Lecture #7:

• Basic Notions of Fracture Mechanics
• Ductile Fracture

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Basic Notions of Fracture Mechanics
Fracture Mechanics

Fracture mechanics is a branch of mechanics that is concerned with the study of the propagation of cracks and growth of flaws. The starting point of a fracture mechanics analysis therefore is a structure with a pre-existing crack or flaw. Central questions in fracture mechanics are for example:

- Under which mechanical loads does a pre-existing crack propagate?
- What is the maximum size of a crack that can be tolerated in a structure that is subject to a known mechanical load such that the crack does not propagate?
Griffith theory – an energy approach

Griffith’s (1921) made an attempt to come up with a fracture criterion for brittle solids by writing down the energy balance during the growth of a crack.

Consider an isotropic linear elastic plate subject to uniaxial tension and let $W_0$ denote the elastic strain energy stored in that system. According to Inglis (1913) analysis, the introduction of a through thickness crack of length $2a$ reduces the elastic strain energy (energy release) by

$$\frac{\pi a^2 t \sigma_\infty^2}{E}$$
Introducing the **free surface energy** per unit area, $\gamma_s$, the total energy of the system then reads

$$U_{tot} = U_0 - a^2 \frac{\pi t \sigma_\infty^2}{E} + 4at\gamma_s$$

After differentiating with respect to $a$, we obtain the rate of change of the total energy as a function of the crack length and the applied stress

$$\frac{dU_{tot}}{da} = -2a \frac{\pi t \sigma_\infty^2}{E} + 4t\gamma_s$$
Griffith theory

\[
\frac{dU_{\text{tot}}}{da} = -2a \frac{\pi t \sigma^2}{E} + 4t \gamma_s
\]

Note that for small flaws and for low applied stresses, the surface energy term dominates, i.e. the total energy of the system would increase as the crack advances. However, when the critical condition \( dU_{\text{tot}}/da = 0 \) is met, the change in total energy becomes negative (energy release).

According to Griffith, this condition defines the onset of unstable crack growth. The critical far field stress for fracture initiation therefore reads

\[
-2a \frac{\pi t \sigma^2}{E} + 4t \gamma_s = 0 \quad \Rightarrow \quad \sigma_c = \sqrt{\frac{2\gamma_s E}{\pi a}}
\]
Linear Elastic Fracture Mechanics (LEFM)

The stress fields in the vicinity of cracks in elastic media can be calculated using linear elastic stress analysis. In particular, the analytical solutions have been developed for three **basic modes of fracture**:

- **Mode I**  
  “opening mode”

- **Mode II**  
  “in-plane shear mode”

- **Mode III**  
  “out-of-plane shear mode”
Linear elastic fracture mechanics

The leading terms of analytical solutions for the crack tip stress fields are typically expressed through the product of a radial and circumferential term. For example, the stress field for Mode I plane stress loading reads

\[
\sigma_x = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)
\]

\[
\sigma_y = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right)
\]

\[
\tau_{xy} = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2}
\]

http://www.fracturemechanics.org
Linear elastic fracture mechanics

\[
\sigma_y = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right)
\]

Note that the governing terms all exhibit a singularity at the crack tip,

\[
\lim_{r \to 0} |\sigma_{ij}| = \infty
\]
Stress Intensity Factor

The limit

\[ K_I := \lim_{r \to 0} \left\{ \sqrt{2\pi r} \sigma_y \right\}_{\theta=0} \]

is called stress intensity factor. It characterizes the magnitude of the stresses at the crack tip. An analog factor can be defined for Mode II and Mode III fracture. In the above example, we have

\[ \sigma_y[\theta = 0] = \frac{\sigma_\infty \sqrt{\pi a}}{\sqrt{2\pi r}} \quad \text{and thus} \quad K_I = \sigma_\infty \sqrt{\pi a} \]

The expressions of the stress intensity factor for different crack shapes and loading conditions can be found in many textbooks.
Fracture toughness

In linear elastic fracture mechanics, it is assumed that a crack propagates if the stress intensity factor reaches a critical value,

\[ K_I \geq K_c \]

The critical value \( K_C \) for Mode I fracture under plane strain conditions is called fracture toughness. Its units are \( MPa\sqrt{m} \).

It is considered as a material property which is measured using Single Edge Notch Bend (SENB) or Compact Tension (CT) specimens, see ASTM E-399 standard.
K-dominance

The stress intensity factor characterizes only the leading term of the stress field near the crack tip. The exact solution for the near-tip fields includes also non-singular higher order terms

\[ \sigma_{ij}[\theta, r] = \frac{K}{\sqrt{2\pi r}} f_{ij}[\theta] + \text{higher order terms} \]

The annular region within which the singular terms (so-called K-fields) dominate is described by the radius \( r_K \) of the zone of K-dominance.
Small scale yielding condition

The K-fields can still be meaningful even if the “real” mechanical system is different from that assumed in the theoretical analysis. Examples include situations where

• the crack is not sharp;
• the material deforms plastically
• micro-cracks are present near the crack tip

The condition of applicability of linear elastic fracture mechanics is that the radius $r_p$ of the zone of inelastic deformation at the crack tip must be well confined inside the region of K-dominance.

$$r_p << r_K$$
An alternative to fracture mechanics

The **classical fracture mechanics approach** is based on the assumption that failure is the outcome of the growth of a pre-existing crack (and that all materials contain flaws).

An alternative approach consists of **assuming that a solid is initially crack-free**. Phenomenological fracture criteria in terms of macroscopic stresses and strains are then often employed to predict the onset of fracture.

A simple example of a **phenomenological criterion for brittle solids** is to assume that fracture initiates when the maximum principal stress exceeds a critical value,

\[ \sigma_I \geq \sigma_{\text{crit}} \]
Ductile Fracture
Fracture in Automotive Applications

- **Shear induced fracture** *(Courtesy of ThyssenKrupp)*

- **Bending under tension** *(Courtesy of US Steel)*

- Fractures on tight radii during stamping cannot be predicted by Forming Limit Diagram (FLD)

- Usually termed as **shear fracture**, presents little necking, shows slant fracture
Interrupted tension experiments

- **Flat** notched tensile specimens (1.4mm initial thickness)
Dynamic behavior of materials and structures

50μm
Dynamic behavior of materials and structures

[Graph showing force vs. displacement]

- Fractured
- 98%
- 95%
- Maximum force

[Micrograph with 50μm scale]

RD

Th
Signature of voids on fracture surface!

... BUT: Almost no voids just below fracture surface!
Define “strain to fracture” as strain at the onset of shear localization.

1. Onset of necking
2. Void volume fraction increases (more nucleation)
3. Shear localization
4. Void sheet failure
A Think Model of the Ductile Fracture Process

1. Initial porosity
2. Growth & nucleation
3. Primary localization
4. Growth & nucleation
5. Secondary localization
6. Nucleation & growth
7. Final fracture
Definition of “strain to fracture”

1. Initial porosity
2. Growth & nucleation
3. Primary localization

Strain to fracture = macroscopic equiv. plastic strain at instant of first localization

RVE
Tomographic observations

T. F. Morgeneyer, L. Helfen, I. Sinclair, H. Proudhon, F. Xu, T. Baumbach
Scripta Materialia, 2011
Void evolution in a plastic solid

.... void evolution depends on stress state
Decomposition of the Stress Tensor

\[ \sigma_m = \frac{\sigma_I + \sigma_{II} + \sigma_{III}}{3} \]

HYDROSTATIC PART
(average stress)

DEVIATORIC PART
(differences among stresses)

\[ (\sigma_{III} - \sigma_m) \]
\[ (\sigma_I - \sigma_m) \]
\[ (\sigma_{II} - \sigma_m) \]
Effect of Stress State on Void Evolution

HYDROSTATIC PART

... controls void growth

\[ \sigma_m \]

DEVIATORIC PART

... controls shape change

\[ \eta = \frac{\sigma_m}{\bar{\sigma}} \]

STRESS TRIAXIALITY

LODE PARAMETER
Definition of Lode Parameter

- Maximum shear stress (radius of biggest circle):
  \[ \tau = \frac{\sigma_1 - \sigma_{III}}{2} \]

- Normal stress on plane of max. shear (center of biggest circle)
  \[ \sigma_N = \frac{\sigma_1 + \sigma_{III}}{2} \]

- Position of the intermediate principal stress:
  \[ L = \frac{\sigma_{II} - \sigma_N}{\tau} \]  LODE PARAMETER
Which states have the same Lode parameter?

- **Uniaxial tension**
  \[ \sigma_m = \sigma / 3 \]
  \[ \bar{\sigma} = \sigma \]  \[ \Rightarrow \eta = 1/3 \]
  \[ \sigma_{II} = \sigma_{III} = 0 \]  \[ \Rightarrow L = -1 \]

- **Plane strain tension**
  \[ \sigma_m = \sigma / 2 \]
  \[ \bar{\sigma} = \sqrt{3} / 2 \sigma \]  \[ \Rightarrow \eta = 1/\sqrt{3} \]
  \[ \sigma_{II} = (\sigma_I + \sigma_{III})/2 \]  \[ \Rightarrow L = 0 \]

- **Pure shear**
  \[ \sigma_m = 0 \]
  \[ \bar{\sigma} = \sqrt{3} / 2 \sigma \]  \[ \Rightarrow \eta = 0 \]
  \[ \sigma_{II} = (\sigma_I + \sigma_{III})/2 \]  \[ \Rightarrow L = 0 \]
Lode angle parameter

- Stress triaxiality:
  \[ \eta = \frac{\sigma_m}{\bar{\sigma}} \]

- Normalized third stress invariant
  \[ \xi = \frac{27}{2} \frac{J_3}{\bar{\sigma}^3} \]

- Lode angle parameter
  \[ \bar{\theta} = 1 - \frac{2}{\pi} \arccos(\xi) \]
Lode angle parameter

- Lode parameter (Lode, 1926)
  \[ L = \frac{2\sigma_{II} - \sigma_I - \sigma_{III}}{\sigma_I - \sigma_{III}} \]

- Lode angle parameter
  \[ \bar{\theta} = 1 - \frac{2}{\pi} \arccos(\xi) \approx -L \]

- Diagram showing the relationship between \( \bar{\theta} \) and the types of loading:
  - \( \bar{\theta} = 1 \) for axisymmetric tension
  - \( \bar{\theta} = 0 \) for generalized shear
  - \( \bar{\theta} = -1 \) for axisymmetric compression
Plane stress states

For isotropic materials, the stress tensor is fully characterized by three stress tensor invariants,

\[ \{I_1, J_2, J_3\} \quad \text{or} \quad \{\sigma_I, \sigma_{II}, \sigma_{III}\} \]

while the stress state is characterized by the two dimensionless ratios of the invariants, e.g.

\[ \{I_1 / \sqrt{J_2}, \ J_3 / J_2^{3/2}\} \quad \text{or} \quad \{\eta, \bar{\theta}\} \quad \text{or} \quad \{\sigma_{II} / \sigma_I, \ \sigma_{III} / \sigma_I\} \]

with

\[ \eta = \frac{I_1}{3\sqrt{3J_2}} \quad \text{and} \quad \bar{\theta} = 1 - \frac{2}{\pi} \arccos \left[ \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right] \]
Under plane stress conditions, one principal stress is zero. The stress state may thus be characterized by the ratio of the two non-zero principal stresses.

As a result, the stress triaxiality and the Lode angle parameter are no longer independent for plane stress, i.e. we have a functional relationship

$$\bar{\theta} = \bar{\theta}[\eta]$$
Results from Localization Analysis

Unit Cell with Central Void

Stresses on Plane of Localization

- Failure strain vs. Lode angle parameter
- Shear stress vs. Normal stress
- Linear Mohr-Coulomb approximation

\( \eta = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 1 \)
Hosford-Coulomb Ductile Fracture Model

Principal stress space \( \{\sigma_1, \sigma_2, \sigma_3\} \)

Haigh-Westergaard space \( \{\eta, \theta, \sigma\} \)

Mixed strain-stress space \( \{\eta, \theta, \epsilon_p\} \)

Hosford-Coulomb

Coordinate transformation

\[
\bar{\sigma}_{Hf} + c(\sigma_I + \sigma_{III}) = b
\]

\[
\bar{\sigma}_f = \bar{\sigma}_f[\theta, \eta]
\]

Isotropic hardening law

\[
\bar{\sigma} = k[\bar{\epsilon}_p]
\]

\[
\bar{\epsilon}_f = k^{-1}[\bar{\sigma}_f[\eta, \theta]]
\]
**Hosford-Coulomb Ductile Fracture Model**

**General form**

von Mises equivalent plastic strain to fracture

\[ \bar{\varepsilon}_f = \bar{\varepsilon}_f [\eta, \bar{\theta}, a, b, c] \]

Stress triaxiality  Lode angle parameter  3 material parameters

**Detailed expressions**

\[ \bar{\varepsilon}_f = b \left( \frac{1 + c}{g_{HC}[\eta, \bar{\theta}]} \right)^{\frac{1}{n}} \]

\[ g_{HC} = \left( \frac{1}{2} | f_I - f_{II} |^a + \frac{1}{2} | f_{II} - f_{III} |^a + \frac{1}{2} | f_I - f_{III} |^a \right)^{\frac{1}{a}} + c(2\eta + f_I + f_{III}) \]

\[ f_I[\bar{\theta}] = \frac{2}{3} \cos \left( \frac{\pi}{6} (1 - \bar{\theta}) \right) \]

\[ f_{II}[\bar{\theta}] = \frac{2}{3} \cos \left( \frac{\pi}{6} (3 + \bar{\theta}) \right) \]

\[ f_{III}[\bar{\theta}] = -\frac{2}{3} \cos \left( \frac{\pi}{6} (1 + \bar{\theta}) \right) \]
• Influence of parameter $b$

$b = \text{strain to fracture for uniaxial tension (or equi-biaxial tension)}$
Hosford-Coulomb Ductile Fracture Model

- Influence of parameter $a$

Can easily adjust the depth of the “plane strain valley”

Compare: Mohr-Coulomb

Influence of parameter $a$
Hosford-Coulomb Ductile Fracture Model

- Influence of parameter $c$

![Graph showing the influence of parameter $c$ on equivalent plastic strain with respect to stress triaxiality. The graph includes multiple curves for different values of $c$: $c=0$, $c=0.05$, $c=0.1$, and $c=0.2$. The parameter $a$ is set to 1.3 and $n$ to 0.1.]
Hosford-Coulomb Ductile Fracture Model

Rapid calibration guideline

**Step I:** Identify \( b \)

\[ a = 2, \quad c = 0 \]

**Step II:** Identify \( a \)

\[ b = 0.4, \quad c = 0 \]

**Step III:** Identify \( c \)

\[ b = 0.4, \quad a = 1.15 \]

Hosford-Coulomb Ductile Fracture Model
Hosford-Coulomb Ductile Fracture Model

Excel program for calibration

**INPUT**

<table>
<thead>
<tr>
<th>loading</th>
<th>EPS_F</th>
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</thead>
<tbody>
<tr>
<td>equi-biaxial tension</td>
<td>0.76</td>
</tr>
<tr>
<td>plane strain tension</td>
<td>0.34</td>
</tr>
<tr>
<td>pure shear</td>
<td>0.67</td>
</tr>
</tbody>
</table>

**OUTPUT**

<table>
<thead>
<tr>
<th>EMC Parameters</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.257</td>
<td>0.76</td>
<td>0.07</td>
</tr>
</tbody>
</table>

**USER ADJUSTED**

<table>
<thead>
<tr>
<th>EMC Parameters</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Hosford-Coulomb Ductile Fracture Model

"heart" of the model:

$$
\bar{\varepsilon}_f = \bar{\varepsilon}_f[\eta, \bar{\theta}]
$$
Damage Accumulation

Define “damage indicator”

\[ D = \int \frac{d\bar{\varepsilon}_p}{\bar{\varepsilon}_f[\eta, \bar{\theta}]} \]

\( D = 0 \) (initial)
\( D = 1 \) (fracture)

• Example: uniaxial tension

![Graphs showing damage accumulation vs. stress triaxiality and equivalent plastic strain]
Damage Accumulation

Define “damage indicator”

\[
\bar{e}_f = \bar{e}_f [\eta, \bar{\Theta}]
\]

\[
D = \int \frac{d\bar{e}_p}{\bar{e}_f [\eta, \bar{\Theta}]}
\]

\[
D = 0 \quad \text{(initial)}
\]

\[
D = 1 \quad \text{(fracture)}
\]

• Example: uniaxial compression followed by tension

VIDEO
Damage Accumulation

Define “damage indicator”

\[
\bar{e}_f = \bar{e}_f [\eta, \bar{\theta}]
\]

\[
D = \int \frac{d\bar{e}_p}{\bar{e}_f [\eta, \bar{\theta}]}
\]

\[D = 0 \quad \text{(initial)}\]
\[D = 1 \quad \text{(fracture)}\]

- **Example:** uniaxial compression followed by tension

\[\int_{0}^{1} \]

\[\int_{0}^{T} \]

Non-linear loading path effect!
Reading Materials for Lecture #7