4_1) \ggcurly^* (x^3_0, x^4_1)$. This is an exhaustive list of revealed relations between distinct elements, there are no others.[13] In particular, $x^1_1 = x^4_1$ and $x^2_1 = x^3_1$ are not related by $\succeq^*_1$ or $\ggcurly^*_1$. From (i)–(iv), we obtain $(x^1_0, x^1_1) \ggcurly^\sharp (x^2_0, x^2_1)$ and $(x^3_0, x^3_1) \ggcurly^\sharp (x^4_0, x^4_1)$ and it is clear that $\succeq^\sharp$ and $\ggcurly^\sharp$ do not generate any cycles on $\mathcal{X}$. However, the data do not admit a consistent family of preferences.

Indeed, since $(x^1_0, x^1_1) \ggcurly^\sharp (x^2_0, x^2_1)$ and $(x^3_0, x^3_1) \ggcurly^\sharp (x^4_0, x^4_1)$, any consistent preference on $\mathcal{X}$ will satisfy $(x^1_0, x^1_1) \succ (x^2_0, x^2_1)$ and $(x^3_0, x^3_1) \succ (x^4_0, x^4_1)$. Let $\succeq_1$ be a preference on $\mathcal{X}_1$ and suppose $x^2_1 \succeq_1 x^1_1$. This leads to a contradiction because (by (ii)) $(x^2_0, x^2_1) \ggcurly^* (x^1_0, x^2_1)$ and so $(x^2_0, x^2_1) \succ (x^1_0, x^1_1)$. We conclude that $x^1_1 \succ_1 x^2_1$; equivalently, $x^4_1 \succ_1 x^3_1$. This also leads to a contradiction since (by (iv)) $(x^4_0, x^4_1) \ggcurly^* (x^3_0, x^4_1)$ and thus $(x^4_0, x^4_1) \succ (x^3_0, x^3_1)$.

Can we guarantee that the utility function be quasiconcave?

In Theorem 4, while the subutility functions U_j can be chosen to be concave, the aggregator function F was only specified as well-behaved, i.e., strongly monotone and continuous, which means that G need not be a quasiconcave function. We know from Afriat's Theorem that any data set that is rationalizable by a well-behaved utility function is rationalizable by a well-behaved and concave utility function, so it is reasonable to ask whether a data set that is rationalizable by a well-behaved and weakly separable utility function can also be rationalizable by such a function with concavity as an added property. The following example provides a data set that can be rationalized by a well-behaved and weakly separable utility function, but such a function *cannot* be quasiconcave.

Example 5: Consider a data set with the following six observations of an agent choosing from two X_1-goods and one X_0-good. Note that w^t_1 and w^t denote the total expenditure on

[13] Notice that no bundle involving x^3_0 or x^4_0 is affordable at p^1 or p^2; similarly, no bundle involving x^1_0 or x^2_0 is affordable at p^3 or p^4.

the X_1-goods and the total expenditure over all goods respectively.
$p^1_{X_1} = (1,2)$, $p^1_{X_0} = 1$, $x^1_1 = (1,1)$, $x^1_0 = 12$, $w^1_{X_1} = 3$, $w^1 = 15$;
$p^2_{X_1} = (1,2)$, $p^2_{X_0} = 1$, $x^2_1 = (4,4)$, $x^2_0 = 3$, $w^2_{X_1} = 12$, $w^2 = 15$;
$p^3_{X_1} = (1,2)$, $p^3_{X_0} = 1$, $x^3_1 = (4,0)$, $x^3_0 = 11$, $w^3_{X_1} = 4$, $w^3 = 15$;
$p^4_{X_1} = (2,1)$, $p^4_{X_0} = 1$, $x^4_1 = (1,1)$, $x^4_0 = 12$, $w^4_{X_1} = 3$, $w^4 = 15$;
$p^5_{X_1} = (2,1)$, $p^5_{X_0} = 1$, $x^5_1 = (4,4)$, $x^5_0 = 3$, $w^5_{X_1} = 12$, $w^5 = 15$;
$p^6_{X_1} = (2,1)$, $p^6_{X_0} = 1$, $x^6_1 = (4,0)$, $x^6_0 = 100$, $w^6_{X_1} = 8$, $w^6 = 108$.

It is straightforward to check that $\mathcal{O}_1 = \{(p^t_{X_1}, x^t_1)\}^6_{t=1}$ obeys GARP. Let U_1 be any well-behaved and concave utility function rationalizing $\mathcal{O}_1$. While there are six observations, there are only three distinct budget sets. At observations 1, 2, and 3, the budget set is $B((1,1,2),15)$, though the choices made are distinct. In x_0-u_1 space (think of x_0 and u_1 on the horizontal and vertical axes respectively), the chosen bundles are $(12, U_1(1,1))$, $(3, U_1(4,4))$, and $(11, U_1(4,0))$. Since U_1 rationalizes $\mathcal{O}_1$, these points must lie on the upper boundary of $K^1 = K^2 = K^3$. (K^t is the image of B^t under U_1 (see (21)).) At observations 4 and 5, the budget set is $B((1,2,1),15)$. In x_0-u_1 space, the chosen bundles are $(12, U_1(1,1))$ and $(3, U_1(4,4))$; they lie on the upper boundary of $K^4 = K^5$. Note that the upper boundaries of K^1 and K^4 intersect at two points at least. The budget set at observation 6 is $B((1,2,1),106)$ and the chosen bundle in x_0-u_1 space is $(100, U_1(4,0))$; the very high level of income guarantees that K^6 contains K^1 and K^4 in its interior.

We claim that $\{(K^t, (x^t_0, u^t_1)\}^6_{t=1}$ obeys GARP, so there is a well-behaved function F rationalizing those observations (and hence $G(x_0,x_1) = F(x_0, U_1(x_1))$ will rationalize $\mathcal{O}$). To check this, we note that K^1 and K^4 have at least two intersections, namely at $(12, U_1(1,1))$ and $(3, U_1(4,4))$; to check that GARP holds, all we need to do is check that $(11, U_1(4,0))$, which is also chosen at $K^3 = K^1$, is *not* in the interior of K^4. Indeed,

$$I_{U_1}((2,1),4) < I_{U_1}((2,1),8) = U_1(4,0) = I_{U_1}((1,2),4),$$

where the strict inequality follows from the strong monotonicity of U_1, the first equality from observation 6, and the second equality from observation 3. So we obtain

$$(11, I_{U_1}((2,1),4)) < (11, U_1(4,0)) \tag{28}$$

where $(11, I_{U_1}((2,1),4))$ is on the upper boundary of K^4 and $(11, U_1(4,0))$ on the upper boundary of K^1.

Therefore there is a well-behaved function F that rationalizes $\widehat{\mathcal{O}} = \{(K^t, (x^t_0, u^t_1)\}^6_{t=1}$. However, G *cannot* be quasiconcave, and thus F cannot be quasiconcave if U_1 is concave.

To see this, we choose $\theta \in (0,1)$ so that $3\theta + 12(1-\theta) = 11$. Then

$$\begin{aligned} G(\theta(3,4,4) + (1-\theta)(12,1,1)) &= F(11, U_1(\theta(4,4) + (1-\theta)(1,1))) \\ &\leqslant F(11, I_{U_1}((2,1),4)) \\ &< F(11, U_1(4,0)) = G(3,4,4), \end{aligned}$$

where the first inequality holds because $\theta(4,4) + (1-\theta)(1,1) \in B((2,1),4)$ and the second inequality from (28). Furthermore, $G(11,4,0) = G(3,4,4) = G(12,1,1)$ because these three bundles are observed choices at $B^1 = B^2 = B^3$. Thus there is a violation of quasiconcavity.

5. Nested weakly separable preferences

The tools we developed in the last section can be adapted to test for more elaborate models of utility-maximization that involve multiple layers of weak separability. These tests take a form similar to that described in Theorem 4 for the basic model: it involves finding preferences *on the data set* that obey certain properties consistent with the hypothesis and then extending those preferences to the entire commodity space. To clarify what we mean, we shall now consider a more elaborate model of weakly separable demand.

We assume the agent chooses from a finite set of goods which can be divided into non-overlapping subsets X_1, X_2,..., $X_{\bar{J}}$, with X_j consisting of ℓ_j goods ($j \in \mathcal{J} = \{1, 2, ..., \bar{J}\}$). In addition, we now assume that the X_j-goods can be grouped into $\bar{K}$ categories; let $Y_1 = \bigcup_{j=1}^{j_1} X_j$, $Y_2 = \bigcup_{j=j_1+1}^{j_2} X_j$,..., $Y_{\bar{K}} = \bigcup_{j=j_{\bar{K}-1}+1}^{\bar{J}} X_j$. We denote the set $\{1, 2, ..., \bar{K}\}$ by $\mathcal{K}$, the set $\{j : X_j \subset Y_k\}$ by $J(k)$, the agent's consumption of goods in Y_k by the vector $y_k = (x_{j_{k-1}+1}, x_{j_{k-1}+2}, ..., x_{j_k})$, and the price of goods in Y_k by $p_{Y_k} = (p_{j_{k-1}+1}, p_{j_{k-1}+2}, ..., p_{j_k})$. The agent's (overall) consumption bundle can be written as $x = (x_1, x_2, ..., x_{\bar{J}})$, which emphasizes the division along the X_j subsets, or it could be written as $y = (y_1, y_2, ..., y_{\bar{K}})$, which emphasizes the division along the Y_k subsets. We adopt a convention which the reader should bear in mind to avoid later confusion: if we denote a particular bundle by (say) x'', then it could correspondingly be written as y'', with y''_k denoting the subvector over the Y_k-goods.

The observer has access to a data set $\mathcal{O} = \{(p^t, x^t)\}_{t \in \mathcal{T}}$. We can segment this data set to focus on prices and demand for certain subsets of goods; let $\mathcal{O}_j = \{(p^t_j, x^t_j)\}_{t \in \mathcal{T}}$ and $\mathcal{O}_{Y_k} = \{(p^t_{Y_k}, y^t_k)\}_{t \in \mathcal{T}}$. We now require that the demand for goods in $Y_k = \bigcup_{j \in J(k)} X_j$ be separately rationalizable by a utility function that is weakly separable along the sets $X_{j_{k-1}+1}$, $X_{j_{k-1}+2}$,..., X_{j_k}. Formally, we wish to test the hypothesis that $\mathcal{O}$ can be rationalized by a utility function $G : R^\ell_+ \to R$ with the following property:

(P) there is a well-behaved function $F : R_+^{\bar{K}} \to R$, such that

$$G(x) = F\left(V_1(y_1), V_2(y_2), ..., V_{\bar{K}}(y_{\bar{K}})\right) \tag{29}$$

at $x = (x_1, x_2, ..., x_{\bar{J}}) = (y_1, y_2, ..., y_{\bar{K}})$ and, for all $k \in \mathcal{K}$,

$$V_k(y_k) = H_k\left(U_{j_{k-1}+1}(x_{j_{k-1}+1}), U_{j_{k-1}+2}(x_{j_{k-1}+2}), ..., U_{j_k}(x_{j_k})\right), \tag{30}$$

where $H_k : R^{|J(k)|} \to R$ and $U_j : R_+^{\ell_j} \to R$ (and thus V_k) are well-behaved functions.

It is certainly possible to test an even more elaborate hypothesis on $\mathcal{O}$ that involve adding one or more layers of weakly separable commodity groups; i.e., with sub-utilities defined over subsets of V_k, sub-utilities defined over subsets of those sub-utilities, and so on. Such an extension poses no difficulty: once the reader understands the one- and two-layer cases covered in this and the previous section, it will be clear how cases with more layers can be handled. We choose to proceed in this step-by-step manner because it is less notation-heavy and easier to follow.

If $\mathcal{O}$ can be rationalized by a function obeying **(P)** in this sense then the following is also true: for all $k \in \mathcal{K}$, $\mathcal{O}_{Y_k}$ can be rationalized by the utility function V_k (as defined by (30)). We know from Theorem 4 that this holds if and only if, for each k, there is a preference $\succeq_{Y_k}$ on $\mathcal{Y}_k = \{y_k^t\}_{t \in \mathcal{T}}$ that is consistent, conditional on $\{\succeq_j\}_{j \in J(k)}$, where $\succeq_j$ is a consistent preference on $\mathcal{X}_j = \{x_j^t\}_{t \in \mathcal{T}}$. For $\mathcal{O}$ to be rationalizable, we require (in addition) a *preference on $\mathcal{X}$ that is consistent, conditional on $\{\succeq_j\}_{j \in \bar{J}}$ and $\{\succeq_{Y_k}\}_{k \in \mathcal{K}}$*. To explain what we mean by this, we first extend the notion of conditional revealed preference introduced in the previous section. For bundles x and $\bar{x}$ in $\mathcal{X}$, x is *directly revealed preferred to $\bar{x}$ ($x \succeq^\star \bar{x}$) conditional on $\{\succeq_j\}_{j \in \mathcal{J}}$ and $\{\succeq_{Y_k}\}_{k \in \mathcal{K}}$* if there are bundles x' and x'' such that:

(a) $x' \in \Pi_{j=1}^{\bar{J}} \mathcal{X}_j$ with $x \succeq^* x'$ and

(b) $x'' = y'' \in \Pi_{k=1}^{\bar{K}} \mathcal{Y}_k$, with $x'_j \succeq_j x''_j$ for all j, and $y''_k \succeq_k \bar{y}_k$ for all $k \in \mathcal{K}$.

(Note that $\mathcal{X} \subset \Pi_{k=1}^{\bar{K}} \mathcal{Y}_k \subset \Pi_{j=1}^{\bar{J}} \mathcal{X}_j$ and x' need not be in $\Pi_{k=1}^{\bar{K}} \mathcal{Y}_k$.) The revealed preference relation is said to be *strict* ($x \ggcurly^\star \bar{x}$) if, in addition, $x \ggcurly^* x'$, $x'_j \succeq_j x''_j$ for some j, or $y''_k \succ_{Y_k} \bar{y}_k$ for some k. Comparing this definition with the definition in the previous section, we see that in both cases we require the existence of a bundle x' that is less costly than x (at the prices prevailing when x was chosen) and then require that x' to be superior to $\bar{x}$ via some route, except that there are in this case, so to speak, more routes because one could pass through the preferences $\succeq_{Y_k}$, rather than simply $\succeq_j$.

The data set $\mathcal{O}$ is said to obey GARP conditional on $\{\succeq_j\}_{j\in\mathcal{J}}$ and $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$ if $\succeq^\star$ and $\succ\!\!\succ^\star$ does not admit strict revealed preference cycles. By Proposition 2, conditional GARP is necessary and sufficient for the existence of a preference $\succeq$ on $\mathcal{X}$ that is consistent with $\succeq^\star$ and $\succ\!\!\succ^\star$; the preference $\succeq$ is said to be *consistent, conditional on* $\{\succeq_j\}_{j\in\mathcal{J}}$ *and* $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$. A data set that can be rationalized by a utility function that obeys **(P)** will admit preferences $\{\succeq_j\}_{j\in\mathcal{J}}$, $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$, and $\succeq$ that satisfy the following nested consistency conditions: for all j, $\succeq_j$ is a consistent preference on $\mathcal{X}_j$; for all k, $\succeq_{Y_k}$ is a consistent preference on $\mathcal{Y}_k$, conditional on $\{\succeq_j\}_{j\in J(k)}$; and $\succeq$ is a consistent preference on $\mathcal{X}$, conditional on $\{\succeq_j\}_{j\in\mathcal{J}}$ and $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$. To confirm this claim, simply choose $\succeq_j$ to be the restriction of U_j to $\mathcal{X}_j$, $\succeq_{Y_k}$ to be the restriction of V_k to $\mathcal{Y}_k$, and $\succeq$ to be the restriction on G to $\mathcal{X}$. Extending the terminology in the previous section, we say that $\{\succeq_j\}_{j\in\mathcal{J}}$, $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$, and $\succeq$ form a *consistent family* of preferences. The next theorem says that the converse is also true.

THEOREM 5 *Suppose that $\mathcal{O}$ admits preferences $\{\succeq_j\}_{j\in\mathcal{J}}$, $\{\succeq_{Y_k}\}_{k\in\mathcal{K}}$, and $\succeq$ that form a consistent family. Then $\mathcal{O}$ can be rationalized by a utility function $G : R^\ell \to R$ that satisfies* **(P)***, with the preference induced by U_j on $\mathcal{X}_j$ agreeing with $\succeq_j$, the preference induced by V_k on $\mathcal{Y}_k$ agreeing with $\succeq_{Y_k}$, and the preference induced by G on $\mathcal{X}$ agreeing with $\succeq$.*

An important special case of a utility function with property **(P)** is the following:

$$G(x) = U_0(x_0, U_1(x_1, U_2(x_2))) \tag{31}$$

where $x = (x_0, x_1, x_2)$, $x_0 \in R_+^{\ell_0}$, $x_1 \in R_+^{\ell_1}$, $x_2 \in R_+^{\ell_2}$ and U_0, U_1, and U_2 are well-behaved functions. This functional form generalizes (20) and is of a type often used to model intertemporal consumption (with x_i being the agent's consumption in period i). If $\mathcal{O}$ is rationalizable by (31), then the segmented data set $\mathcal{O}_{12} = \{((p_1^t, p_2^t), (x_1^t, x_2^t))\}_{t\in\mathcal{T}}$ is rationalizable by $V(x_1, x_2) = U_1(x_1, U_2(x_2))$ and thus admits a consistent family of preferences, consisting of $\succeq_2$ on $\mathcal{X}_2 = \{x_2^t\}_{t\in\mathcal{T}}$ and $\succeq_{12}$ on $\mathcal{X}_{12} = \{(x_1^t, x_2^t)\}_{t\in\mathcal{T}}$. Conversely, suppose $\mathcal{O}_{12}$ admits a consistent family, comprising $\succeq_2$ and $\succeq_{12}$ on $\mathcal{X}_2$ and $\mathcal{X}_{12}$ respectively. Conditional on this family, the revealed preference relations on $\mathcal{X} = \{x^t\}_{t\in\mathcal{T}}$ takes the following form: for $x = (x_0, x_1, x_2)$ and $\bar{x} = (\bar{x}_0, \bar{x}_1, \bar{x}_2)$ in $\mathcal{X}$, $x \succeq^\star \bar{x}$ if there are bundles $(x_1', x_2') \in \mathcal{X}_{12}$ and $x_2'' \in \mathcal{X}_2$ such that $x \succeq^* (\bar{x}_0, x_1', x_2'')$, $x_2'' \succeq_2 x_2'$, and $(x_1', x_2') \succeq_{12} (\bar{x}_1, \bar{x}_2)$. If, in addition, either $x \succ\!\!\succ^* (\bar{x}_0, x_1', x_2'')$ or $x_2'' \succ_2 x_2'$, or $(x_1', x_2') \succ_{12} (\bar{x}_1, \bar{x}_2)$, then $x \succ\!\!\succ^\star \bar{x}$. If the revealed preference relations obey conditional GARP, then there is a preference $\succeq$ on $\mathcal{X}$ that is consistent with those relations and $\mathcal{X}$ admits a consistent family, consisting of $\succeq_2$ on $\mathcal{X}_2$, $\succeq_{12}$ on $\mathcal{X}_{12}$, and $\succeq$ on $\mathcal{X}$. By Theorem 5, $\mathcal{O}$ is rationalizable by a utility function of the form (31).

The proof of Theorem 5 is in the Appendix. The proof structure is essentially the same as that of Theorem 4. Just as the proof of Theorem 4 uses Lemma 2, so a version of this lemma is needed in the proof of Theorem 5. Given the data set $\mathcal{O}$, we define $\widehat{\mathcal{O}} = \{(K^t, v^t)\}_{t\in\mathcal{T}}$, where $v^t = \big(V_1(y_1^t), V_2(y_2^t), ..., V_{\bar{K}}(y_{\bar{K}}^t)\big)$ and K_V is the constraint set induced by $\{V_k\}_{k\in\mathcal{K}}$, i.e.

$$K^t = \{(V_1(y_1), V_2(y_2), ..., V_{\bar{K}}(y_{\bar{K}})) : y \in B(p^t, p^t \cdot x^t)\} \tag{32}$$

We denote the direct revealed preference relations of $\widehat{\mathcal{O}}_V$ by $\succeq^*$ and $\succ\!\!\succ^*$. We know, from Theorem 4, that there are functions U_j and V_k, related by (30), such that U_j agrees with $\succeq_j$ on $\mathcal{X}_j$ and V_k agrees with $\succeq_{Y_k}$ on $\mathcal{Y}_k$. It is clear that with any such choice of functions,

$$y^t \succeq^\star y^s \Longrightarrow v^t \succeq^* v^s.$$

What is needed is a lemma, similar to Lemma 2, saying that the U_j and V_k can be chosen such that the reverse implication holds as well. Then one could prove Theorem 5 with this new lemma, by an argument similar to the one that establishes Theorem 4 with Lemma 2.

Our characterization of data sets that are rationalizable by weakly separable utility functions uses the key concept of a consistent family of preferences. It is obvious that this concept, along with the related concept of conditional revealed preference, can be recursively extended. It is also intuitive that the existence of a consistent preference family should characterize data sets that are rationalizable by weakly separable utility functions, for *any* finite hierarchy of separable commodity groups; in other words, Theorems 4 and 5, which cover the one- and two-layer cases, should be extendable in a natural way. The proofs we provide for those two theorems in the Appendix, which uses an inductive argument, will make it clear that that intuition is correct.

Appendix

Proof of Proposition 2

The proof of this proposition uses the following well-known result. The standard proof is an application of Szpilrajn's Theorem (see, for example, Ok (2007, Corollary 1, Chapter A)), which in turn uses the axiom of choice. In the case where S is a finite set (which is the case in our application of this result, where S equals the set of observed bundles $\mathcal{X}$), there is a straightforward inductive proof on the number of elements of S which we shall omit.

Lemma 3 *Let S be a set and R a reflexive and transitive relation on S with P its asymmetric part, i.e., xPy if xRy but not yRx. Then there is a reflexive, transitive and complete relation (i.e., a preference relation) $\hat{R}$ that extends R in the following sense: if xRy then $x\hat{R}y$ and xPy then $x\hat{P}y$, where $\hat{P}$ is the asymmetric part of $\hat{R}$.*

Proof of Proposition 2: Suppose that $\mathcal{O}$ admits a preorder $\geq$ that is consistent with its revealed relations. To establish that GARP holds, let there be observations satisfying (3). The consistency of $\geq$ (in particular, with $\geq^*$), guarantees that

$$x^{t_1} \geq x^{t_2} \geq x^{t_3} \geq \ldots x^{t_{n-1}} \geq x^{t_n} \geq x^{t_1}, \tag{33}$$

and hence $x^{t_i} \sim x^{t_{i+1}}$ for $i = 1, 2, ..., (n-1)$. Furthermore, $\geq$ is consistent with $\gg^*$ and so $x^{t_i} \not\gg^{**} x^{t_{i+1}}$ in (3), as required by GARP.

For the "if" part of this proposition, first note that $\geq^*$ is a reflexive and transitive relation, on $\mathcal{X}$. We denote the asymmetric part of $\geq^*$ by $>^*$, i.e., $x^t >^* x^r$ if $x^t \geq^* x^r$ and $x^r \not\geq^* x^t$. Furthermore, GARP guarantees that if $x^t \gg^* x^r$ then $x^r \not\geq^* x^t$; therefore, if $x^t \gg^* x^r$ then $x^t >^* x^r$. By Lemma 3, there is a preference $\geq$ (with asymmetric part $>$) on $\mathcal{X}$ that extends $\geq^*$ in the following sense: for any x^t, $x^r \in \mathcal{X}$, $x^t \geq x^r$ if $x^t \geq x^r$ and $x^t > x^r$ if $x^t >^* x^r$. The latter implies that $x^t > x^r$ if $x^t \gg^* x^r$. **QED**

Proof of Theorem 3

We organize the proof of this result around several lemmas. Note

Lemma 4 *Suppose $\mathcal{O}$ admits a consistent preference $\geq$ on $\mathcal{X}$. Given a regular set $K \subset R^\ell_+$ with gauge function g, there exist $\hat{x} \in \partial K$ (so $g(\hat{x}) = 1$) and a preference $\geq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}\}$ such that (a) $\geq'$ is consistent with the revealed relations of $\mathcal{O}' = \mathcal{O} \cup \{(K, \hat{x})\}$; (b) $\geq'$ is an extension of $\geq$; and (c) $\overline{\beta}(K) >' \hat{x} \geq' \beta(K)$.*

If there is $\underline{x} \in \beta(K)$ such that $g(\underline{x}) < 1$, then we can let $\hat{x}$ be any element such that $g(\hat{x}) = 1$ and $\hat{x} > \underline{x}$; in this case, we obtain $\overline{\beta}(K) >' \hat{x} >' \beta(K)$. If $g(\underline{x}) = 1$ for all $\underline{x} \in \beta(K)$, then we can choose $\hat{x}$ to be any element in $\beta(K)$; in this case, $\mathcal{X}' = \mathcal{X}$ and $\geq' = \geq$.

Proof: We first consider the case where there is $x^{t'} \in \beta(K)$ such that $g(x^{t'}) < 1$ and choose $\hat{x} \in \partial K$ (so $g(\hat{x}) = 1$) with $\hat{x} > x^{t'}$. Let $\geq'$ be the extension of $\geq$ such that $x^{t''} >' \hat{x}$ and $\hat{x} >' x^{t'}$ (where $x^{t''} \in \overline{\beta}(K)$). We need to show the consistency of $\geq'$ with the relations $\geq^*$ and $\gg^*$ induced by $\mathcal{O}'$.

It is clear from our construction that if $\hat{x} \succeq^{**} x^t$ then $\hat{x} \succ' x^t$, since $\hat{x} \succ' x^{t'}$ and $x^{t'} \succeq' x^t$. Suppose that for some s, $x^s \succeq^{**} \hat{x}$. Then $x^s \ggcurly^{**} x^{t'}$, since $\hat{x} > x^{t'}$. By the consistency of $\succeq$ (with respect to the revealed relations of $\mathcal{O}$), $x^s \succ x^{t'}$. Therefore, $x^s \notin K$ and we obtain $x^s \succeq x^{t''}$ (by the definition of $x^{t''}$). Since $\succeq'$ is an extension of $\succeq$, we also have $x^s \succeq' x^{t''}$ and, by construction of $\succeq'$, $x^{t''} \succ' \hat{x}$. By the transitivity of $\succeq'$, we obtain $x^s \succ' \hat{x}$.

We now turn to the case where there is no $\underline{x} \in \beta(K)$ such that $g(\underline{x}) = 1$. Let $\hat{x} = x^{t'}$, where $x^{t'}$ is some element in $\beta(K)$. We need to show that $\succeq$ remains consistent with the revealed relations of $\mathcal{O}'$. If $\hat{x} \succeq^{**} x^t$, then $\hat{x} = x^{t'} \succeq x^t$ by definition of $x^{t'}$. If $\hat{x} = x^{t'} \ggcurly^{**} x^t$ then $\hat{x} = x^{t'} \succ x^t$ since there does not exist $x^t \sim x^{t'}$ with $g(x^t) < 1$. Suppose that for some s, $x^s \succeq^{**}$ $(\ggcurly^{**})\,\hat{x} = x^{t'}$. Then $x^s \succeq$ $(\succ)\,\hat{x} = x^{t'}$, by the consistency of $\succeq$ (with respect to the revealed relations of $\mathcal{O}$). **QED**

LEMMA 5 *Suppose $\mathcal{O}$ admits a preference $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. Let K be a regular set with gauge function g, and construct $\hat{x} \in K$ and the preference $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}\}$ in the way specified in Lemma 4. For any regular set C, let*

$$\begin{aligned} \beta'(C) &= \{\tilde{x} \in C \cap \mathcal{X}' : \tilde{x} \succeq' x \, \forall x \in C \cap \mathcal{X}'\} \text{ and} \\ \overline{\beta}'(C) &= \{\tilde{x} \in \mathcal{X}' : \tilde{x} \succ' \beta'(C) \text{ and if } y \in \mathcal{X}' \text{ obeys } y \succ' \beta'(C) \text{ then } y \succeq' \tilde{x}\ \}. \end{aligned}$$

Then $\beta'(C) \succeq' \beta(C)$ and $\overline{\beta}(C) \succeq' \overline{\beta}'(C)$.

Proof: In the case where $\hat{x} \in \mathcal{X}$, these claims are trivially true since $\succeq = \succeq'$. So consider the case where $\hat{x} \notin \mathcal{X}$. It is obvious that $\beta'(C) \succeq' \beta(C)$. So we focus on proving that $\overline{\beta}(C) \succeq' \overline{\beta}'(C)$. This has to be true if $\hat{x} \notin C$, because then $\beta(C) = \beta'(C)$. We make the following claim: *if $\hat{x} \in C$, then $\overline{\beta}(C) \succ' \hat{x}$*. By definition of $\overline{\beta}$ and since $\succeq'$ is an extension of $\succeq$, $\overline{\beta}(C) \succ' \beta(C)$; this observation, together with the claim, tells us that $\overline{\beta}(C) \succ' \beta'(C)$. Lastly, by definition of $\overline{\beta}'$, we obtain $\overline{\beta}(C) \succeq \overline{\beta}'(C)$.

It remains for us to show that the claim is true. Recall that $\hat{x}$ was chosen to satisfy $\hat{x} > x^t$ for some $x^t \in \beta(K)$. Since C is regular, $x^t \in C$ and so $\beta(C) \succeq \beta(K)$. Consequently, $\overline{\beta}(C) \succeq \overline{\beta}(K)$ and since $\succeq'$ extends $\succeq$ we obtain $\overline{\beta}(C) \succeq' \overline{\beta}(K)$. Since, by construction, $\overline{\beta}(K) \succ' \hat{x}$, we obtain $\overline{\beta}(C) \succ' \hat{x}$. **QED**

LEMMA 6 *Suppose $\mathcal{O}$ admits a preference $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. For any finite collection of regular sets $\mathcal{C} = \{K^m\}_{m \in M}$ (with corresponding gauge functions $\{g^m\}_{m \in M}$, there exist $\{\hat{x}^m\}_{m \in M}$, where $\hat{x}^m \in \partial K^m$, and a preference $\succeq'$ on $\mathcal{X}' =$*

$\mathcal{X} \cup \{\hat{x}^m\}_{m \in M}$ such that (a) $\succeq'$ is consistent with the revealed relations induced by $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^m\}_{m \in M}$; (b) $\succeq'$ is an extension of $\succeq$; (c)

$$\bar{\beta}(K^m) \succ' \hat{x}^m \succeq' \beta(K^m) \text{ for all } m \in M, \tag{34}$$

where we can choose $\hat{x}^m \in \beta(K^m)$ if $g^m(\underline{x}) = 1$ for all $\underline{x} \in \beta(K^m)$.[14]

Proof: Let $\widetilde{M} = \{m \in M : g^m(\underline{x}) = 1 \,\forall\, \underline{x} \in \beta(K^m)\}$. Choose $m' \in \widetilde{M}$ and let $\hat{x}^{m'} = x^{t'}$ for some $x^{t'} \in \beta(K^{m'})$. By Lemma 4, $\succeq$ remains consistent with the revealed relations of $\mathcal{O}' = \mathcal{O} \cup \{(K^{m'}, \hat{x}^{m'})\}$. Now choose another element m' in $\widetilde{M}$ (if one exists), $\hat{x}^{m''} \in \beta(K^{m''})$ and Lemma 4 again guarantees that $\succeq$ is consistent with the revealed relations of $\mathcal{O}'' = \mathcal{O}' \cup \{(K^{m''}, \hat{x}^{m''})\}$. Repeating this procedure until all the elements of $\widetilde{M}$ are exhausted, we conclude that $\succeq$ is consistent with the revealed relations of $\widetilde{\mathcal{O}} = \mathcal{O} \cup \{(K^m, \hat{x}^m)\}_{m \in \widetilde{M}}$.

Now choose an element $n \in M \backslash \widetilde{M}$. By Lemma 4, there is a bundle $\hat{x}^n$ and a preference $\succeq^1$ on $\mathcal{X}^1 = \mathcal{X} \cup \{\hat{x}^n\}$ such that $\succeq^1$ extends $\succeq$, $\succeq^1$ is consistent with the revealed relations of $\mathcal{O}^1 = \widetilde{\mathcal{O}} \cup \{(K^n, \hat{x}^n)\}$, and the following holds:

$$\bar{\beta}(K^n) \succ^1 \hat{x}^n \succeq^1 \beta(K^n). \tag{35}$$

Suppose there exists $n' \neq n$ in $M \backslash \tilde{M}$; by Lemma 4, there are $\hat{x}^{n'}$ and a consistent preference $\succeq^2$ on $\mathcal{O}^2 = \mathcal{O}^1 \cup \{(K^{n'}, \hat{x}^{n'})\}$ that extends $\succeq^1$ and that satisfy

$$\bar{\beta}^1(K^{n'}) \succ^2 \hat{x}^{n'} \succeq^2 \beta^1(K^{n'}) \tag{36}$$

where, by definition,

$$\begin{aligned} \beta^1(C) &= \{\tilde{x} \in C \cap \mathcal{X}^1 : \tilde{x} \succeq^1 x \,\forall x \in C \cap \mathcal{X}^1\} \text{ and} \\ \bar{\beta}^1(C) &= \{\tilde{x} \in \mathcal{X}^1 : \tilde{x} \succ^1 \beta^1(C) \text{ and if } y \in \mathcal{X}^1 \text{ obeys } y \succ^1 \beta^1(C) \text{ then } y \succeq^1 \tilde{x}\ \} \end{aligned}$$

for any regular set C. By Lemma 5, $\beta^1(K^{n'}) \succeq^1 \beta(K^{n'})$ and $\bar{\beta}(K^{n'}) \succeq^1 \bar{\beta}^1(K^{n'})$ and since $\succeq^2$ is an extension of $\succeq^1$, we also have $\beta^1(K^{n'}) \succeq^2 \beta(K^{n'})$ and $\bar{\beta}(K^{n'}) \succeq^2 \bar{\beta}^1(K^{n'})$. Thus

$$\bar{\beta}(K^{n'}) \succ^2 \hat{x}^{n'} \succeq^2 \beta(K^{n'}). \tag{37}$$

Suppose there is $n'' \in M \backslash \tilde{M}$, $n'' \neq n'$, n. By an argument analogous to the one we just made, we obtain $\hat{x}^{n''}$ and a consistent preference $\succeq^3$ on $\mathcal{O}^3 = \mathcal{O}^2 \cup \{(K^{n''}, \hat{x}^{n''})\}$ that extends $\succeq^2$ and satisfy

$$\bar{\beta}^2(K^{n''}) \succ^3 \hat{x}^{n''} \succeq^3 \beta^2(K^{n''})$$

[14] We deem $\bar{\beta}(K^m) \succ' \hat{x}^m$ to be satisfied if $\bar{\beta}(K^m)$ is empty. Note that β and $\bar{\beta}$ are defined with respect to $\mathcal{X}$.

(where β^2 and $\bar{\beta}^2$ are defined in a similar way to β^1 and $\bar{\beta}^1$, but with $\succeq^2$ and $\mathcal{X}^2$ replacing $\succeq^1$ respectively). By Lemma 5, $\beta^2(K^{n''}) \succeq^2 \beta^1(K^{n''})$, $\bar{\beta}^1(K^{n''}) \succeq^2 \bar{\beta}^2(K^{n''})$, $\beta^1(K^{n''}) \succeq^1 \beta(K^{n''})$, and $\bar{\beta}(K^{n''}) \succeq^1 \bar{\beta}^1(K^{n''})$. Since $\succeq^2$ extends $\succeq^2$ which extends $\succeq^1$, we obtain $\beta^2(K^{n''}) \succeq^3 \beta(K^{n''})$ and $\bar{\beta}(K^{n''}) \succeq^3 \bar{\beta}^2(K^{n''})$ and thus

$$\bar{\beta}(K^{n''}) \succ^3 \hat{x}^{n''} \succeq^3 \beta(K^{n''}). \tag{38}$$

This argument can be repeated until we eventually obtain (34). **QED**

Proof of Theorem 3: By Lemma 6, there is a preorder $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^m\}_{m\in M}$ satisfying properties (a), (b), and (c) in the lemma. By Theorem 2, there exists U concave, continuous, and strongly monotone that rationalizes U and the preorder induced by U on $\mathcal{X}'$ coincides with $\succeq'$. Therefore, (17) follows from (34) since $\hat{x}^m$ maximizes U in K^m and so $I_U(K^m) = U(\hat{x}^m)$. If $p^m \cdot \underline{x} = w^m$ for all $\underline{x} \in \beta(p^m, w^m)$, we can choose $\hat{x}^m \in \beta(p^m, w^m)$ and so $I_U(p^m, w^m) = U(\beta(p^m, w^m))$. **QED**

Proof of Theorem 4

To complete the proof of Theorem 4, we need only prove Lemma 2. We first introduce a few useful concepts. Suppose that the set of goods is partitioned into subsets along the lines described in Section 4 and, for each j, there is a consistent preference $\succeq_j$ on $\mathcal{X}_j$. Let $S \subseteq \mathcal{J}$ and x'' and x' elements in $\Pi_{j\in S}\mathcal{X}_j$. We write $x'' \succeq_J x'$ if $x''_j \succeq_j x'_j$ for all $j \in S$ and $x'' \sim_J x'$ if $x''_j \sim_j x'_j$ for all $j \in S$. Given a price vector $p \in R^\ell_{++}$ and income $w > 0$, we define

$$B_{\mathcal{X}}(p, w) = \{x \in \mathcal{X} : \exists\ x'' \in \Pi_{j\in\mathcal{J}}\mathcal{X}_j \text{ such that } x'' \in B(p, w) \text{ and } x'' \succeq_J x\}. \tag{39}$$

We say that $x^0 \in B_{\mathcal{X}}(p, w)$ is in $\partial B_{\mathcal{X}}(p, w)$, *the upper boundary of* $B_{\mathcal{X}}(p, w)$, if $x^0 \in B_{\mathcal{X}}(p, w)$ and, for any $x'' \in \Pi_{j\in J}\mathcal{X}_j$ such that $p \cdot x'' \leqslant w$ and $x'' \succeq_J x^0$, we have $p \cdot x'' = w$ and $x'' \sim_J x^0$. The following result is needed in the proof of Lemma 2.

Lemma 7 (i) *Suppose $x^s \in \mathcal{X}\backslash B_{\mathcal{X}}(p, w)$. Then there are scalars $w^s_j \geqslant 0$ with $\sum_{j=1}^{\bar{J}} w^s_j = w$, such that for $x^s_j \succ_j \beta_{\mathcal{X}_j}(p_j, w^s_j)$ for all j.*
(ii) *Suppose $x^s \in \partial B_{\mathcal{X}}(p, w)$. Then there are scalars $w^s_j \geqslant 0$ with $\sum_{j=1}^{\bar{J}} w^s_j = w$ such that $x^s_j \sim_j \beta_{\mathcal{X}_j}(p_j, w^s_j)$ and $\beta_{\mathcal{X}_j}(p_j, w^s_j) \subseteq \partial B(p_j, w^s_j)$ for all $j \in \mathcal{J}$.*

Note: By definition, $\beta_{\mathcal{X}_j}(p_j, w^s_j) = \{x'_j \in B(p_j, w^s_j) \cap \mathcal{X}_j : x'_j \succeq_j x_j \forall x_j \in B(p_j, w^s_j) \cap \mathcal{X}_j\}$.

Proof: We first choose $\hat{x}^s \in \arg\min\{p^t \cdot x' : x' \in \Phi(x^s)\}$, where

$$\Phi(x^s) = \{x' \in \Pi_{j\in\mathcal{J}}\mathcal{X}_j : x' \succeq_J x^s\}. \tag{40}$$

Note that $\hat{x}^s \in \arg\min\{p^t \cdot x' : x' \in \Phi(x^s)\}$ if and only if

$$\hat{x}^s_j = \arg\min\{p^t_j \cdot x'_j : x'_j \in \Phi_j(x^s)\} \text{ for all } j \in \mathcal{J}, \text{ where} \tag{41}$$

$$\Phi_j(x^s) = \{x_j \in \mathcal{X}_j : x'_j \succeq_j x^s_j\}. \tag{42}$$

If $x^s \in \mathcal{X}\backslash B_{\mathcal{X}}(p,w)$, then $p \cdot \hat{x}^s > w$. Therefore, we can choose w^s_j such that $p_j \cdot \hat{x}^s_j > w^s_j$ for all $j \in \mathcal{J}$ and $\sum_{j=1}^{\bar{J}} w^s_j = w$. Because of (41), we obtain $x^s_j \succ_j \beta_{\mathcal{X}_j}(p_j, w^s_j)$.

If $x^s \in \partial B_{\mathcal{X}}(p,w)$, $p \cdot \hat{x}^s = w$ and $\hat{x}^s_j \sim_j x^s_j$ for all $j \in \mathcal{J}$. Let $w^s_j = p_j \cdot \hat{x}^s_j$; then $\sum_{j=1}^{\bar{J}} w^s_j = w$. Again because $x^s \in \partial B_{\mathcal{X}}(p,w)$, there does not exist $\tilde{x}_j \in \mathcal{X}_j$ such that $\tilde{x}_j \succ \hat{x}^s_j \sim_j x^s_j$ and $p_j \cdot \tilde{x}_j \leqslant w^s_j$. In other words, $\hat{x}^s_j \in \beta_{\mathcal{X}_j}(p_j, w^s_j)$. Now let $\bar{x}_j \in \beta_{\mathcal{X}_j}(p_j, w^s_j)$; since $\hat{x}^s_j \sim_j x^s_j$, we obtain $\bar{x}_j \sim_j x^s_j$. It follows from the definition of $\hat{x}^s_j$ that $p_j \cdot \bar{x}_j = w^s_j$. Thus $\beta_{\mathcal{X}_j}(p, w^s_j) \subseteq \partial B(p_j, w^s_j)$. **QED**

Proof of Lemma 2: First we note that property (25) is equivalent to the following pair of conditions which are more convenient to prove: (I) if $x^t \not\succeq^\star x^s$, then $u^t \not\succeq^* u^s$ and (II) if $x^t \succeq^\star x^s$ and $x^t \not\succ^\star x^s$, then $u^t \succeq^* u^s$ and $u^t \not\succ^* u^s$.

Let $H = \{(t,s) : x^t \not\succeq^\star x^s\}$ and $H' = \{(t,s) :\ x^t \succeq^\star x^s \text{ and } x^t \not\succ^\star x^s\}$.
Claim: *There are scalars $f^{ts}_j \geqslant 0$ with $\sum_{j=1}^{\bar{J}} f^{ts}_j = p^t \cdot x^t$, such that for $(t,s) \in H$, $x^s_j \succ_j \beta_{\mathcal{X}_j}(p^t_j, f^{ts}_j)$ and, for $(t,s) \in H'$, $x^s_j \sim_j \beta_{\mathcal{X}_j}(p^t_j, f^{ts}_j)$ and $\beta_{\mathcal{X}_j}(p^t_j, f^{ts}_j) \subseteq \partial B(p^t_j, f^{ts}_j)$.*

This claim follows immediately from Lemma 7. We need only note that for $(t,s) \in H$, $x^s \in \mathcal{X}\backslash B_{\mathcal{X}}(p^t, p^t \cdot x^t)$, so application part (i) of the lemma guarantees the existence of f^{ts}_j with the desired properties. If $(t,s) \in H'$, then $x^s \in \partial B_{\mathcal{X}}(p^t, p^t \cdot x^t)$, and so part (ii) of the lemma guarantees that the claim holds.

For each $j \in \mathcal{J}$, we can apply Theorem 3 to $\mathcal{O}_j$, with

$$\mathfrak{C}_j = \left\{B(p^t_j, f^{ts}_j) : (t,s) \in H \cup H'\right\}. \tag{43}$$

This guarantees that there is a well-behaved and concave function U_j that rationalizes $\mathcal{O}_j$ and agrees with $\succeq_j$ on $\mathcal{X}_j$; furthermore, we obtain (a) $I_{U_j}(p^t_j, f^{ts}_j) < U_j(x^s_j)$ for all $(t,s) \in H$ and (b) $I_{U_j}(p^t_j, f^{ts}_j) = U_j(\hat{x}^{ts}_j) = U_j(x^s_j)$ for all $(t,s) \in H'$. For a given $(t,s) \in H$, (a) holds for every j; in other words, there is $f^{ts}_j \geqslant 0$ such that $\sum_{j=1}^{\bar{J}} f^{ts}_j = p^t \cdot x^t$ and $I_{U_j}(p^t_j, f^{ts}_j) < U_j(x^s_j)$ for all j. Therefore, $u^s \notin K^t$ as required by (I). For a given $(t,s) \in H'$, (b) holds for every j; in other words, there is f^{ts}_j such that $\sum_{j=1}^{\bar{J}} f^{ts}_j = p^t \cdot x^t$ and $I_{U_j}(p^t_j, f^{ts}_j) = U_j(x^s_j)$ for all j. Thus, $u^s \in \partial K^t$, as required by (II). **QED**

Proofs of Theorems 5 and related results.

Our objective here is not just to prove Theorem 5, but to prove it in a way that makes it clear that analogous results for testing models with more nested layers of separability are true and can be established by induction. For that purpose, the right results to prove are not Theorems 4 and 5 as such, but stronger versions of those results, Theorems 6 and 7 respectively, that includes the property (♣). This property controls the indirect utility at some finite collection of linear budget sets $\{B^a\}_{a\in A}$ that are not part of the set of observations, i.e., similar to the way Theorem 3 strengthens Theorem 2. It is this family of stronger results that are extendable by induction in the manner depicted below:

Theorem 3 → Lemma 8 → Theorem 6 → Lemma 10 → Theorem 7 ...

So we use Theorem 3 to prove Theorem 6 (via Lemma 8), which covers the single-layer weakly separable case. We then use Theorem 6 to prove Theorem 7 (via Lemma 10), thus establishing the result for the two-layer weakly separable case. Theorem 7 can in turn be used to establish the analogous rationalizability result for the three-layer separable case, etc.

Theorem 6 *Suppose that the data set $\mathcal{O}$ admits a consistent family of preferences: $\succeq_j$ on $\mathcal{X}_j$ for all j and $\succeq$ on $\mathcal{X}$. In addition, let $\mathfrak{C} = \{B^a\}_{a\in A}$ be a finite collection of linear budget sets, where $B^a = B(p^a, w^a)$, for $p^a \gg 0$ and $w^a > 0$. Then $\mathcal{O}$ can be rationalized by a utility function G of the form (19), satisfying properties* (i) *and* (ii) *in Theorem 4 and following property: (♣) for any $x \in \mathcal{X}$ and $a \in A$ such that $x \succ \beta_{\mathcal{X}}(p^a, w^a)$, we have*

$$G(x) > I_G(p^a, w^a) \geqslant G(\beta_{\mathcal{X}}(p^a, w^a)) \tag{44}$$

and $I_G(p^a, w^a) = G(\beta_{\mathcal{X}}(p^a, w^a))$ if $\beta_{\mathcal{X}}(p^a, w^a) \subseteq \partial B_{\mathcal{X}}(p^a, w^a)$.

Note: $I_G(p, w)$ refers to the indirect utility at (p, w), given the utility function G, while $\beta_{\mathcal{X}}(p^a, w^a) = \{x' \in B_{\mathcal{X}}(p^a, w^a) : x' \succeq x \ \forall \ x \in B_{\mathcal{X}}(p^a, w^a)\}$.

The proof of Theorem 6 uses Lemma 8 below, which is a stronger version of Lemma 2.

Lemma 8 *Suppose that the data set $\mathcal{O}$ admits consistent preferences $\succeq_j$ on $\mathcal{X}_j$ for all j. Then there are well-behaved and concave utility functions U_j that satisfy condition (i) in Theorem 4, property (25), and the following properties:*

$$u^t \in K^a \Longleftrightarrow x^t \in B_{\mathcal{X}}(p^a, w^a); \tag{45}$$

$$x^t \in \partial B_{\mathcal{X}}(p^a, w^a) \Longrightarrow u^t \in \partial K^a \tag{46}$$

NOTE: K^a is the image of $B(p^a, w^a)$ under $\{U_j\}_{j\in\mathcal{J}}$.

Proof: The proof of this result is an extension of the proof of Lemma 2. Recall that the proof of that result involves applying Theorem 3 to a suitably constructed collection of budget sets $\mathfrak{C}_j$. We now have to construct U_j to satisfy a longer list of conditions, which means that $\mathfrak{C}_j$ has to be larger than the one specified in (54).

Let $S = \{(a,s) : x^s \in \mathcal{X}\backslash B_{\mathcal{X}}(p^a, w^a)\}$ and $S' = \{(a,s) :\ x^s \in \partial B_{\mathcal{X}}(p^a, w^a)\}$. The following claim follows immediately from Lemma 7.

CLAIM: *There are scalars* $w_j^{as} \geqslant 0$ *with* $\sum_{j=1}^{\bar{J}} w_j^{as} = p^t \cdot x^t$*, such that for* $(a,s) \in S$, $x_j^s \succ_j \beta_{\mathcal{X}_j}(p_j^a, w_j^{as})$ *and, for* $(a,s) \in S'$, $x_j^s \sim_j \beta_{\mathcal{X}_j}(p_j^a, w_j^{as})$ *and* $\beta_{\mathcal{X}_j}(p_j^a, w_j^{as}) \subseteq \partial B(p_j^a, w_j^{as})$.

For each j, we can apply Theorem 3 to $\mathcal{O}_j$, with

$$\mathfrak{C}_j = \left\{B(p_j^t, f_j(t,s)) : (t,s) \in H \cup H'\right\} \bigcup \left\{B(p_j^a, w_j^{as}) : (a,s) \in S \cup S'\right\}.$$

This guarantees that there is a well-behaved and concave function U_j that rationalizes $\mathcal{O}_j$, with its restriction to $\mathcal{X}_j$ agreeing with $\succeq_j$. We also obtain

(a) $I_{U_j}(p_j^t, f_j^{ts}) < U_j(x_j^s)$ for all $(t,s) \in H$ and

(b) $I_{U_j}(p_j^t, f_j^{ts}) = U_j(x_j^s)$ for all $(t,s) \in H'$

from which one could show that (25) holds (see the proof of Lemma 2). Furthermore, we obtain the following:

(c) $I_{U_j}(p_j^a, w_j^{as}) < u_j^s$ for all $(a,s) \in S$ and

(d) $I_{U_j}(p_j^a, w_j^{as}) = u_j^s$ for all $(a,s) \in S'$.

It is clear that $u^s \in K^a$ if $x^s \in B_{\mathcal{X}}(p^a, w^a)$, so the nontrivial part of (45) is the claim that $u^s \notin K^a$ if $x^s \notin B_{\mathcal{X}}(p^a, w^a)$. Indeed, if $(a,s) \in S$, then (c) holds and $(I_{U_j}(p_j^a, w_j^{as}))_{j=1}^{\bar{J}} \in \partial K^a$ because $\sum_{j=1}^{\bar{J}} w_j^{as} = w^a$; thus $u^s \notin K^a$. Lastly, it is clear that (d) implies that $u^s \in \partial K^a$ whenever $x^s \in \partial B_{\mathcal{X}}(p^a, w^a)$. **QED**

Proof of Theorem 6: The proof is a more elaborate version of the proof of Theorem 4. First we choose U_j to obey the properties listed in Lemma 8. Following the proof of Theorem 4, we know that, with this choice of U_js, the preference $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}}$ (as defined by (24)) is consistent, i.e., $u^t \widehat{\gtrsim} u^s$ if $u^t \geq^{**} u^s$ and $u^t \widehat{>} u^s$ if $u^t \gg^{**} u^s$. We also obtain

$$u^t \,\widehat{\gtrsim}\, u^s \iff x^t \succeq x^s. \tag{47}$$

(by combining (24) and (26)). We claim that, with this choice of U_js, the following two properties are also satisfied: (A) $u^t \in \beta(K^a)$ if and only if $x^t \in \beta_{\mathcal{X}}(p^a, w^a)$ and (B) if $\beta_{\mathcal{X}}(p^a, w^a) \subseteq \partial B_{\mathcal{X}}(p^a, w^a)$, then $\beta(K^a) \subseteq \partial K^a$.

To prove (A), suppose $u^t \in \beta(K^a)$, which means that $u^t \widehat{\gtrsim} u^s$ for all $u^s \in K^a$, where $u^s \in \widehat{\mathcal{X}}$. If $x^{\hat{s}} \in B_{\mathcal{X}}(p^a, w^a)$, then $u^{\hat{s}} \in K^a$ (by (45)) and so $u^t \widehat{\gtrsim} u^{\hat{s}}$. By (47), we obtain $x^t \succeq x^{\hat{s}}$. Thus $x^t \in \beta_{\mathcal{X}}(p^a, w^a)$. Similarly, suppose $x^t \in \beta_{\mathcal{X}}(p^a, w^a)$. For $u^{\hat{s}} \in \widehat{\mathcal{X}}$ such that $u^{\hat{s}} \in K^a$, then $x^{\hat{s}} \in B_{\mathcal{X}}(p^a, w^a)$ (by (45)), which means that $x^t \succeq x^{\hat{s}}$ and so $u^t \succeq u^{\hat{s}}$ (by (47)). Thus $u^t \in \beta(K^a)$. To prove (B), let $u^t \in \beta(K^a)$. Then $x^t \in \beta_{\mathcal{X}}(p^a, w^a)$ (by (A)) and, by assumption, in $\partial B_{\mathcal{X}}(p^a, w^a)$. This implies that $u^t \in \partial K^a$ (by Lemma 8), as required.

Theorem 2 guarantees the existence of a well-behaved function F that rationalizes $\widehat{\mathcal{O}}$ and induces $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}}$. Furthermore, for all $a \in A$,

$$F(\bar{\beta}(K^a)) > I_F(K^a) \geqslant F(\beta(K^a)) \tag{48}$$

and $F(\beta(K^a)) = I_F(K^a)$ if $\beta(K^a) \subseteq \partial K^a$. It follows from Proposition 3 that the resulting function G (as defined by (19)) rationalizes $\mathcal{O}$ and induces $\succeq$ on $\mathcal{X}$. Lastly, we need to show that property (iii) (as specified in the theorem) holds. Suppose $x^t \succ \beta_{\mathcal{X}}(p^a, w^a)$; then by property (A) and (47), we obtain $u^t \widehat{\succ} \beta(K^a)$, which implies that $u^t \widehat{\gtrsim} \bar{\beta}(K^a)$. It follows from (48) that $F(u^t) > I_F(K^a) \geqslant F(\beta(K^a))$ which implies (44). Now suppose that $\beta_{\mathcal{X}}(p^a, w^a) \subseteq \partial B_{\mathcal{X}}(p^a, w^a)$; property (B) tells us that $\beta(K^a) \subset \partial K^a$. Thus $F(\beta(K^a)) = I_F(K^a)$, which implies $G(\beta_{\mathcal{X}}(p^a, w^a)) = I_G(p^a, w^a)$. So we have shown (♣). **QED**

Instead of proving Theorem 5, we shall prove a stronger result, Theorem 7, which controls for indirect utility at some finite family of budget sets $\{B^a\}_{a \in A}$.

Suppose that goods are partitioned in the manner described in Section 5, with two layers of separability. Assume that, for all j, $\succeq_j$ is a consistent preference on $\mathcal{X}_j$ and, for all k, $\succeq_{Y_k}$ is a consistent preference on $\mathcal{Y}_k$, conditional on $\{\succeq_j\}_{j \in J(k)}$. Let $C \subset \mathcal{K}$ and y'' and y' be two elements in $\Pi_{k \in C} \mathcal{Y}_k$. We write $y'' \succeq_K y'$ if $y''_k \succeq_k y'_k$ for all $k \in C$. For x' and x in $\mathcal{X}$, we say that $x' \succeq_{JK} x''$ if there exists $x'' = y'' \in \Pi_{k=1}^{\bar{K}} \mathcal{Y}_k$ such that $x' \succeq_J x''$ and $y'' \succeq_K x$. Given a price vector $p \in R^{\ell}_{++}$ and income $w > 0$, we define

$$B_{\mathcal{X}}(p, w) = \{x \in \mathcal{X} : \exists\ x' \in \Pi_{j=1}^{\bar{J}} \mathcal{X}_j \text{ such that } p^a \cdot x' \leqslant w^a \text{ and } x' \succeq_{JK} x\ \}. \tag{49}$$

(This definition is the natural extension of (39) to this context.) We say that x^0 is in $\partial B_{\mathcal{X}}(p, w)$, *the boundary of* $B_{\mathcal{X}}(p, w)$, if $x^0 \in B_{\mathcal{X}}(p, w)$ and, for any $x' \in \Pi_{j=1}^{\bar{J}} \mathcal{X}_j$ and $x'' =$

$y'' \in \Pi_{k=1}^{\bar{K}} \mathcal{Y}_k$ such that $p \cdot x' \leqslant w$, $x' \succeq_J x''$ and $y'' \succeq_K x$, we must have $p \cdot x' = w$, $x' \sim_J x''$ and $y'' \sim_K x$.

THEOREM 7 *Suppose that $\mathcal{O}$ admits a consistent family of preferences: $\{\succeq_j\}_{j=1}^{\bar{J}}$, $\{\succeq_{Y_k}\}_{k=1}^{\bar{K}}$, and $\succeq$. Let $\mathfrak{C} = \{B^a\}_{a \in A}$ be a finite collection of linear budget sets, where $B^a = B(p^a, w^a)$, for $p^a \gg 0$ and $w^a > 0$. Then $\mathcal{O}$ can be rationalized by a utility function G obeying the properties listed in Theorem 5 as well as the following: (♣) for any $x \in \mathcal{X}$ and $a \in A$ such that $x \succ \beta_{\mathcal{X}}(p^a, w^a)$,*

$$G(x) > I_G(p^a, w^a) \geqslant G(\beta_{\mathcal{X}}(p^a, w^a)) \tag{50}$$

and $I_G(p^a, w^a) = G(\beta_{\mathcal{X}}(p^a, w^a))$ if $\beta(p^a, w^a) \subseteq \partial B_{\mathcal{X}}(p^a, w^a)$.

NOTE: By definition, $\beta_{\mathcal{X}}(p^a, w^a) = \{x' \in B_{\mathcal{X}}(p^a, w^a) : x' \succeq x \, \forall \, x \in B_{\mathcal{X}}(p^a, w^a)\}$ with $B_{\mathcal{X}}(p^a, w^a)$ defined by (49).

LEMMA 9 (i) *Suppose $x^s = y^s \in \mathcal{X} \backslash B_{\mathcal{X}}(p, w)$. Then there are scalars $w_k^s \geqslant 0$ with $\sum_{k=1}^{\bar{K}} w_k^s = p^t \cdot x^t$, such that for $x_{J(k)}^s \succ_k \beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$ for all k.*
(ii) *Suppose $x^s \in \partial B_{\mathcal{X}}(p, w)$. Then there are scalars $w_k^s \geqslant 0$ with $\sum_{k=1}^{\bar{K}} w_k^s = w$ such that $x_{J(k)}^s \sim_k \beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$ and $\beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s) \in \partial B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$ for all k.*

NOTE: By definition, $\beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s) = \{y_k \in B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s) : y_k' \succeq_k y_k \, \forall \, y_k \in B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)\}$, where $B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s) = \{y \in \mathcal{Y}_k : \exists \, x''_{J(k)} \in \left(\Pi_{j \in J(k)} \mathcal{X}_j\right) \cap B(p_{J(k)}, w_k^s)$ such that $x''_{J(k)} \succeq_J y\}$ (i.e., following the definition (39)).

Proof: We first choose $\hat{x}^s \in \arg\min\{p \cdot x' : x' \in \Phi(x^s)\}$, where

$$\Phi(x^s) = \{x' \in \Pi_{j=1}^{\bar{J}} \mathcal{X}_j : x' \succeq_{JK} x^s\}.$$

Note that $\hat{x}^s \in \arg\min\{p \cdot x' : x' \in \Phi(x^s)\}$ if and only if

$$\hat{x}_{J(k)}^s \in \arg\min\{p_{J(k)} \cdot x'_{J(k)} : x'_{J(k)} \in \Phi_k(x^s)\} \text{ for all } k, \text{ where}$$

$$\Phi_k(x^s) = \{x'_{J(k)} \in \Pi_{j \in J(k)} \mathcal{X}_j : \exists \, x''_{J(k)} = y''_k \in \mathcal{Y}_k \text{ with } x'_{J(k)} \succeq_J x''_{J(k)} \text{ and } y'' \succeq_k x_{J(k)}^s = y_k^s\}$$

If $x^s \in \mathcal{X} \backslash B_{\mathcal{X}}(p, w)$, then $p \cdot \hat{x}^s > w$. So there are scalars $w_k^s \geqslant 0$ such that $\sum_k w_k^s = w$ and $p_{J(k)} \cdot \hat{x}_{J(k)}^s > w_k^s$ for all k. Note that there does not exist $\tilde{x}_{J(k)} \in \Phi_k(x^s)$ such that $p_{J(k)} \cdot \tilde{x}_{J(k)} \leqslant w_k^s$ since this will mean that $\hat{x}_{J(k)}^s$ does not minimize $p_{J(k)} \cdot x'_{J(k)}$ subject to $x'_{J(k)} \in \Phi_k(x^s)$. Hence $x_{J(k)}^s = y_k^s \succ_k \beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$.

If $x^s \in \partial B_{\mathcal{X}}(p, w)$, $p \cdot \hat{x}^s = w$ and there is $y'' \in \Pi_{k=1}^{\bar{K}} \mathcal{Y}_k$ such that $\hat{x}_{J(k)}^s \sim_J x''_{J(k)} = y''_k \in \mathcal{Y}_k$, and $y''_k \sim_k y_k^s = x_{J(k)}^s$. Let $w_k^s = p_{J(k)} \cdot \hat{x}_{J(k)}^s$. Furthermore, there does not exist $\tilde{y}_k \in \mathcal{Y}_k$

such that $\tilde{y}_k \succ_k y_k^s = x_{J(k)}^s$ with $\tilde{y}_k \in B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$. In other words, $y_k'' \in \beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$, as required. Now let $\bar{y}_k \in \beta_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$; since $y_k'' \sim_k x_{J(k)}^s$, we have $\bar{y}_k \sim_k x_{J(k)}^s$. Suppose $\bar{x}_{J(k)} \in \Pi_{j \in J(k)} \mathcal{X}_j$ satisfies $p_{J(k)} \cdot \bar{x}_{J(k)} \leqslant w_k^s$ and $\bar{x}_{J(k)} \succeq_J \bar{y}_k$ for $\bar{y}_k \in \mathcal{Y}_k$; then $\bar{x}_{J(k)} \in \Phi_k(x^s)$ and $p_{J(k)} \cdot \bar{x}_{J(k)} = w_k^s$ by the definition of w_k^s. Furthermore, $\bar{x}_{J(k)} \sim_J \bar{y}_k$ since $x^s \in \partial B_{\mathcal{X}}(p, w)$. Hence $\bar{y}_k \in \partial B_{\mathcal{Y}_k}(p_{J(k)}, w_k^s)$. **QED**

LEMMA 10 *Suppose that $\succeq_j$ is a consistent preference on $\mathcal{X}_j$ and $\succeq_{Y_k}$ is a consistent preference on $\mathcal{Y}_k$, conditional on $\{\succeq_j\}_{j \in J(k)}$. Then there exists U_j, H_k such that V_k, given by (30), rationalizes $\mathcal{O}_{\mathcal{Y}_k}$ with the preference induced by U_j on $\mathcal{X}_j$ agreeing with $\succeq_j$, the preference induced by V_k on $\mathcal{Y}_k$ agreeing with $\succeq_{Y_k}$. Furthermore, the following properties hold:*

$$v^t \succeq_V^* [\ggcurly_V^*]\, v^s \Longrightarrow x^t \succeq^\star [\ggcurly^\star]\, x^s\,; \tag{51}$$

$$v^t \in K^a \Longleftrightarrow x^t \in B_{\mathcal{X}}(p^a, w^a)\,; \tag{52}$$

$$x^t \in \partial B_{\mathcal{X}}^a \Longrightarrow v^t \in \partial K^a \tag{53}$$

NOTE:; $K^a \in R_+^K$ is the image of B^a under $\{V_j\}_{k=1}^K$.

Proof: Our proof strategy is identical to the one used to prove Lemma 8. Just as the proof of that result uses Theorem 3 to find functions U_j obeying the specified conditions, so this proof uses Theorem 6 to obtain V_k so that properties (51) – (53) are satisfied. First, note that property (51) is equivalent to the following pair of conditions which are more convenient to prove: (I) if $x^t \not\succeq^\star x^s$, then $v^t \not\succeq_V^* v^s$ and (II) if $x^t \succeq^\star x^s$ and $x^t \not\ggcurly^\star x^s$, then $v^t \succeq_V^* v^s$ and $v^t \not\ggcurly_V^* x^s$. Let $H = \{(t,s) : x^t \not\succeq^\star x^s\}$ and $H' = \{(t,s) : x^t \succeq^\star x^s \text{ and } x^t \not\ggcurly^\star x^s\}$.

CLAIM 1: *There are scalars $f_k^{ts} \geqslant 0$ with $\sum_k f_k^{ts} = p^t \cdot x^t$ such that for $(s,t) \in H$, $x_{J(k)}^s \succ_k \beta(p_{J(k)}^t, f_k^{ts})$ and, for $(s,t) \in H'$, $x_{J(k)}^s = y_k \sim_k \beta(p_{J(k)}^t, f_k(t,s))$ and $\beta(p_{J(k)}^t, f_k(t,s)) \subseteq \partial B_{\mathcal{Y}_k}(p_{J(k)}^t, f_k(t,s))$.*

This claim follows immediately from Lemma 9. We need only note that for $(t,s) \in H$, $x^s \in \mathcal{X} \backslash B_{\mathcal{X}}(p^t, p^t \cdot x^t)$, so application part (i) of the lemma guarantees the existence of f_j^{ts} with the desired properties. If $(t,s) \in H'$, then $x^s \in \partial B_{\mathcal{X}}(p^t, p^t \cdot x^t)$, and so part (ii) of the lemma guarantees that the claim holds.

Let $S = \{(a,s) : x^s \in \mathcal{X} \backslash B_{\mathcal{X}}(p^a, w^a)\}$ and $S' = \{(a,s) : x^s \in \partial B_{\mathcal{X}}^a\}$. The following claim also follows immediately from Lemma 9.

CLAIM 2: *There are scalars $w_k^{as} \geqslant 0$ with $\sum_{k \in L} w_k^{as} = w^a$, such that for $(a,s) \in S$, $x_{J(k)}^s = y_k^s \succ_k \beta(p_{J(k)}^a, w_k^{as})$ and, for $(a,s) \in S'$, $x_{J(k)}^s = y_k^s \sim_k \beta(p_{J(k)}^a, w_k^{as})$ and $\beta(p_{J(k)}^a, w_k^{as}) \subseteq \partial B_{\mathcal{Y}_k}(p_{J(k)}^t, w_k^{as})$.*

For each k, we can apply Theorem 6 to the data set $\mathcal{O}_{\mathcal{Y}_k} = \{(p^t_{\mathcal{Y}_k}, y^t_k)\}_{t\in\mathcal{T}}$, with

$$\mathfrak{C}_k = \left\{B(p^t_{J(k)}, f_k(t,s)) : (t,s) \in H \cup H'\right\} \bigcup \{B(p^a_{J(k)}, w^{as}_k) : (a,s) \in S \cup S'\}. \qquad (54)$$

We obtain U_j, H_k such that V_k, given by (30), rationalizes $\mathcal{O}_{\mathcal{Y}_k}$ with the preference induced by U_j on $\mathcal{X}_j$ agreeing with $\succeq_j$, the preference induced by V_k on $\mathcal{Y}_k$ agreeing with $\succeq_{Y_k}$. Furthermore, the following holds:

(a) $I_{V_k}(p^t_{J(k)}, f_k(t,s))) < V_k(x^s_{J(k)}) = V_k(y^s_K) = v^s_k$ for all $(t,s) \in H$;

(b) $I_{V_k}(p^t_{J(k)}, f_k(t,s))) = V_k(x^s_{J(k)}) = V_k(y^s_K) = v^s_k$ for all $(t,s) \in H'$;

(c) $I_{V_k}(p^t_{J(k)}, w^{as}_k) < V_k(x^s_{J(k)}) = V_k(y^s_K) = v^s_k$ for all $(a,s) \in S$; and

(d) $I_{V_k}(p^t_{J(k)}, w^{as}_k) = V_k(x^s_{J(k)}) = V_k(y^s_K) = v^s_k$ for all $(a,s) \in S'$.

For a given $(t,s) \in H$, (a) holds for every k; in other words, there is $f^{ts}_k \geqslant 0$ such that $\sum_{k=1}^{\bar{K}} f^{ts}_k = p^t \cdot x^t$ and $I_{V_k}(p^t_{J(k)}, f^{ts}_k) < V_k(x^s_{J(k)})$ for all k. Therefore, $v^s \notin K^t$ as required by (I). For a given $(t,s) \in H'$, (b) holds for every k; in other words, there is $\sum_{k=1}^{\bar{K}} f^{ts}_k = p^t \cdot x^t$ such that $I_{V_k}(p^t_{J(k)}, f^{ts}_{J(k)}) = V_k(x^s_{J(k)})$ for all k. Therefore, $v^s \in \partial K^t$, as required by (II). This establishes (51).

By an argument similar to the one we used for $(t,s) \in H$, we obtain $v^s \notin K^a$ for $(a,s) \in S$, i.e., if $x^s \notin B_{\mathcal{X}}(p^a, w^a)$, then $v^s \notin K^a$. It is straightforward that if $x^s \in B_{\mathcal{X}}(p^a, w^a)$, then $v^s \in K^a$. Therefore, we obtain (52). By an argument similar to the one we used for $(t,s) \in H'$, we conclude that $v^s \in \partial K^a$ if $(a,s) \in S'$, i.e., (53) holds. **QED**

Proof of Theorem 7: The final step in the proof is essentially the same as that for Theorem 6 so we shall only sketch the argument. First, we choose V_k to obey the properties listed in Lemma 10. With this choice, we define the preference $\widehat{\gtrsim}$ on $\widehat{\mathcal{O}} = \{(K^t, v^t\}_{t\in\mathcal{T}}$ by $v^t \widehat{\gtrsim} v^s$ if $x^t \succeq x^s$. Applying Lemma 10 we can establish that $\widehat{\gtrsim}$ is consistent with the revealed relations of $\widehat{\mathcal{O}}$, i.e., $v^t \widehat{\gtrsim} v^s$ if $v^t \succeq^{**} v^s$ and $v^t \widehat{>} v^s$ if $v^t \ggg^{**} v^s$, and we also obtain

$$v^t \,\widehat{\gtrsim}\, v^s \iff x^t \succeq x^s.$$

Furthermore, the following two properties are satisfied: (A) $v^t \in \beta(K^a)$ if and only if $x^t \in \beta_{\mathcal{X}}(p^a, w^a)$ and (B) if $\beta_{\mathcal{X}}(p^a, w^a) \subseteq \partial B_{\mathcal{X}}(p^a, w^a)$, then $\beta(K^a) \subseteq \partial K^a$. Theorem 2 guarantees the existence of a well-behaved function F that rationalizes $\widehat{\mathcal{O}}$ and induces $\widehat{\gtrsim}$ on $\widehat{\mathcal{X}}$, with $F(\bar{\beta}(K^a)) > I_F(K^a) \geqslant F(\beta(K^a))$ (for all a) and $F(\beta(K^a)) = I_F(K^a)$ if $\beta(K^a) \subseteq \partial K^a$. The resulting function G (as defined by (29)) rationalizes $\mathcal{O}$ and induces $\succeq$ on $\mathcal{X}$. Lastly, we can use (A) and (B) to show that ($\clubsuit$) holds. **QED**

It is clear that there is an analog to Theorem 7 for a model with three layers of nested weak separability. The argument for that will rely on the analogous version of Lemmas 9 and 10, with the proof of the latter relying on the result for the two-layer case, Theorem 7. And so our main result can be extended any finite number of times.

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