Problem Solutions

Vehicle Energy and Fuel Consumption

Vehicle Energy Losses and Performance Analysis

Problem 2.1

For a vehicle with \( m_v = 1500 \text{ kg} \), \( A_f \cdot c_d = 0.7 \text{ m}^2 \), \( c_r = 0.012 \), a vehicle speed \( v = 120 \text{ km/h} \) and an acceleration \( a = 0.027 \text{ g} \), calculate the traction torque required at the wheels and the corresponding rotational speed level (tires 195/65/15T). Calculate the road slope that is equivalent to that acceleration.

• Solution

\[
F_t = m \cdot c_r \cdot g + \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot a = \\
= 1500 \cdot 0.012 \cdot 9.81 + \frac{1}{2} \cdot 0.7 \cdot \left( \frac{120}{3.6} \right)^2 + 1500 \cdot 0.027 \cdot 9.81 = 1041 \text{ N}
\]

The information about the tires is explained by

<table>
<thead>
<tr>
<th>195</th>
<th>/</th>
<th>65</th>
<th>/</th>
<th>15</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>width of the tire in cm</td>
<td>ratio of sidewall height to tire width in %</td>
<td>wheel diameter in inch</td>
<td>max. 190 km/h</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus
\[ r_w = \frac{15^\circ}{2} + 0.65 \cdot 0.195 = 15 \cdot \frac{0.0254}{2} + 0.65 \cdot 0.195 = 0.317 \text{ m} \]

\[ T_t = r_w \cdot F_t = 0.317 \cdot 1041 = 330 \text{ Nm} \]

\[ \omega_w = \frac{v}{r_w} = \frac{120}{3.6} \cdot 0.317 = 10.57 \text{ rad/s} = 101 \text{ rpm} \]

\[ \alpha = \frac{a}{g} = 0.027 \text{ rad} \]

\[ \alpha_{\%} = 100 \cdot \tan(0.027) = 100 \cdot 0.027 = 2.7 \% \]

For the requested velocity, a traction torque of 330 Nm at a rotational speed of 101 rpm is required. This is equivalent to the acceleration caused by a slope of 2.7%.

**Problem 2.2**

Find the road slope \( \alpha \) that is equivalent to a step of height \( h \) for a car with (a rigid) wheel radius \( r_w \) on a flat terrain. Calculate the result for \( h/2r_w = \{0.01, 0.02, 0.05, 0.1, 0.2\} \).

- **Solution**

  When in contact with the step, the car wheel will rotate around the contact point. Thus the reaction force \( R \) will be directed from the contact point to the wheel center (neglect slip here) and the reaction force at the contact point on the terrain becomes null. \( R \) is balanced by the weight \( N \) and the traction force \( F_t \) that has to be calculated. One can write two equations

\[ F_t = R_x \]
\[ N = R_y \]

where \( R_x \) and \( R_y \) are the two components of \( R \), with \( R_x = R_y \cdot \tan(\alpha) \).

But \( \cos(\alpha) = (r_w-h)/r_w = 1-h/r_w \) (see figure). Thus \( \sin(\alpha) = \sqrt{1 - (1-h/r_w)^2} \)

and finally

\[ F_t = N \cdot \tan(\alpha) = N \cdot \frac{\sin(\alpha)}{\cos(\alpha)} = N \cdot \sqrt{1 - \left(1 - \frac{h}{r_w}\right)^2} \cdot \frac{r_w}{r_w-h} \]

One could obtain the same result with a balance of torques acting around the contact point:

\[ T_t = F_t \cdot (r_w - h) = N \cdot a \]

where \( a^2 + (r_w-h)^2 = r_w^2 \).

The equivalent slope is calculated by equating

\[ m_v \cdot g \cdot \sin(\alpha) \cdot r_w = T_t, \]
from whence
\[
\alpha = \arcsin \left( N \cdot \frac{a}{r_w \cdot m_w \cdot g} \right)
\]
In a first approximation, \( N = m_w \cdot g / 2 \) (for center of gravity centered along the wheelbase) and thus
\[
\alpha = \arcsin \left( \frac{a}{2 \cdot r_w} \right) = \arcsin \left( \sqrt{z \cdot (1 - z)} \right), \quad \text{where} \quad z = \frac{h}{2 \cdot r_w} \quad (z \leq \frac{1}{2})
\]
The exact result for rigid wheels would be slightly different,
\[
\alpha = \arcsin \left( a \cdot \frac{b}{2 \cdot (a + b) \cdot r_w} \right)
\]
that tends to the former value if \( a \ll b \), where \( b \) is the wheelbase. For flexible wheel, the result is lower and decreases with the decrease of the wheel stiffness.

<table>
<thead>
<tr>
<th>( z [m] )</th>
<th>( \alpha [\text{rad}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.41 (44%)</td>
</tr>
<tr>
<td>0.10</td>
<td>0.30 (31%) (exact solution for ( b/r_w = 6.67 ): 0.27 rad)</td>
</tr>
<tr>
<td>0.05</td>
<td>0.22 (22%)</td>
</tr>
<tr>
<td>0.02</td>
<td>0.14 (14%)</td>
</tr>
<tr>
<td>0.01</td>
<td>0.10 (10%)</td>
</tr>
</tbody>
</table>

**Problem 2.3**

Find an equation to evaluate the speed profile of an ICE vehicle under maximum engine torque. Assume a maximum torque curve of the type
\[
T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c.
\]
Include the engine inertia and a traction efficiency \( \eta_t \).

- **Solution**
\[
\dot{v} = \left( \frac{\gamma}{r_w} \cdot \eta_t \cdot T_e - m_w \cdot g \cdot c_r - \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 \right) \cdot \frac{1}{m_{eq}}
\]
where
\[ m_{eq} = m_v + \Theta_e \cdot \left( \frac{\gamma}{r_w} \right)^2 \]
\[ T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c \]

where \( \omega_e = (\gamma/r_w) \cdot v \) (if the clutch is closed). Thus the law of motion is of the type \( \dot{v} = A \cdot v^2 + B \cdot v + C \), where

\[ A = \left( -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \left( \frac{\gamma}{r_w} \right)^3 \cdot \eta_r \cdot a \right) \cdot \frac{1}{m_{eq}} \]
\[ B = \left( \frac{\gamma}{r_w} \right)^2 \cdot \eta_r \cdot b \cdot \frac{1}{m_{eq}} \]
\[ C = \left( -g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_r \cdot c \right) \cdot \frac{1}{m_{eq}} \]

To solve the differential equation an integral of the type

\[ t - t_0 = \int_{t_0}^{t} d\tau = \int_{v_0}^{v(t)} \frac{d\nu}{A \cdot \nu^2 + B \cdot \nu + C} \]

must be generally solved between \( v_0 = v(t_0) \) and \( v(t) \). As it is to find in any integral table, the solution to this integral depends on the sign of \( D = B^2 - 4 \cdot A \cdot C \):

If \( D > 0 \),
\[ v(t) = \frac{S \cdot (1 + K(t)) + B \cdot (K(t) - 1)}{2 \cdot A \cdot (1 - K(t))} \]
where
\[ K(t) = \frac{2 \cdot A \cdot v_0 + B - S}{2 \cdot A \cdot v_0 + B + S} \cdot e^{S \cdot (t-t_0)} = K_0 \cdot e^{S \cdot (t-t_0)} \]
and
\[ S = \sqrt{D} \]

If \( D < 0 \),
\[ v(t) = \frac{S \cdot \tan \left( \arctan \left( \frac{2 \cdot A \cdot v_0 + B}{S} \right) + \frac{S \cdot (t-t_0)}{2} \right) - B}{2 \cdot A} \]
where
\[ S = \sqrt{-D} \]

If \( B = 0 \) and \( C < 0 \) (coasting), then \( D < 0 \) and one finds back (2.18) for the coasting velocity. Note that \( \gamma/r_w \) generally varies along the acceleration, so a solution must be obtained step by step.
**Problem 2.4**

Evaluate the 0-100 km/h time precisely using the result of Problem 2.3. Use the following data: engine launch speed = 2500 rpm, engine upshift speed $\omega_{e,\text{max}} = 6500$ rpm, $\gamma/r_w = \{46.48, 29.13, 20.39, 15.04, 11.39\}$, $a = -4.38 \cdot 10^{-4}$ Nm s$^2$, $b = 0.3514$ Nm s, $c = 80$ Nm, and the following vehicle data: $m_v = 1240$ kg, $A_f \cdot c_d = 0.65$ m$^2$, $c_r = 0.009$, $\eta_t = 0.9$. Further assume that a slipping clutch transmits all the torque.

- **Solution**

Since $A$, $B$, and $C$ depend on $\gamma$, which changes along the speed trajectory, the calculation of $v(t)$ must be separated in segments according to the gear and the clutch status. The target speed will be reached in the gear whose $\gamma/r_w$ is immediately lower than $\omega_{e,\text{max}} = 6500 \cdot 2 \cdot \pi/60 = 24.5$ km/h.

Thus the target speed is reached in the third gear. Four segments (including takeoff) must be considered.

1st segment: takeoff, $v_0 = 0$, $t_0 = 0$

The result of Problem 2.6 is used, with $\omega_e = 2500$ rpm, and

\[
T_e = a \cdot \omega_e^2 + b \cdot \omega_e + c = 142 \text{ Nm}
\]

\[
A = -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot \frac{1}{m_v} = -\frac{1}{2} \cdot \frac{1.2 \cdot 0.65}{1240} = -3.15 \cdot 10^{-4}
\]

$B = 0$

\[
C = \left( -g \cdot m_v \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_t \cdot T_e \right) \cdot \frac{1}{m_v} = -9.81 \cdot 1240 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 142
\]

\[
S = \sqrt{-4 \cdot A \cdot C^2} = \sqrt{4 \cdot 3.15 \cdot 10^{-4} \cdot 4.70} = 0.077
\]

The synchronization time between the engine and the vehicle (clutch closed) is when

\[
v = \frac{2500 \cdot \pi \cdot r_w}{30 \cdot \gamma} = 5.63 \text{ m/s} = 20.275 \text{ km/h},
\]

thus when

\[
K = \frac{2 \cdot A \cdot v - S}{2 \cdot A \cdot v + S} = \frac{-2 \cdot 3.15 \cdot 10^{-4} \cdot 5.36 - 0.077}{-2 \cdot 3.15 \cdot 10^{-4} \cdot 5.36 + 0.077} = -1.09
\]

The synchronize time is thus

\[
t = \frac{\ln(1.09)}{0.077} = 1.14 \text{ s}
\]
Here $\gamma/r_w = 46.48$, thus

$$m_{eq} = m_w + \Theta_k \cdot \left( \frac{\gamma}{r_w} \right)^2 = 1240 + 0.128 \cdot 46.48^2 = 1517 \text{ kg}$$

$$A = \left( -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d + \frac{\gamma}{r_w} \gamma \cdot a \right) \cdot \frac{1}{m_{eq}} =$$

$$= \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 46.48^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \cdot \frac{1}{1517} = 0.0264$$

$$B = \left( \frac{\gamma}{r_w} \right)^2 \cdot \eta e \cdot b \cdot \frac{1}{m_{eq}} = 46.48^2 \cdot 0.9 \cdot 0.3514 \cdot \frac{1}{1517} = 0.45$$

$$C = \left( -g \cdot m_w \cdot c_r + \frac{\gamma}{r_w} \gamma \cdot c \right) \cdot \frac{1}{m_{eq}} =$$

$$= -9.81 \cdot 1270 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 80 \cdot \frac{1}{1517} = 2.13$$

$$D = B^2 - 4 \cdot A \cdot C = 0.45^2 + 4 \cdot 0.0264 \cdot 2.13 = 0.428$$

$$S = \sqrt{D} = 0.654$$

$$K_0 = \frac{2 \cdot A \cdot v_0 + B - S}{2 \cdot A \cdot v_0 + B + S} = -\frac{2 \cdot 0.0264 \cdot 5.36 + 0.45 - 0.654}{2 \cdot 0.0264 \cdot 5.36 + 0.45 + 0.654} = -0.5932$$

This phase ends when $\omega_e = \omega_{e,max}$, thus when

$$v = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 46.48} = 14.64 \text{ m/s}$$

and

$$K = \frac{2 \cdot A \cdot v + B - S}{2 \cdot A \cdot v + B + S} = -\frac{2 \cdot 0.0264 \cdot 14.64 + 0.45 - 0.654}{2 \cdot 0.0264 \cdot 14.64 + 0.45 + 0.654} = -2.9516$$

thus at time

$$t = t_0 + \ln \left( \frac{K}{K_0} \right) \cdot \frac{1}{S} = 1.14 + \ln \left( \frac{2.9516}{0.5932} \right) \cdot \frac{1}{0.654} = 3.6 \text{ s}$$

Here $\gamma/r_w = 29.13$
\[ m_{eq} = m_{e} + \Theta_{e} \cdot \left( \frac{\gamma}{r_{w}} \right)^2 = 1240 + 0.128 \cdot 29.13^2 = 1349 \text{ kg} \]

\[ A = \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 29.13^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \frac{1}{1349} = -0.0075 \]

\[ B = \frac{29.13^2 \cdot 0.9 \cdot 0.3514}{1349} = 0.199 \]

\[ C = \frac{-9.81 \cdot 1270 \cdot 0.009 + 29.13 \cdot 0.9 \cdot 80}{1349} = 1.48 \]

\[ D = 0.199^2 + 4 \cdot 0.0075 \cdot 1.48 = 0.084 \]

\[ S = 0.29 \]

\[ K_0 = \frac{-2 \cdot 0.0075 \cdot 14.64 + 0.199 - 0.290}{-2 \cdot 0.0075 \cdot 14.64 + 0.199 + 0.290} = -1.15 \]

This phase ends when \( \omega_e = \omega_{e,\text{max}} \), thus when \( v = 6500 \cdot 2 \cdot \pi \cdot 29.13 = 23.37 \text{ m/s} = 84.1 \text{ km/h} \)

and \[ K = \frac{-2 \cdot 0.0075 \cdot 23.37 + 0.199 - 0.290}{-2 \cdot 0.0075 \cdot 23.37 + 0.199 + 0.290} = -3.1892 \]

thus at time \[ t = 3.6 + \frac{\ln \left( \frac{3.1892}{0.29} \right)}{0.29} = 7.1 \text{ s} \]

4th segment, 3rd gear, \( v_0 = 23.37 \text{ m/s}, t_0 = 7.1 \text{ s} \)

Here \( \gamma/r_w = 20.39 \)

\[ m_{eq} = m_{e} + \Theta_{e} \cdot \left( \frac{\gamma}{r_{w}} \right)^2 = 1240 + 0.128 \cdot 20.39^2 = 1293 \text{ kg} \]

\[ A = \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 20.39^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \frac{1}{1293} = -0.0029 \]

\[ B = \frac{20.39^2 \cdot 0.9 \cdot 0.3514}{1293} = 0.1017 \]

\[ C = \frac{-9.81 \cdot 1270 \cdot 0.009 + 20.39 \cdot 0.9 \cdot 80}{1293} = 1.05 \]

\[ D = 0.102^2 + 4 \cdot 0.0029 \cdot 1.05 = 0.225 \]

\[ S = 0.15 \]

\[ K_0 = \frac{-2 \cdot 0.0029 \cdot 23.37 + 0.102 - 0.15}{-2 \cdot 0.0029 \cdot 23.37 + 0.102 + 0.15} = -1.58 \]

We want to calculate the time to reach \( v = 100 \text{ km/h} = 27.8 \text{ m/s} \) that will take place in this segment. Thus \[ K = \frac{-2 \cdot 0.0029 \cdot 27.8 + 0.102 - 0.15}{-2 \cdot 0.0029 \cdot 27.8 + 0.102 + 0.15} = -2.305 \]
thus at time
\[ t = 7.1 + \frac{\ln \left( \frac{2.305}{0.15} \right)}{0.15} = 9.6 \text{ s} \]

**Problem 2.5**

Consider again Problem 2.4 for an engine with negligible inertia \( \Theta_e \). Compare the result with (2.16).

- Solution

1\(^{st}\) segment \( v_0 = 0, t_0 = 0 \)

The same result as in Problem 2.4

2\(^{nd}\) segment, 1\(^{st}\) gear \( v_0 = 5.36 \text{ m/s}, t_0 = 1.14 \text{ s} \)

Here \( \gamma/r_w = 46.48 \), thus

\[
A = \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 46.48^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \cdot \frac{1}{1240} = -0.0322
\]

\[
B = \frac{46.48^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.551
\]

\[
C = \frac{-9.81 \cdot 1270 \cdot 0.009 + 46.48 \cdot 0.9 \cdot 80}{1240} = 2.608
\]

\[
D = 0.551^2 + 4 \cdot 0.0322 \cdot 2.608 = 0.6395
\]

\( S = 0.8 \)

\[
K_0 = \frac{-2 \cdot 0.0322 \cdot 5.36 + 0.551 - 0.8}{-2 \cdot 0.0322 \cdot 5.36 + 0.551 + 0.8} = -0.591
\]

This phase ends when \( \omega_e = \omega_{e,max} \), thus when
\[
v = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 46.48} = 14.64 \text{ m/s}
\]

and

\[
K = \frac{-2 \cdot 0.0322 \cdot 14.64 + 0.551 - 0.8}{-2 \cdot 0.0322 \cdot 14.64 + 0.551 + 0.8} = -2.92
\]

thus at time
\[
t = 1.14 + \frac{\ln \left( \frac{2.92}{0.8} \right)}{0.8} = 3.14 \text{ s}
\]

3\(^{rd}\) segment, 2\(^{nd}\) gear, \( v_0 = 14.64 \text{ m/s}, t_0 = 3.14 \text{ s} \)

Here \( \gamma/r_w = 29.13 \)
\[ A = \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 29.13^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \cdot \frac{1}{1240} = -0.0082 \]

\[ B = \frac{29.13^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.216 \]

\[ C = \frac{-9.81 \cdot 1270 \cdot 0.009 + 29.13 \cdot 0.9 \cdot 80}{1240} = 1.601 \]

\[ D = 0.216^2 + 4 \cdot 0.0082 \cdot 1.601 = 0.099 \]

\[ S = 0.315 \]

\[ K_0 = -2 \cdot 0.0082 \cdot 14.64 + 0.216 - 0.315 = -1.166 \]

This phase ends when \( \omega_e = \omega_{e,\text{max}} \), thus when

\[ v = \frac{6500 \cdot 2 \cdot \pi}{60 \cdot 29.13} = 23.37 \text{ m/s} = 84.1 \text{ km/h} \]

and

\[ K = \frac{-2 \cdot 0.0082 \cdot 23.37 + 0.216 - 0.315}{2 \cdot 0.0082 \cdot 23.37 + 0.216 + 0.315} = -3.265 \]

Thus at time

\[ t = 3.14 + \frac{\ln \left( \frac{4.205}{0.315} \right)}{0.315} = 6.4 \text{ s} \]

4th segment, 3rd gear, \( v_0 = 23.37 \text{ m/s}, t_0 = 6.4 \text{ s} \)

Here \( \gamma/r_w = 20.39 \)

\[ A = \left( -\frac{1}{2} \cdot 1.2 \cdot 0.65 - 20.39^3 \cdot 0.9 \cdot 4.38 \cdot 10^{-4} \right) \cdot \frac{1}{1240} = -0.003 \]

\[ B = \frac{20.39^2 \cdot 0.9 \cdot 0.3514}{1240} = 0.106 \]

\[ C = \frac{-9.81 \cdot 1270 \cdot 0.009 + 20.39 \cdot 0.9 \cdot 80}{1240} = 1.09 \]

\[ D = 0.106^2 + 4 \cdot 0.003 \cdot 1.09 = 0.243 \]

\[ S = 0.156 \]

\[ K_0 = \frac{-2 \cdot 0.003 \cdot 23.37 + 0.106 - 0.156}{-2 \cdot 0.003 \cdot 23.37 + 0.106 + 0.156} = -1.562 \]

We want to calculate the time to reach \( v = 100 \text{ km/h} = 27.8 \text{ m/s} \) that will take place reasonably in this segment. Thus

\[ K = \frac{-2 \cdot 0.003 \cdot 27.8 + 0.106 - 0.156}{-2 \cdot 0.003 \cdot 27.8 + 0.106 + 0.156} = -2.277 \]

Thus at time

\[ t = 6.4 + \frac{\ln \left( \frac{2.277}{0.156} \right)}{0.156} = 8.82 \text{ s} \]
Comparison

Equation (2.16) gives

\[ t = \frac{v^2}{P_{e,max}} = \frac{27.8^2}{80.4 \cdot 10^3} = 11.9 \text{ s}, \]

which means a substantial overestimation. In reality, the relation \( \bar{P} = P_{\text{max}}/2 \)
should be corrected in this case as

\[ \bar{P} = \frac{P_{e,max} + P_{e,min}}{2} \]

where

\[ P_{e,min} = \omega_{\text{e,launch}} \cdot T_{e,\text{launch}} = 2500 \cdot \frac{\pi}{30} \cdot 142 = 37.2 \text{ kW} \]

With this formula,

\[ \bar{P} = 58.8 \text{ kW} \quad \text{and} \quad t = 8.1 \text{ s} \]

which is an underestimation of 9%.

Problem 2.6

Find an equation to calculate the takeoff time (=time to synchronise the speed before and after the clutch) as a function of engine launch speed and torque. Assume that the clutch is slipping but transmitting the whole engine torque.

- Solution

The motion law of the vehicle is

\[ \dot{v} = \left( \frac{\gamma}{r_w} \cdot \eta_T \cdot T_e - m_w \cdot g \cdot c_r - \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 \right) \cdot \frac{1}{m_{eq}} \]

Thus the law of motion is of the type \( \dot{v} = A \cdot v^2 + C \), where

\[ A = -\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot \frac{1}{m_{eq}} \quad C = \left( -g \cdot m_w \cdot c_r + \frac{\gamma}{r_w} \cdot \eta_T \cdot T_e \right) \cdot \frac{1}{m_{eq}} \]

The solution can be obtained from the result of Problem 2.3. Now \( A < 0 \) and \( B = 0 \), thus \( D > 0 \). Moreover, \( t_0 = 0 \) and \( v_0 = 0 \). Therefore,

\[ v(t) = \frac{1 + K(t)}{2 \cdot A \cdot (1 - K(t))} \]

where

\[ K(t) = -e^{S \cdot t} \quad \text{and} \quad S = \sqrt{-4 \cdot A \cdot C} \]
The synchronization time is obtained by imposing that \( v = \omega_e \cdot r_w / \gamma \), and then calculating \( K \) as

\[
K = \frac{2 \cdot A \cdot v - S}{2 \cdot A \cdot v + S}
\]

and subsequently

\[
t_{\text{takeoff}} = \frac{\ln(-K)}{S}
\]

**Problem 2.7**

Evaluate the coasting speed and the roll-out time without acting on the brakes for a vehicle with an initial speed \( v_0 = 50 \) km/h and \( m_v = 1200 \) kg, \( A_f \cdot c_d = 0.65 \) m², \( c_r = 0.009 \). Assume the clutch open (no engine friction).

- Solution

The coasting speed as a function of time (2.18 and Problem 2.3) is

\[
v(t) = \frac{\beta}{\alpha} \cdot \tan \left\{ \arctan \left( \frac{\alpha}{\beta} \cdot v_0 \right) - \alpha \cdot \beta \cdot t \right\}
\]

where

\[
\alpha = \sqrt{\frac{1}{2} \cdot \frac{\rho_a \cdot A_f \cdot c_d}{m_v}} = \sqrt{\frac{1.2}{2} \cdot \frac{0.65}{1200}} = 0.0180
\]

\[
\beta = \sqrt{g \cdot c_r} = \sqrt{9.81 \cdot 0.009} = 0.297
\]

The braking time at which \( v = 0 \) is calculated as

\[
t_{\text{rollout}} = \frac{1}{\alpha \cdot \beta} \cdot \arctan \left( \frac{\alpha}{\beta} \cdot v_0 \right) = \frac{1}{0.018 \cdot 0.297} \cdot \arctan \left( \frac{0.018}{0.297} \cdot \frac{50}{3.6} \right) = 130.9 \text{ s}
\]

**Mechanical Energy Demand in Driving Cycles**

**Problem 2.8**

Evaluate the traction energy and the recuperation energy for the MVEG–95 for the vehicle examples of Fig. 2.8, left and right, assuming perfect recuperation.

- Solution
The traction energy is given by (2.31). With the data in Fig. 2.8 left, one obtains
\[
\bar{E} = (0.7 \cdot 1.9 \cdot 10^4 + 1500 \cdot 0.012 \cdot 8.4 \cdot 10^2 + 1500 \cdot 10) = \\
= 43 \text{ MJ/100 km} \cdot x_{tot} = 43 \cdot 0.114 = 4.78 \text{ MJ}.
\]
The total energy is given by (2.35),
\[
\bar{E}_{rec} = (0.7 \cdot 2.2 \cdot 10^4 + 1500 \cdot 0.012 \cdot 9.81 \cdot 10^2) = \\
= 33 \text{ MJ/100 km} \cdot x_{tot} = 3.64 \text{ MJ}
\]
\[
\Delta \bar{E} = \bar{E} - \bar{E}_{rec} = 1.14 \text{ MJ} \quad (24\% \text{ of } \bar{E}).
\]
For the data of Fig. 2.8 right,
\[
\bar{E} = (0.4 \cdot 1.9 \cdot 10^4 + 750 \cdot 0.008 \cdot 8.4 \cdot 10^2 + 750 \cdot 10) = \\
= 20 \text{ MJ/100 km} \cdot x_{tot} = 2.21 \text{ MJ}
\]
\[
\bar{E}_{rec} = (0.4 \cdot 2.2 \cdot 10^4 + 750 \cdot 0.008 \cdot 9.81 \cdot 10^2) = \\
= 15 \text{ MJ/100 km} \cdot x_{tot} = 1.61 \text{ MJ}
\]
\[
\Delta \bar{E} = 2.21 - 1.61 = 0.60 \text{ MJ} \quad (27\% \text{ of } \bar{E})
\]
The potential of regenerative braking is more important for the smaller vehicle here.

**Problem 2.9**

Calculate the mean force and fuel consumption data shown in Fig. 2.8 left.

*Solution*

**Case 1: No recuperation**

\[
\bar{F}_{trac,a} = \frac{1}{2} \rho_a \cdot A_f \cdot c_d \cdot 319 = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 319 = 134 \text{ N}
\]
\[
\bar{F}_{trac,r} = m_v \cdot g \cdot c_r \cdot 0.856 = 1500 \cdot 9.81 \cdot 0.012 \cdot 0.856 = 151 \text{ N}
\]
\[
\bar{F}_{trac,m} = m_v \cdot 0.101 = 1500 \cdot 0.101 = 150 \text{ N}
\]
The mechanical energy per 100 km that corresponds to 1 N is
\[
1 \text{ N} \cdot 10^5 \text{ m} = 10^5 \text{ J} = 10^5/3600 = 27.78 \text{ Wh}
\]
Therefore,
\[
\bar{V}_a = \frac{27.78 \text{ Wh/N} \cdot 100 \text{ km}}{10^5 \text{ Wh/l}} \cdot \bar{F}_{trac,a} = \frac{27.78}{10^4} \cdot 134 = 0.372 \text{ l/100 km}
\]
\[
\bar{V}_r = \frac{27.78}{10^4} \cdot 151 = 0.420 \text{ l/100 km}
\]
\[
\bar{V}_m = 0.417 \text{ l/100 km}
\]
\[
\bar{V} = 0.372 + 0.420 + 0.417 = 1.2 \text{ l/100 km}
\]
Case 2: Perfect recuperation

\[ \bar{F}_a = \frac{1}{2} \rho_a \cdot A_f \cdot c_d \cdot 363 = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 363 = 152 \text{ N} \]

\[ \bar{F}_r = m_v \cdot g \cdot c_r \cdot 1 = 1500 \cdot 9.81 \cdot 0.012 \cdot 1 = 177 \text{ N} \]

\[ \bar{V}_a = \frac{27.78}{10^4} \cdot 152 = 0.422 \text{ l/100 km} \]

\[ \bar{V}_r = \frac{27.78}{10^4} \cdot 177 = 0.492 \text{ l/100 km} \]

\[ \bar{V} = 0.422 + 0.492 = 0.91 \text{ l/100 km} \]

Evaluation of the differences results in

\[ \bar{F}_{m,r} = 152 + 177 - (134 + 151) = 44 \text{ N} \]

\[ \bar{F}_{m,b} = 150 - 44 = 106 \text{ N} \]

Problem 2.10

Calculate the data in Fig. 2.9, left and right.

• Solution

Full-sized vehicle

The cycle energy assuming no recuperation is given by (2.31),

\[ \bar{E} = 0.7 \cdot 1.9 \cdot 10^4 + 1500 \cdot 0.012 \cdot 8.4 \cdot 10^2 + 1500 \cdot 10 = 43 \cdot 10^3 \text{ kJ/100 km} \]

thus

\[ S(A_f \cdot c_d) = \frac{1.9 \cdot 10^4 \cdot 0.7}{43 \cdot 10^3} = 0.31 \]

\[ S(c_r) = \frac{1500 \cdot 8.4 \cdot 10^2 \cdot 0.012}{43 \cdot 10^3} = 0.35 \]

\[ S(m_v) = \frac{0.012 \cdot 8.4 \cdot 10^2 + 10}{43 \cdot 10^3} = 0.70 \]

Light-weight vehicle

\[ \bar{E} = 0.4 \cdot 1.9 \cdot 10^4 + 750 \cdot 0.008 \cdot 8.4 \cdot 10^2 + 750 \cdot 10 = 20 \cdot 10^3 \text{ kJ/100 km} \]

thus

\[ S(A_f \cdot c_d) = \frac{1.9 \cdot 10^4 \cdot 0.4}{20 \cdot 10^3} = 0.38 \]

\[ S(c_r) = \frac{750 \cdot 8.4 \cdot 10^2 \cdot 0.008}{20 \cdot 10^3} = 0.25 \]

\[ S(m_v) = \frac{0.008 \cdot 8.4 \cdot 10^2 + 10}{20 \cdot 10^3} = 0.63 \]
Problem 2.11

Calculate which constant vehicle speed on a flat road is responsible for the same energy demand at the wheels along a MVEG-95 cycle, in the case of no recuperation and of perfect recuperation, respectively. Assume the light-weight vehicle data of Fig. 2.8, left and right: \( \{A_f \cdot c_d, m_v, c_r\} \) = \{0.7 \text{ m}^2, 1500 \text{ kg}, 0.012\}.

- Solution

No recuperation

The cycle energy is calculated in Problem 2.8 and it is \( \bar{E} = 20 \cdot 10^3 \text{ kJ/100 km} \).

The mean traction force is

\[
\bar{F} = \frac{\bar{E}}{100 \text{ km}} = 200 \text{ N}
\]

To have the same \( \bar{F} \), find \( v \) such that

\[
\frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot g \cdot c_r = \bar{F}
\]

\[
v = \sqrt{\frac{2 \cdot \bar{F} - m_v \cdot g \cdot c_r}{\rho_a \cdot A_f \cdot c_d}} = 24.2 \text{ m/s} = 87 \text{ km/h}
\]

Perfect recuperation

Here, \( \bar{E} = 15 \cdot 10^3 \text{ kJ/100 km} \), thus

\[
F = 150 \text{ N}
\]

\[
v = 19.5 \text{ m/s} = 70 \text{ km/h}
\]

The equivalent speed is found by equating

\[
\bar{F}(m_v) = \frac{\bar{E}}{100} = A_f \cdot c_d \cdot 2.2 \cdot 10^2 + m_v \cdot c_r \cdot g \quad \text{(from 2.35)}
\]

\[
F_v(m_v) = \frac{1}{2} \cdot \rho_a \cdot A_f \cdot c_d \cdot v^2 + m_v \cdot g \cdot c_r.
\]

The two second right-hand side terms are equal, thus the equivalent constant speed is

\[
\frac{1}{2} \cdot \rho_a \cdot v^2 = 2.2 \cdot 10^2 \quad \Rightarrow \quad v = \sqrt{\frac{2 \cdot 2.2 \cdot 10^2}{1.2}} = 19.5 \text{ m/s}.
\]

Problem 2.12

Calculate the maximum mass allowed for a recuperation system with \( \eta_{rec} = 40\% \). Use the vehicle parameters of Fig. 2.12: \( \{A_f \cdot c_d, m_v, c_r\} \) = \{0.7 \text{ m}^2, 1500 \text{ kg}, 0.012\}. 
Solution

The maximum weight is the one that leads to an energy demand equal to the energy demand without recuperation. Thus, from (2.41)

\[
\bar{E}(\eta_{rec}, m_{rec}) = 0.7 \cdot (2.2 \cdot 10^4 - 0.6 \cdot 3 \cdot 10^3) + 0.012 \cdot \bar{m}_v \cdot (9.8 \cdot 10^2 - 0.6 \cdot 1.4 \cdot 10^2) + 0.6 \cdot 10 \cdot \bar{m}_v = 14.1 \cdot 10^3 + (10.75 + 6) \cdot \bar{m}_v \quad \text{(kJ/100 km)}
\]

where \(\bar{m}_v = m_v + m_{rec}\). That number must equal \(\bar{E}\), that has been calculated in Problem 2.8 as \(43.4 \cdot 10^3 \text{ kJ/100 km}\). Thus

\[
\bar{m}_v = \frac{(43.4 - 14.1) \cdot 10^3}{16.75} = 1750 \text{ kg}
\]

\[m_{rec} = 1720 \text{ kg} - 1500 \text{ kg} = 250 \text{ kg}.
\]

IC-Engine-Based Propulsion Systems

Gear-box Models

Problem 3.1

Improve (3.9) in order to take into account the engine inertia and the transmission efficiency.

Solution

Including the engine inertia

\[
m_{v,eq} = m_v + m_{e,eq}
\]

\[
m_{e,eq} = \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2
\]

\[
F_t = T_e \cdot \eta_t \cdot \frac{\gamma}{r_w}
\]

Neglecting other forces,

\[
a = g \cdot \sin(\alpha) = \frac{F_t}{m_{v,eq}} = \frac{T_e \cdot \eta_t \cdot \frac{\gamma}{r_w}}{m_v + \Theta_e \cdot \left(\frac{\gamma}{r_w}\right)^2}
\]

Thus

\[
\left(\frac{\gamma}{r_w}\right)^2 - \frac{T_e \cdot \eta_t \cdot \frac{\gamma}{r_w}}{\Theta_e \cdot a} + \frac{m_v}{\Theta_e} = 0,
\]

a quadratic equation in \(\gamma\). Evaluate \(\gamma_1\) by imposing that \(a = a_{max}\). With \(\Theta_e = 0\) and \(\eta_t = 1\), obtain back equation (3.9).
Problem 3.2

Dimension the first gear of an ICE-based powertrain not based on a given gradability as in (3.9) but in order to obtain a given acceleration at vehicle take-off. Do the calculations according to Problem 3.1, using the following data: \( m_v = 1100 \) kg, payload \( m_p = 100 \) kg, equivalent mass of the wheels \( m_{r,w} = 1/30 \) of \( m_v \), \( c_r = 0.009 \), transmission efficiency \( \eta_t = 0.9 \), \( T_e = 142 \) Nm at \( \omega_{\text{takeoff}} \), desired acceleration \( a = 4 \) m/s\(^2\), engine inertia \( \Theta_e = 0.128 \) kgm\(^2\), \( r_w = 30 \) cm. Compare with the approximate solution of (3.9).

Solution

Substitute \( a \) to \( g \cdot \sin(\alpha) \) in the result of Problem 3.1. Then solve

\[
\left( \frac{\gamma}{r_w} \right)^2 - \frac{142 \cdot 0.9}{0.128 \cdot 4} \cdot \frac{\gamma}{r_w} + \frac{1100 \cdot 1.033 + 100}{0.128} = 0.
\]

The two mathematical solutions are \( \gamma/r_w = 201.7 \) and 47.9. Choose the latter solution as the physical one (the former would lead to a too short gear with lower fuel efficiency; it would be too distant from \( \gamma_5 \) and would cause a bad sizing; during vehicle launch it would be held for a too short time in detriment of drivability). For a common wheel with \( r_w = 30 \) cm, \( \gamma_1 = 14.4 \). Using (3.9), thus neglecting the engine inertia, one would have \( \gamma/r_w = m_v \cdot a/(T_e \cdot \eta_t) = 38.7 \) (instead of 47.9), or \( \gamma_1 = 11.6 \) (instead of 14.4).

Problem 3.3

Dimension the fifth gear in a six-gear transmission for maximum power using (3.10) for a vehicle with the following characteristics: curb \( m_v = 1100 \) kg, performance mass = 100 kg, \( c_r = 0.009 \), \( A_f \cdot c_d = 0.65 \) m\(^2\), \( r_w = 30 \) cm, transmission efficiency = 0.9, \( P_{e,max} = 80.4 \) kW at 6032 rpm.

Solution

The maximum power at the wheels is

\[ 80.4 \cdot 0.9 = 72.4 \text{ kW}. \]

Equation (3.10) is written as

\[ P_{e,max} = v_{max} \cdot \left( \frac{1}{2} \cdot 1.2 \cdot 0.65 \cdot v_{max}^2 + 9.81 \cdot 1200 \cdot 0.009 \right), \]

which is solved by \( v_{max} = 55.5 \) m/s = 199.7 km/h. The fifth gear ratio is

\[ \frac{\gamma_5}{r_w} = \frac{\pi \cdot 6032}{30 \cdot 55.5} = 11.4. \]

For a common wheel radius of 30 cm, \( \gamma_5 = 3.4 \).
Problem 3.4

Consider again the system of Problems 3.2–3.3. Calculate the vehicle speed values, \( v_j \) at which the engine is at its maximum speed, for all the gears \( j \).

- Solution

The 2\textsuperscript{nd} to 4\textsuperscript{th} gear can be chosen according to a geometric law with

\[
R = \frac{47.9}{11.4} = 4.2,
\]

\[
\frac{1}{\kappa} = R^{1/4} = 1.43,
\]

\[
\frac{\gamma_2}{\gamma_3} = \frac{\gamma_4}{\gamma_5} = \frac{\gamma_4}{\gamma_5} = \frac{1}{\kappa},
\]

from whence

\[
\gamma_2 = \frac{\gamma_1}{\gamma_1 + R} \cdot \kappa = 33.5,
\]

\[
\gamma_3 = \frac{\gamma_2}{\gamma_2 + R} \cdot \kappa = 23.4,
\]

\[
\gamma_4 = \frac{\gamma_3}{\gamma_3 + R} \cdot \kappa = 16.3,
\]

\[
\gamma_5 = \frac{\gamma_4}{\gamma_4 + R} \cdot \kappa = 11.4 \text{ (confirmed)}.
\]

An alternative to the geometric sizing law is the arithmetic law with

\[
R = \left( \frac{1}{11.4} - \frac{1}{47.9} \right) \cdot \frac{1}{4} = 0.0167,
\]

from whence

\[
\gamma_2 = \frac{1}{\gamma_1 + R} = 26.6,
\]

\[
\gamma_3 = \frac{1}{\gamma_2 + R} = 18.4,
\]

\[
\gamma_4 = \frac{1}{\gamma_3 + R} = 14.1,
\]

\[
\gamma_5 = \frac{1}{\gamma_4 + R} = 11.4 \text{ (confirmed)}.
\]

The shifting speeds are

\[
v_k = \frac{\pi \cdot 6032}{30 \cdot \gamma_1},
\]

thus the values in the following table.
Problem 3.5

Calculate the approximate efficiency of a clutch during a vehicle takeoff maneuver.

Solution

In a first approximation, the transmitted torque could be considered as equal to the engine torque while the clutch is slipping. The engine speed is approximately constant at the launch value $\omega_e$, while the speed downstream of the clutch is given by the differential equation

$$\Theta_{gb} \cdot \frac{d\omega_{gb}}{dt} = T_e - T_{losses},$$

with $\Theta_{gb} = \Theta_v / \gamma^2$, which can be approximated by neglecting the losses. Thus

$$\omega_{gb}(t) = \frac{T_e}{\Theta_{gb}} \cdot t.$$

The launch maneuver ends when $\omega_{gb}(t_f) = \omega_e$. The corresponding synchronization speed is

$$v = \frac{\omega_e}{\gamma}.$$

The launch time is

$$t_f = \omega_e \cdot \frac{\Theta_{gb}}{T_e}.$$

The energy provided by the engine during this time is

$$E_e = \int_0^{t_f} T_e \cdot \omega_e \, dt = T_e \cdot \omega_e \cdot t_f = \Theta_{gb} \cdot \omega_e^2.$$

The energy transferred to the vehicle is the kinetic energy at the end of launch, which is

$$E_v = \frac{1}{2} \cdot \Theta_{gb} \cdot \omega_e^2.$$

Thus the energy lost is $E_e - E_v$ which coincides with (3.16) since $\omega_{w,0} = \omega_e / \gamma$ and therefore, the efficiency is $E_v / E_e = 0.5$. 

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>13.2 m/s = 47.5 km/h</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_2$</td>
<td>18.9 m/s = 68.0 km/h</td>
</tr>
<tr>
<td>$v_3$</td>
<td>27.0 m/s = 97.2 km/h</td>
</tr>
<tr>
<td>$v_4$</td>
<td>38.8 m/s = 139.7 km/h</td>
</tr>
<tr>
<td>$v_5$</td>
<td>55.4 m/s = 199.4 km/h</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v_1$</th>
<th>23.7 m/s = 85.5 km/h</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_2$</td>
<td>34.3 m/s = 123.6 km/h</td>
</tr>
<tr>
<td>$v_3$</td>
<td>44.8 m/s = 161.3 km/h</td>
</tr>
</tbody>
</table>
Fuel Consumption of IC Engine Powertrains

Problem 3.6

Find the CO₂ emission factor (g/km) as a function of the fuel consumption rate (l/100km) for gasoline and diesel fuels. Use these average fuels (gasoline, diesel) data: density $\rho = \{0.745, 0.832\}$ kg/l, carbon dioxide to fuel mass fraction $m = \{3.17, 3.16\}$.

• Solution

Consider the stoichiometric fuel burning reaction

$$\text{CH}_a + \left(1 + \frac{a}{4}\right)\text{O}_2 \rightarrow \text{CO}_2 + \left(\frac{a}{2}\right)\text{H}_2\text{O}$$

where a fuel of molar composition CHₐ is assumed. The mol CO₂/mol emission factor is 1. The kg CO₂/kg emission factor is thus

$$m = \frac{M_{\text{CO}_2}}{M_{\text{fuel}}} = \frac{12 + 2 \cdot 16}{12 + a} = \frac{44}{12 + a}$$

The factor $a$ can be found by using $m$ for gasoline,

$$a = \frac{44}{m} - 12 = \frac{44}{3.17} - 12 = 1.88,$$

and for diesel,

$$a = \frac{44}{3.16} - 12 = 1.92.$$ 

The kg/l factor is $m \cdot \rho$, where $\rho$ (kg/l) is the fuel density. For gasoline we get

$$m \cdot \rho = 3.17 \cdot 0.745 = 2.36 \text{ kg/l}$$

and for diesel,

$$m \cdot \rho = 3.16 \cdot 0.832 = 2.63 \text{ kg/l}.$$ 

The overall factor is

$$\text{g CO}_2$/km = \text{g CO}_2$/l \cdot C \cdot \frac{1}{100} = 23.6 \cdot C$$

for gasoline, and

$$\text{g CO}_2$/km = 26.3 \cdot C$$

for diesel, where $C$ is the fuel consumption (l/100 km).
Problem 3.7

Calculate the fuel consumption and the CO₂ emission rate for the MVEG–95 cycle for a vehicle having the following characteristics: \( m_v = 1100 \text{kg} \), payload \( c_d \cdot A_f = 0.7 \), \( e_r = 0.013 \), \( e_g = 0.98 \), \( P_{0,gb} = 3\% \), \( P_{aux} = 250 \text{W} \), \( v_{launch} = 3 \text{m/s} \), \( e = 0.4 \), \( P_{e,0} = 1.26 \text{kW} \), \( P_{e,max} = 66 \text{kW} \), diesel fuel \( (H_f = 43.1 \text{MJ/kg}, \rho_f = 832 \text{g/l}) \), idle consumption \( V_{f, idle} = 150 \text{g/h} \). The declared CO₂ emission rate for this car is 99 g/km.

- Solution

The mean traction force is calculated as

\[
\bar{F}_{trac, r} = 1100 \cdot 9.81 \cdot 0.013 \cdot 0.856 = 120 \text{ N}
\]
\[
\bar{F}_{trac, a} = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 319 = 134 \text{ N}
\]
\[
\bar{F}_{trac, m} = 1200 \cdot 0.101 = 120 \text{ N}
\]
\[
\bar{F}_{trac} = 120 + 134 + 120 = 374 \text{ N}
\]

From this result the mean engine power can be calculated as in (3.23)–(3.25),

\[
\bar{P}_{trac} = \frac{374 \cdot 9.5}{0.6} = 5.9 \text{ kW}
\]
\[
\bar{P}_1 = \frac{5.9 \cdot 10^3 \cdot 1.03}{0.98} = 6.2 \text{ kW}
\]
\[
\bar{P}_{start} = \frac{1}{2} \cdot \frac{1200 \cdot 3^2}{105} = 51 \text{ W}
\]
\[
\bar{P}_e = 6200 \cdot 10^3 + 250 + 51 = 6.5 \text{ kW}
\]

Using the given Willans approach, the average fuel power can be calculated as

\[
\eta_e = \frac{e \cdot \bar{P}_e}{\bar{P}_e + P_{0,e}} = \frac{0.4 \cdot 6.5}{6.5 + 1.26} = 0.34
\]
\[
\bar{P}_f = \frac{0.6 \cdot \bar{P}_e}{\eta_e} = 11.5 \text{ kW}
\]

Remember: The average velocity on the MVEG–95 cycle is 9.5 m/s, the engine idle time is approximately 300 seconds and the cycle length is 11.4 kilometers. The equivalent fuel flow is therefore calculated as in (3.28)–(3.32),

\[
\dot{V}_{trac} = \frac{\bar{P}_f}{H_f \cdot \rho_f} = \frac{11.7 \cdot 10^3}{43.1 \cdot 10^6 \cdot 0.832} = 3.2 \cdot 10^{-4} \text{ l/s} = \frac{3.4 \cdot 10^{-4}}{9.5} \cdot 10^5 \text{ l/100 km} = 3.5 \text{ l/100 km}
\]
\[
\dot{V}_{idle} = \frac{150}{3600 \cdot 0.832} \cdot \frac{300}{11.4} \cdot \frac{100}{11.4} = 0.1 \text{ l/100 km}
\]
\[
\dot{V}_f = 3.5 \text{ l/100 km} + 0.1 \text{ l/100 km} = 3.6 \text{ l/100 km}
\]
\[
= 3.6 \cdot 26.3 = 95 \text{ g/km}
\]
The additional CO\textsubscript{2} not taken into account in this method amounts to 4 g/km (env. 5%).

**Problem 3.8**

Evaluate in a first approximation the contribution of stop-and-start, regenerative braking ($m_{rec} = 20\% \cdot m_v, \eta_{rec} = 0.5$) and optimization of power flows in reducing the fuel consumption when the system of Problem 3.7 is hybridized. Assume an 80% charge-discharge efficiency for the reversible storage system.

- **Solution**

  From the result of Problem 3.7,
  \[ \frac{V_{idle}}{V_{tot}} = 4\% \]

  Now consider regenerative braking. Redo calculations of Problem 3.7 with $m_v = 1100 \cdot 1.2 = 1320$ kg.

  \[
  \bar{F}_{\text{trac,r}} = 1320 \cdot 9.81 \cdot 0.013 \cdot 0.856 = 144 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac,a}} = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 319 = 134 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac,m}} = 1420 \cdot 0.101 = 142 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac}} = 144 + 134 + 142 = 420 \text{ N}
  \]

  For ideal regenerative braking,
  \[
  \bar{F}_{\text{trac,r}} = 1320 \cdot 9.81 \cdot 0.013 \cdot 1 = 168 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac,a}} = \frac{1}{2} \cdot 1.2 \cdot 0.7 \cdot 363 = 152 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac,m}} = 0 \text{ N}
  \]
  \[
  \bar{F}_{\text{trac}} = 168 + 152 = 320 \text{ N}
  \]

  For $\eta_{rec} = 0.5$, the traction force is
  \[
  F_{\text{trac}} = 320 + (1 - 0.5) \cdot (420 - 320) = 370 \text{ N}
  \]

  Thus the potential gain due to regenerative braking seems rather limited,
\[
\bar{P}_{\text{trac}} = \frac{370 \cdot 9.5}{0.6} = 5.8 \text{ kW}
\]
\[
\bar{P}_1 = \frac{5.8 \cdot 10^3 \cdot 1.03}{0.98} = 6.1 \text{ kW}
\]
\[
\bar{P}_{\text{start}} = \frac{1}{2} \cdot \frac{1200 \cdot 3^2}{105} = 51 \text{ kW}
\]
\[
\bar{P}_e = 6.1 \cdot 10^3 + 0.25 + 0.05 = 6.4 \text{ kW}
\]
\[
\eta_e = \frac{\bar{e} \cdot \bar{P}_e}{\bar{P}_e + \bar{P}_0,e} = \frac{0.4 \cdot 6.4}{6.4 + 1.26} = 0.33
\]
\[
\bar{P}_f = \frac{0.6 \cdot \bar{P}_e}{\eta_e} = 11.6 \text{ kW}
\]
\[
\bar{V}_{\text{trac}} = \frac{11.6 \cdot 10^3}{43.1 \cdot 10^6 \cdot 0.832} = 3.2 \cdot 10^{-4} \text{ l/s} = \frac{3.2 \cdot 10^{-4} \cdot 10^5}{9.5} \text{ l/100 km} = 3.4 \text{ l/100 km}
\]

In summary we have 3.6 l/100 km w.r.t. 3.4 l/100 km. The benefit due to regenerative braking is equal to the benefit due to idle consumption suppression and they amount to 3\%. To evaluate the potential benefit of engine operating point shifting, assume that the engine could be able to work always at its maximum efficiency point, thus at \(P_{\text{e, max}} = 66 \text{ kW}\). The efficiency is \(\eta_e = 0.4 \cdot 66/(66 + 1.26) = 0.39\). Moreover, during a time \(t_1\), the engine delivers its surplus power to the battery, to be reused later. An energy balance across the traction phase yields \(\bar{P}_e \cdot 0.6 = P_{\text{e, max}} \cdot t_1 + (P_{\text{e, max}} - \bar{P}_e) \cdot t_1 \cdot \eta_{\text{acc}}\), where \(\eta_{\text{acc}}\) is the efficiency of the accumulation system (to be charged and then discharged). Assuming \(\eta_{\text{acc}} = 0.8\), from the latter one calculates
\[
\begin{align*}
t_1 &= \frac{6.4 \cdot 0.6}{66 + (66 - 6.4) \cdot 0.8} = 0.034,
\end{align*}
\]
thus
\[
\begin{align*}
\bar{P}_f &= \frac{0.034 \cdot 66 \cdot 10^3}{0.39} = 5.7 \text{ kW}
\end{align*}
\]
\[
\bar{V}_{\text{trac}} = \frac{5.7 \cdot 10^3}{43.1 \cdot 10^6 \cdot 0.832} = 1.6 \cdot 10^{-4} \text{ l/s} = \frac{1.6 \cdot 10^{-4} \cdot 10^5}{9.5} \text{ l/100 km} = 1.7 \text{ l/100 km}
\]

Ideal gain due to optimization of power flows = \((3.4 - 1.7)/3.6 = 47\%\).
**Problem 4.1**

Design an electric powertrain for a small city car having the following characteristics: curb mass = 840 kg, payload = 2 · 75 kg, tires: 155/65/14T, $c_d \cdot A_f = 1.85 m^2$, rolling resistance coefficient = 0.009, to meet the following performance criteria: (i) max speed = 65 km/h, (ii) max grade = 16%, (iii) 100 km range. Assume perfect recuperation, overall efficiency of 0.6, and 85% SoC window. Choose motor size in a class with a maximum speed of 6000 rpm and the number of battery modules having a capacity of 1.2 kWh each.

- **Solution**

  For $v_{max} = 65 \text{ km/h} = 18.1 \text{ m/s}$, the required power is
  
  $$P_{max} = m_v \cdot g \cdot c_r \cdot v_{max} + 0.5 \cdot 1.2 \cdot A_f \cdot v_{max}^2 = 8 \text{ kW}.$$  

  The max speed of the motor is $\omega_{m,max} = v_{max} \cdot \gamma/r_w$ where $\gamma$ is the reduction ratio and $r_w$ is the wheel radius. The wheel radius is obtained from the tire specifications (see Problem 2.1) as
  
  $$\frac{14'}{2} + 0.65 \cdot 0.155 \text{ m} = \frac{14 \cdot 0.0254}{2} + 0.65 \cdot 0.155 = 0.28 \text{ m}.$$  

  If one fixes $\omega_{m,max} = 6000 \text{ rpm} = 628 \text{ rad/s}$, then $\gamma = 628/18.05 \cdot 0.28 = 9.7$.

  The max torque is
  
  $$T_{m,max} = r_w/\gamma \cdot m_v \cdot g \cdot \sin(\alpha) = 0.28/9.7 \cdot (840 + 150) \cdot 9.8 \cdot 0.16 = 45 \text{ Nm}.$$  

  Thus the base speed is $P_{m,max}/T_{m,max} = 8000/45 = 178 \text{ rad/s} = 1700 \text{ rpm}$. The base to max speed ratio is 1:3.5, which is a reasonable design choice.

  Assuming an efficiency $\eta = 0.6$ and perfect recuperation, the mean traction force for an ECE drive cycle is (see (2.34))
  
  $$\bar{F} = m_v \cdot g \cdot c_r + \frac{1}{2} \cdot 1.2 \cdot A_f \cdot c_d \cdot 100 + 840 \cdot 0.14 = 303 \text{ N}.$$  

  Thus the energy required is $303/0.6 \cdot 100 \cdot 10^3 = 50.5 \text{ MJ} = 14 \text{ kWh}$. Add an unused 15% range and obtain 16.1 kWh. Using 6V/200 Ah (1.2 kWh) modules, 14 modules would be needed for a stored energy of 16.8 kWh.

**Problem 4.2**

Find an equation for the AER $D_ev$ of a full electric vehicle as a function of its vehicle parameters, battery capacity and powertrain efficiency. Then evaluate the $D_ev$ for a bus with the following characteristics: $\eta_{rec} = 100\%$, $\eta_{sys} = 0.45$ (including unused SoC), $c_r = 0.006$, $A_f \cdot c_d = 6.8 \cdot 0.62$, $Q_{bat} = 89 \text{ Ah}$, $U_{bat} = 600 \text{ V}$, $m_v = 14.6 \text{ t}$, without payload and with a load of 60 passengers, respectively. Assume a MVEG-type speed profile.
Equation (2.30) is used for the energy at the wheels. To have energy demand in Wh/km, divide the outcome of (2.30) by the factor 100·3.6. Now, if the battery capacity is expressed in Ah,

\[ D_{ev} = \frac{Q_{bat} \cdot U_{bat}}{E_{rec, MV EG} - 95 - 1000 \cdot 3.6 \cdot \eta_{sys}}. \]

For the numerical case without payload,

\[ \bar{E} = 6.8 \cdot 0.62 \cdot 2.2 \cdot 10^4 + 14620 \cdot 0.006 \cdot 9.81 \cdot 100 = 1.8 \cdot 10^5 \text{kJ}/100\text{km} = 1 \cdot 10^7 \text{kJ}/1000\text{km} = 500 \text{Wh/km}, \]

\[ D_{ev} = \frac{89 \cdot 600}{0.45} = 48 \text{ km}. \]

For a payload of 60·75 kg = 4500 kg,

\[ \bar{E} = 6.8 \cdot 0.62 \cdot 2.2 \cdot 10^4 + 19120 \cdot 0.006 \cdot 9.81 \cdot 100 = 2.05 \cdot 10^5 \text{kJ/km} = 2 \text{050 Wh/km}, \]

\[ D_{ev} = \frac{89 \cdot 600}{0.45} = 42 \text{ km}. \]

**Problem 4.3**

The 2011 Nissan Leaf electric vehicle has been rated by the EPA as achieving 99 mpg equivalent or 34 kWh/100 miles. Justify this rating.

- **Solution**

Just consider that the energy content of one U.S. liquid gallon of gasoline is 33.41 kWh. Then

\[ 33.4 \frac{kWh}{gal} \cdot \frac{1}{99} \frac{gal}{miles} \cdot 100 = 34 \frac{kWh}{100 miles}. \]

**Hybrid-Electric Propulsion Systems**

**Problem 4.4**

Classify the five different parallel hybrid architectures, (1) single-shaft with single clutch between engine and electric machine (E-c-M-T-V), (2) single-shaft with single clutch between engine–electric machine and transmission (E-M-c-T-V) or (M-E-c-T-V), (3) two-clutches single-shaft (E-c-M-c-T-V), (4) double-shaft (E-c-T-M-V), (5) double-drive, with respect to the following features:
- regenerative braking: optimized (without unnecessary losses) / not optimized
- ZEV mode: optimized (without unnecessary losses) / not optimized
- stop-and-start: optimized (independent from vehicle motion) / not optimized / not possible
- battery recharge at vehicle stop: possible / not possible
- gear synchronization: optimized (no additional inertia on the primary shaft) / not optimized
- compensation of the torque “holes” during gear changes: possible / not possible
- active dampening of engine idle speed oscillations: possible / not possible
- Solution

Architecture (1):
- compensation of the torque ”holes” during gear changes not possible
- reg. braking ideal
- ZEV mode ideal
- stop/start compromised
- battery recharge at vehicle stop impossible
- gear synchronization compromised (but compensation through the electric machine itself possible)
- active dampening impossible

Architecture (2), e.g., an Honda IMA-type system (E-M-c-T-V), or a belt starter-alternator case (M-E-c-T-V):
- reg. braking compromised
- ZEV mode compromised
- stop/start ideal
- active dampening possible
- battery recharge at vehicle stop possible
- gear synchronization ideal

Architecture (3):
- reg. braking ideal
- ZEV mode ideal
- stop/start ideal
- active dampening possible
- battery recharge at vehicle stop possible
- gear synchronization ideal

Architecture (4):
- reg. braking compromised
- ZEV mode compromised
- stop/start compromised
- active dampening impossible
- battery recharge at vehicle stop impossible
- gear synchronization ideal
- compensation of the torque "holes" during gear changes possible

Architecture (5):
- reg. braking ideal
- ZEV mode not ideal
- stop/start impossible (would need an additional starter machine)
- active dampening impossible (see stop/start)
- battery recharge at vehicle stop impossible
- gear synchronization ideal
- compensation of the torque "holes" during gear changes possible

Find a summary of these features in the table below.

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**Problem 4.5**

Determine the overall degrees of freedom \( n \) in modeling (i) a parallel hybrid, (ii) a series hybrid, (iii) a combined hybrid, with the quasistatic approach. For (ii) and (iii) use both the generator causality depicted in Figs. 4.11 – 4.13 and the alternative causality introduced in Sect. 4.4.

- **Solution**

*Parallel hybrid* There are 6 blocks \( \{V, T, E, M, P, B\} \) and 7 relationships:

1. \( f_V(v, F_t) = 0 \)
2. \( f_{T,1}(F_t, T_e, T_m, \gamma) = 0 \)
3. \( f_{T,2}(v, \omega_e, \gamma) = 0 \)
4. \( f_{T,3}(v, \omega_m, \gamma) = 0 \)
5. \( f_E(u_e, T_e, \omega_e) = 0 \) (\( u_e \) is the engine control vector)
6. \( f_M(T_m, \omega_m, P_m) = 0 \)
7. \( f_P(P_m, P_b) = 0 \)
between the 10 variables \(v, \dot{T}_{e}, \omega_{e}, \omega_{m}, \gamma, \dot{P}_{m}, \dot{P}_{b}\). Consider \(\gamma\) as fixed. Thus there are two independent variables. In the quasistatic approach, \(v\) is known, thus the remaining degree of freedom is, e.g., the torque split ratio \(u\) (needed to solve the \(f_{T, 1}\) equation).

**Series hybrid** There are 7 blocks and relationships

1. \(f_{V}(v, \dot{F}_{t}) = 0\)
2. \(f_{T, 1}(\dot{F}_{t}, \dot{T}_{m}) = 0\)
3. \(f_{T, 2}(v, \omega_{m}) = 0\)
4. \(f_{M}(\dot{T}_{m}, \omega_{m}, \dot{P}_{m}) = 0\)
5. \(f_{P}(\dot{P}_{m}, P_{g}, \dot{P}_{b}) = 0\)
6. \(f_{G}(P_{g}, \omega_{g}, \dot{T}_{g}) = 0\)
7. \(f_{E}(u_{e}, \dot{T}_{e}, \omega_{e}) = 0\)

between the 10 variables \(v, \dot{F}_{t}, \dot{T}_{m}, \omega_{m}, \dot{P}_{m}, P_{g}, \dot{T}_{g} = \dot{T}_{e}, \omega_{g} = \omega_{e}, u_{e}\). Thus there are three independent variables. In the quasistatic approach, \(v\) is known, thus the remaining degrees of freedom are the power split ratio \(u\) (needed to solve the \(f_{P}\) equation) and the generator speed \(\omega_{g}\) (needed to solve the \(f_{G}\) equation). In the alternative causality of the generator block, generator speed and torque \(\dot{T}_{g}\) are used to solve the \(f_{G}\) equation.

**Combined hybrid** There are 8 blocks and 11 relationships

1. \(f_{V}(v, \dot{F}_{t}) = 0\)
2. \(f_{T, 1}(\dot{F}_{t}, \dot{T}_{f}) = 0\)
3. \(f_{T, 2}(v, \omega_{f}) = 0\)
4. \(f_{PSD, 1}(\omega_{f}, \omega_{g}, \omega_{e}) = 0\)
5. \(f_{PSD, 2}(\omega_{f}, \omega_{g}, \omega_{m}) = 0\)
6. \(f_{PSD, 3}(\dot{T}_{f}, \dot{T}_{g}, \dot{T}_{e}) = 0\)
7. \(f_{PSD, 4}(\dot{T}_{f}, \dot{T}_{g}, \dot{T}_{m}) = 0\)
8. \(f_{M}(\dot{T}_{m}, \omega_{m}, \dot{P}_{m}) = 0\)
9. \(f_{P}(\dot{P}_{m}, P_{g}, \dot{P}_{b}) = 0\)
10. \(f_{G}(P_{g}, \omega_{g}, \dot{T}_{g}) = 0\)
11. \(f_{E}(u_{e}, \dot{T}_{e}, \omega_{e}) = 0\)

between the 14 variables \(v, \dot{F}_{t}, \omega_{f}, T_{f}, \dot{T}_{m}, \omega_{m}, \dot{P}_{m}, \dot{T}_{g}, \omega_{g}, P_{g}, \dot{P}_{b}, \dot{T}_{e}, \omega_{e}, u_{e}\). Thus there are three independent variables. The degrees of freedom are the same as for the series hybrid case.

**Problem 4.6**

Perform the same analysis as in Problem 4.5 with the dynamic approach. Calculate the number \(n_{v}\) of variables in the flowcharts of Figs. 4.11 – 4.13. Then calculate the number \(n_{e}\) of the equations available using the simple models presented in this chapter. Finally evaluate the manipulated variables that are necessary to realize the degrees of freedom (DOF) determined in Problem 4.5.

- Solution
Parallel hybrid There are 6 blocks and $n_e = 10$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_e, T_m, \gamma) = 0$
3. $f_{T,2}(v, \omega_e, \gamma) = 0$
4. $f_{T,3}(v, \omega_m, \gamma) = 0$
5. $f_{M,1}(T_m, I_m) = 0$
6. $f_{M,2}(\omega_m, U_m, u_m) = 0$ ($u_m$ is the motor control vector)
7. $f_{P,1}(U_m, U_b) = 0$
8. $f_{P,2}(I_m, I_b) = 0$
9. $f_G(u_e, T_e, \omega_e) = 0$
10. $f_B(I_b, U_b) = 0$

between the $n_e = 10$ variables represented in the figure. If $\gamma$ is fixed, the control inputs $u_e$, $u_m$ determine the vehicle speed and the torque split ratio.

Series hybrid There are 6 blocks and $n_e = 12$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_m) = 0$
3. $f_{T,2}(v, \omega_m) = 0$
4. $f_{M,1}(T_m, I_m) = 0$
5. $f_{M,2}(\omega_m, U_m, u_m) = 0$
6. $f_{P,1}(U_m, U_b, U_g) = 0$
7. $f_{P,2}(I_m, I_b) = 0$
8. $f_{P,3}(I_m, I_g) = 0$
9. $f_{G,1}(T_g, I_g) = 0$
10. $f_{G,2}(\omega_g, U_g, u_g) = 0$ ($u_g$ is the generator control vector)
11. $f_E(u_e, T_e, \omega_e) = 0$
12. $f_B(I_b, U_b) = 0$

between the $n_e = 12$ variables represented in the figure. The control inputs $u_e$, $u_m$, and $u_g$ determine the vehicle speed, the power split ratio, and the generator speed.

Combined hybrid There are 8 blocks and $n_e = 16$ relationships:

1. $f_V(v, F_t) = 0$
2. $f_{T,1}(F_t, T_f) = 0$
3. $f_{T,2}(v, \omega_f) = 0$
4. $f_{PS,1}(\omega_f, \omega_g, \omega_e) = 0$
5. $f_{PS,2}(\omega_f, \omega_g, \omega_m) = 0$
6. $f_{PS,3}(T_f, T_g, T_e) = 0$
7. $f_{PS,4}(T_f, T_g, T_m) = 0$
8. $f_{M,1}(T_m, I_m) = 0$
9. $f_{M,2}(\omega_m, U_m, u_m) = 0$
10. $f_{P,1}(U_m, U_b, U_g) = 0$
11. $f_{P,2}(I_m, I_b) = 0$
12. $f_{P,3}(I_m, I_g) = 0$
13. \( f_{G,1}(T_g, I_g) = 0 \)
14. \( f_{G,2}(\omega_g, U_g, u_g) = 0 \)
15. \( f_E(u_e, T_e, \omega_e) = 0 \) (\( u_e \) is the engine control vector)
16. \( f_B(I_b, U_b) = 0 \)

between the \( n_v = 16 \) variables represented in the figure. The control inputs \( u_e, u_m, \) and \( u_g \) determine the vehicle speed, the power split ratio, and the generator speed.

**Problem 4.7**

Perform the same analysis as in Problems 4.5 – 4.6 for an electric powertrain powered by a battery and a supercapacitor.

- **Solution**

  **Quasistatic approach** There are 6 blocks \{\( V, T, M, P, B, SC \)\} and 5 relationships:
  1. \( f_V(v, F_t) = 0 \)
  2. \( f_T, 1(F_t, T_m) = 0 \)
  3. \( f_T, 2(v, \omega_m) = 0 \)
  4. \( f_M(T_m, \omega_m, P_m) = 0 \)
  5. \( f_P(P_m, P_b, P_{sc}) = 0 \)

  between the \( n_v = 10 \) variables. If there is only one control input \( u_m \) the power split ratio cannot be chosen. Thus a second controllable component is needed, typically a DC–DC converter on either the supercapacitor or the battery side.

  **Dynamic approach** There are 6 blocks and \( n_v = 10 \) relationships
  1. \( f_V(v, F_t) = 0 \)
  2. \( f_T, 1(F_t, T_m) = 0 \)
  3. \( f_T, 2(v, \omega_m) = 0 \)
  4. \( f_M(T_m, I_m) = 0 \)
  5. \( f_M, 2(\omega_m, U_m, u_m) = 0 \)
  6. \( f_P, 1(U_m, U_b, U_{sc}) = 0 \)
  7. \( f_P, 2(I_m, I_b) = 0 \)
  8. \( f_P, 3(I_m, I_{sc}) = 0 \)
  9. \( f_B(I_b, U_b) = 0 \)
  10. \( f_{SC}(I_{sc}, U_{sc}) = 0 \)

  between the \( n_v = 10 \) variables. If is only one control input \( u_m \) the power split ratio cannot be chosen. Thus a second controllable component is needed, typically a DC–DC converter on either the supercapacitor or the battery side.
Problem 4.8

For a plug-in hybrid, the fuel consumption according to UN/ECE regulation [91] is

\[ C = \frac{D_e \cdot C_1 + D_{av} \cdot C_2}{D_e + D_{av}}, \]

where \( C_1 \) is the fuel consumption in charge-depleting mode, \( C_2 \) is the consumption in charge-sustaining mode, \( D_e \) is the electric range, and \( D_{av} \) is 25 km, the assumed average distance between two battery recharges. Estimate the fuel consumption of the electric system of Problem 4.1 equipped with a range extender having a max power of 5 kW and an efficiency of 0.4.

- Solution

\( D_e = 100 \text{ km}, \ C_1 = 0, \ D_{av} = 25 \text{ km}. \) To evaluate the fuel consumption in charge-sustaining mode, divide the cycle into two phases, with (i) APU on, and (ii) APU off. The mean force is the same. During phase (i),

\[ E_{bat} = -F_r \cdot e_b \cdot x_{on}, \]

where \( F_r \) is the mean traction force to recharge the battery, \( e_b = \sqrt{\varepsilon} \) is the battery efficiency, and \( x_{on} \) is the distance covered during the phase (i). During phase (ii)

\[ E_{bat} = F_e \cdot \left( x_{tot} - x_{on} \right). \]

By equalizing these two energy terms (charge sustaining),

\[ x_{on} = \frac{x_{tot} \cdot F}{F + F_r \cdot \sqrt{\varepsilon}}. \]

The APU mean power during phase (i) is

\[ P_{apu} = \left( F_r + \frac{F}{\sqrt{\varepsilon}} \right) \cdot \frac{x_{on}}{t_{tot}} = \frac{F + F_r \cdot \sqrt{\varepsilon}}{F + F_r \cdot e \cdot \sqrt{\varepsilon}} \cdot \bar{v} \]

and the average fuel power is

\[ P_f = \frac{P_{apu}}{x_{apu} \cdot \bar{v}} \]

Numerically,

\[ F_r = \frac{P_{apu,max}}{\bar{v}} - \frac{F}{\sqrt{\varepsilon}} = \frac{5 \cdot 10^3}{9.5} - \frac{303}{\sqrt{0.6}} = 135 \text{ Nm} \]

\[ P_{apu} = \frac{303 + 135 \cdot \sqrt{0.6}}{303 + 135 \cdot 0.6 \cdot \sqrt{0.6}} \cdot 9.5 = 4.1 \text{ kW} \]

\[ P_f = \frac{4.1 \cdot 10^3}{0.4} = 10.4 \text{ kW} \]

\[ V_f = \frac{10.4 \cdot 10^3}{43.5 \cdot 10^6 \cdot 0.75} = 3.2 \cdot 10^{-4} \text{ l/s} = \frac{3.2 \cdot 10^{-4}}{9.5} \cdot 1 \cdot 10^5 = 3.4 \text{ l/100 km} = C_2 \]
Thus the combined fuel consumption is

\[
C = \frac{D_e \cdot C_1 + D_{av} \cdot C_2}{D_e + D_{av}} = \frac{100 \cdot 0 + 25 \cdot 3.4}{125} = 0.68 \text{ l/100 km}
\]

Motor and Motor Controller

Problem 4.9

Consider a separately-excited DC motor having the following characteristics:

\[ R_a = 0.05 \Omega, \text{ battery voltage} = 50 \text{ V (neglect battery resistance), rated power} = 4 \text{ kW, nominal torque constant} \kappa_a = \kappa_i = 0.25 \text{ Wb, aimed at propelling a small city vehicle. Calculate the motor voltage and current limits, then the flux weakening region limit (maximum torque and base speed). Calculate the step-down chopper duty-cycle } \alpha \text{ for the following operating points: (i) } \omega_m = 100 \text{ rad/s and } T_m = 15 \text{ Nm; (ii) } \omega_m = 300 \text{ rad/s and } T_m = 8 \text{ Nm.}
\]

- Solution

The maximum voltage is \( U_{max} = 50 \text{ V.} \) The maximum admissible current is calculated by forcing \( U_a = U_{max} \) and \( \omega_m \cdot T_m = P_{max}. \) The following quadratic equation is obtained,

\[
U_{max} \cdot I_{max} = R_a \cdot I_{max}^2 + P_{max}, \quad \text{or}
\]

\[
0.05 \Omega \cdot I_{max}^2 - 50 \text{ V} \cdot I_{max} + 4000 \text{ W} = 0,
\]

whose solution is \( I_{max} = 88 \text{ A.} \) Thus the maximum torque is \( 88 \cdot 0.25 = 22 \text{ Nm.} \)

The flux weakening region limit occurs when \( U_a = U_{max} \) thus

\[
\frac{R_a \cdot T_m}{\kappa_a} + \kappa_a \cdot \omega_m = U_{max}, \quad \text{or}
\]

\[
0.2 \cdot T_m + 0.25 \cdot \omega_m = 50,
\]

extending from \( T_m = 250 \text{ Nm on the torque axis to } \omega_m = 200 \text{ rad/s on the speed axis.} \) The base speed is \( \omega_b = 4000/22 = 182 \text{ rad/s.} \) For the first operating point,

\[
I_a = \frac{T_m}{\kappa_a} = \frac{15}{0.25} = 60 \text{ A}
\]

\[
U_a = R_a \cdot I_a + \kappa_a \cdot \omega_m = 0.05 \cdot 60 + 0.25 \cdot 100 = 3 + 25 = 28 \text{ V.}
\]

Both current and voltage limits are respected. The chopper duty-cycle is \( \alpha = 28/50 = 56\%. \) The second operating point belongs to the flux weakening region. In fact, if one were to calculate the current and voltage with the above equations, \( I_a = 8/0.25 = 32 \text{ A} \) would be obtained, but \( U_a = 0.05 \cdot 32 + 0.25 \cdot 300 = 77 \text{ V} \) that is beyond the admissible voltage. Thus \( \kappa_a \) must be reduced.

To find \( \kappa_a \) such that \( U_a = 50 \text{ V (} \alpha = 100\% \), the following equation is used
\[ U_{\text{max}} = \frac{R_a \cdot T_m}{\kappa_a} + \kappa_a \cdot \omega_m, \]

which leads to \( \kappa_a = 0.16 \) Wb. An approximated value is obtained by neglecting the resistance as \( \kappa_a = U_{\text{max}} / \omega_m = 50 / 300 = 0.17 \) Wb.

**Problem 4.10**

For the DC motor of Problem 4.9, evaluate the approximation of mirroring the efficiency from the first to the fourth quadrant, for the two operating points (i) \( \omega_m = 50 \) rad/s and \( T_m = 22 \) Nm; (ii) \( \omega_m = 300 \) rad/s, \( T_m = 8 \) Nm. Assume further that \( P_{l,c} = 0 \).

- **Solution**

  From (4.14), for \( T_m > 0 \)

  \[
  \frac{1}{\eta_m} = 1 + \frac{R_a \cdot T_m}{\kappa_a^2 \cdot \omega_m},
  \]

  \[
  P_l = \frac{R_a \cdot T_m^2}{\kappa_a^2}.
  \]

  For \( T_m < 0 \)

  \[
  \eta_m = 1 + \frac{R_a \cdot T_m}{\kappa_a^2 \cdot \omega_m} = 0.88.
  \]

  For the point (i)

  \[
  \eta_m(50, 22) = \frac{1}{1 + \frac{0.05 \cdot 22}{0.25^2 \cdot 50}} = 0.74,
  \]

  \[
  \eta_m(50, -22) = 1 - \frac{0.05 \cdot 22}{0.25^2 \cdot 50} = 0.65,
  \]

  \[
  P_l = 0.05 \cdot \left( \frac{22}{0.25} \right)^2 = 387 \text{ W}.
  \]

  In the field weakening region, for the point (ii),

  \[
  \eta_m(300, 8) = \frac{1}{1 + \frac{0.05 \cdot 8}{0.25^2 \cdot 300}} = 0.98,
  \]

  \[
  \eta_m(300, -8) = 1 - \frac{0.05 \cdot 8}{0.25^2 \cdot 300} = 0.98,
  \]

  \[
  P_l = 0.05 \cdot \left( \frac{8}{0.25} \right)^2 = 51 \text{ W}.
  \]

  For the given values, the approximation of mirroring the efficiency is better as the losses decrease.
Problem 4.11

Using the PMSM model 4.40–4.43 calculate a static control law, i.e., a selection of reference values $I_d$, $I_q$ as a function of torque and speed, such that the stator current intensity is minimized. Do the calculation in the case (i) $I_s \leq I_{max}$, $U_s \leq U_{max} = mU_m$ (maximum torque region) and (ii) when the voltage constraint is active (flux weakening region). Neglect the stator resistance $R_s$ and consider a machine with $p = 1$. The stator current and voltage intensities are defined as

$$I_s^2 = I_d^2 + I_q^2, \quad U_s^2 = U_q^2 + U_d^2.$$ 

Evaluate the base speed.

Solution

If $R_s$ is neglected, the static counterparts of (4.26)-(4.28) are

$$U_q = \omega_m \cdot (\varphi_m + L_s \cdot I_d),$$
$$U_d = -\omega_m \cdot L_s \cdot I_q, $$
$$T'_m = \frac{2}{3}T_m = \varphi_m \cdot I_q.$$ 

To obtain the desired torque, set $I_q = T'_m/\varphi_m$. To minimize $I_s$ without constraints, $I_d = 0$. Under these conditions, $I_s = I_q = T'_m/\varphi_m$ and

$$U_s^2 = \omega_m^2 \cdot \left(\varphi_m^2 + \left(\frac{L_s \cdot T'_m}{\varphi_m}\right)^2\right).$$ 

Such a situation is valid in (i), i.e., if $I_s \leq I_{max}$, i.e., if $T'_m \leq \varphi_m \cdot I_{max}$, and if

$$U_s^2 = \omega_m^2 \cdot \left(\varphi_m^2 + \left(\frac{L_s \cdot T'_m}{\varphi_m}\right)^2\right) \leq U_{max}^2.$$ 

The base speed is obtained from the intersection of the latter limits, i.e., for

$$\omega_b = \frac{U_{max}}{\sqrt{\varphi_m^2 + (L_s \cdot I_{max})^2}}.$$ 

If a negative $I_d$ is allowed, points above the base speed are obtained. In case (ii), $U_s = U_{max}$ and $T'_m = \varphi_m \cdot I_q$ and $I_d$ is calculated (a second-order equation is obtained) as

$$I_d = \sqrt{\left(\frac{U_{max}}{L_s \omega_m}\right)^2 - \left(\frac{T'_m}{\varphi_m}\right)^2 \frac{\varphi_m}{L_s}}.$$
Problem 4.12

Evaluate the torque limit curve for a PMSM both (i) in the maximum torque region and (ii) in the flux weakening region, see Problem 4.11, assuming \( R_s = 0 \). Evaluate the transition curve between these two regions. Assume that \( \varphi_m > L_s I_{max} \) (why is that important?).

- Solution

The torque limit in the maximum torque region is simply \( T'_{max} = \varphi_m I_{max} \).
In the flux weakening region, the torque limit is generally lower than \( \varphi_m I_{max} \) and is calculated using a graphical construction. The maximum current ("I") curve is a circle in the \( I_d-I_q \) plane, with center at the origin and radius \( I_{max} \). The maximum voltage ("U") curve is a circle with center \([−\varphi_m/L_s, 0]\) and radius \( U_{max}/(\omega_m L_s) \). Under the assumption that \( \varphi_m > L_s I_{max} \), the center of U-curve is found outside the I-curve.

Thus the largest value of \( I_q \) that fulfills both constraints is where the two curves I and U intersect. In this case, the coordinates of the intersection are

\[
I_d = \frac{\left( \frac{U_{max}}{\omega_m} \right)^2 - \varphi_m^2 - L_s^2 I_{max}^2}{2 \cdot \varphi_m \cdot L_s},
\]

\[
I_q = \sqrt{I_{max}^2 - \frac{1}{4 \cdot \varphi_m^2 \cdot L_s^2} \left( \left( \frac{U_{max}}{\omega_m} \right)^2 - \varphi_m^2 - (L_s \cdot I_{max})^2 \right)^2},
\]

from whence \( T_{max}(\omega_m) = \varphi_m \cdot I_q(\omega_m) \).

The maximum speed at which the maximum torque is null is

\[
\omega_{max} = \frac{U_{max}}{\varphi_m - L_s \cdot I_{max}}.
\]

The transition curve is the locus of the torque points that can be still achieved with \( I_d = 0 \). It is given by the intersection of U-curve with the y-axis, resulting in

\[
L_s^2 \cdot I_q^2 = \left( \frac{U_{max}}{\omega_m} \right)^2 - \varphi_m^2,
\]

that is the transition curve sought with \( T_{max}' = \varphi_m I_q \). In particular, for \( I_q = I_{max} \), obtain the base speed as

\[
\omega_b^2 = \frac{U_{max}^2}{\varphi_m^2 + L_s^2 \cdot I_{max}^2}.
\]

Problem 4.13

Using the same assumptions as in Problem 4.11, evaluate the maximim power curve as a function of speed.
Solution

The general expression for the maximum power is

\[ P_{\text{max}} = \omega_m \cdot T_{\text{max}}. \]

Let us calculate the maximum of \( P_{\text{max}} \).

\[
\frac{dP_{\text{max}}}{d\omega_m} = 0 \Rightarrow T_{\text{max}} + \omega_m \cdot \frac{dT_{\text{max}}}{d\omega_m} = 0.
\]

But \( T_{\text{max}} = \varphi_m \cdot I_q \) and \( I_q^2 = f(X) \) as given by Problem 4.11, where \( X = \frac{U_{\text{max}}}{\omega_m} \). Consequently, the maximum condition is given by

\[
2 \cdot I_q \cdot \frac{dI_q}{d\omega_m} = \frac{df}{dX} \cdot X \Rightarrow \frac{df}{dX} = 2 \cdot \frac{I_q}{2 \cdot I_q} \cdot X.
\]

After having calculated the derivative \( df/dX \), obtain a 2nd-order equation in the variable \( X \), whose solution is \( X^2 = \varphi_m^2 - L_s^2 I_{\text{max}}^2 \), from whence

\[
\omega_p = \frac{U_{\text{max}}}{\sqrt{\varphi_m^2 - L_s^2 I_{\text{max}}^2}}.
\]

For this speed,

\[
f = 4 \cdot L_s^2 \cdot I_{\text{max}}^2 (\varphi^2 - L_d^2 I_{\text{max}}^2) = 4 \cdot L_s^2 \cdot I_{\text{max}}^2 \left( \frac{U_{\text{max}}}{\omega_m} \right)^2,
\]

and finally \( T_m = \frac{U_{\text{max}} \cdot I_{\text{max}}}{\omega_m} \) or \( P_{\text{max}} = U_{\text{max}} \cdot I_{\text{max}} \).

\[ \text{Problem 4.14} \]

Equation 4.42 is only valid when \( L_d = L_q = L_s \). In the general case in which \( L_d \neq L_q \), the correct equation is

\[
T_m = \frac{3}{2} \cdot p \cdot I_q \cdot (\varphi_m - p \cdot (L_q - L_d) \cdot I_d).
\]

Consider again Problem 4.11 and derive a static control law \( I_d, I_q \) that minimizes the current intensity (MTPA), assuming that the constraints over current and voltage are not active, for a motor where \( p = 4, R_s = 0.07 \Omega, L_q = 5.4 \cdot 10^{-3} \text{H}, L_d = 1.9 \cdot 10^{-3} \text{H}, \varphi_m = 0.185 \text{Wb} \) and \( T = 50 \text{Nm} \). Then compare the result with that obtained for \( \Delta L = L_q - L_d = 0 \).

Solution
To evaluate $I_d$, a procedure similar to that of Problem 4.11 is used. Now minimize $I_d^2 + I_q^2$, subject to the condition

$$\varphi_m \cdot I_q = \Delta L \cdot I_d \cdot I_q = T_m'.$$

This is a parameter optimization problem in the two parameters $I_d, I_q$. Build the Hamiltonian

$$H = I_d^2 + I_q^2 + \mu \cdot (\varphi_m \cdot I_q - \Delta L \cdot I_d \cdot I_q - T_m').$$

Pontryagin’s Minimum Principle reads

$$\frac{dH}{dI_d} = 2 \cdot I_d - \mu \cdot \Delta L \cdot I_q = 0,$$

$$\frac{dH}{dI_q} = 2 \cdot I_q + \mu \cdot \varphi_m - \mu \cdot \Delta L \cdot I_d = 0,$$

from whence

$$\Delta L \cdot I_d^2 - \varphi_m \cdot I_d - \Delta L \cdot I_q^2 = 0,$$

that is, a quadratic equation is found. Now combine this equation with the torque equation to have $I_d$ and $I_q$ as a function of $T_m'$:

$$\Delta L \cdot (T_m')^2 = (\Delta L \cdot I_d^2 - \varphi_m \cdot I_d) \cdot (\varphi_m - \Delta L \cdot I_d)^2 =$$

$$= \Delta L^3 \cdot I_d^3 - 3 \cdot \varphi_m \cdot \Delta L^2 \cdot I_d^2 + 3 \cdot \varphi_m^2 \cdot \Delta L \cdot I_d^2 - \varphi_m^3 \cdot I_d =$$

$$= I_d \cdot (\Delta L \cdot I_d - \varphi_m)^3.$$

For the numerical case, $\Delta L = 3.5 \cdot 10^{-3}$ H, $T_m' = 50/(3/2 \cdot 4) = 8.33$ Nm, thus

$$0 = (3.5 \cdot 10^{-3})^3 \cdot I_d^3 - 3 \cdot 0.185 \cdot (3.5 \cdot 10^{-3})^2 \cdot I_d^2 +$$

$$+ 3 \cdot 0.185^2 \cdot 3.5 \cdot 10^{-3} \cdot I_d^2 - 0.185^3 \cdot I_d -$$

$$+ 3.5 \cdot 10^{-3} \cdot 8.33^2 \Rightarrow I_d = -17 \text{ A},$$

$$0 = 3.5 \cdot 10^{-3} \cdot 17^2 + 0.185 \cdot 17 - 3.5 \cdot 10^{-3} \cdot I_q^2 \Rightarrow I_q = 34 \text{ A}.$$

Verify that

$$T_m = 4 \cdot \frac{3}{2} \cdot 17 \cdot 34 \cdot (0.185 + 3.5 \cdot 10^{-3} \cdot 17) = 50 \text{ Nm}.$$

With $\Delta L = 0$, one would have obtained $I_d = 0$ and $I_q = 50/4/(3/2)/0.185 = 45 \text{ A}.$

**Problem 4.15**

Calculate the torque characteristic curve $T_m(\omega_m, U_s)$ of a PMSM having the following characteristics: $R_s = 0.2 \Omega, L = 0.003 \text{ H}, \frac{2}{3} \omega_m = 0.89 \text{ Wb}, p = 1$, for a voltage intensity (see definition in Problem 4.11) $U_s = 30 \text{ V}$. Derive an affine approximation of the DC-motor type, $T_{m,lin}(\omega_m, U_s)$. Evaluate the torque error $\varepsilon(\omega_m) \triangleq U_s^2(T_m) - U_s^2(T_{m,lin})$ and calculate its maximum value.
Solution

If \( L_d = L_q \), the MTPA (maximum torque per ampere) control is (see Problem 4.11) \( I_d = 0, I_q = T_m'/\varphi_m \). Consequently, the voltage is

\[
U_q = \omega_m \cdot \varphi_m + R_s \cdot I_q = \omega_m \cdot \varphi_m + \frac{R_s \cdot T_m'}{\varphi_m},
\]

\[
U_d = -\omega_m \cdot L_s \cdot I_q = -\omega_m \cdot \frac{L_s \cdot T_m'}{\varphi_m}.
\]

The torque characteristic curve \( T_m' = T_m'(\omega_m, U_s) \) for a given \( U_s = \sqrt{U_d^2 + U_q^2} \) is given by

\[
U_s^2 = \omega_m^2 \cdot \varphi_m^2 + \frac{R_s^2 \cdot (T_m')^2}{\varphi_m^2} + 2 \cdot \omega_m \cdot R_s \cdot T_m' + \omega_m^2 \cdot \frac{L_s^2 \cdot (T_m')^2}{\varphi_m^2} \Rightarrow
\]

\[
(\varphi_m \cdot U_s)^2 - \omega_m^2 \cdot \varphi_m^4 = (T_m')^2 \cdot \left( (\omega_m \cdot L_s)^2 + R_s^2 \right) + 2 \cdot R_s \cdot \omega_m \cdot \varphi_m^2 \cdot T_m'.
\]

For \( \omega_m = 0 \), the breakaway torque is

\[
T_{br}' = \frac{U_s \cdot \varphi_m}{R_s}.
\]

The zero torque speed is

\[
\omega_0 = \frac{U_s}{\varphi_m}.
\]

The affine approximation of the characteristic curve is

\[
T_{lin}'(\omega_m, U_s) = T_{br}' + \left. \frac{\partial T_m'}{\partial \omega_m} \right|_{\omega_m=0} \cdot \omega_m
\]

where \( T_m' = T_m'(\omega, U_s) \) is derived from the equation above. From a comparison with the DC-motor characteristic curve, one can make the equivalence \( \kappa_a = \varphi_m \), and

\[
T_{lin}'(\omega_m, U_s) = \frac{\varphi_m}{R_s} \cdot U_s - \frac{\varphi_m^2}{R_s} \cdot \omega_m.
\]

At \( \omega_m = \omega_0 \) the approximated torque is

\[
T_0' = \frac{\varphi_m}{R_s} \cdot U_s - \frac{\varphi_m}{R_s} \cdot U_s = 0.
\]

Thus, at \( \omega_0 \) the error is zero with respect to the nonlinear characteristic curve. To generally evaluate this error, calculate \( U_s(T_m) \) from the nonlinear curve and \( U_s(T_{m,lin}) \) from the affine curve. One obtains

\[
U_s^2(T_m) - U_s^2(T_{m,lin}) = \varepsilon(\omega_m).
\]

This term can be calculated using the results above, such that
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\[ \varepsilon(\omega_m) = \left( R_s^2 + \omega_m^2 \cdot L_s^2 \right) \frac{(T'_m)^2}{\varphi_m^2} + 2 \cdot \omega_m \cdot R_s \cdot T'_m - \]

\[ + \left( 1 + \frac{\omega_m^2 \cdot L_s^2}{R_s^2} \right) \cdot (U_s - \omega_m \cdot \varphi_m)^2 - 2 \cdot \omega_m \cdot \varphi_m \cdot (U_s - \omega_m \cdot \varphi_m) \]

\[ = \frac{R_s^2 \cdot (T'_m)^2}{\varphi_m^2} + \frac{\omega_m^2 \cdot L_s^2 \cdot (T'_m)^2}{\varphi_m^2} + \frac{\omega_m^2 \cdot \varphi_m^2}{\varphi_m^2} + 2 \cdot \omega_m \cdot R_s \cdot T'_m - U_s^2 - \]

\[ + \left( \frac{\omega_m \cdot L_s}{R_s} \right)^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2 \]

\[ = - \left( \frac{\omega_m \cdot L_s}{R_s} \right)^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2, \]

which is zero for \( \omega_m = 0 \) and \( \omega_m = \omega_0 \). Define

\[ E(\omega_m) = \omega_m^2 \cdot (U_s - \omega_m \cdot \varphi_m)^2. \]

The maximum value for \( E(\omega_m) \) is obtained by differentiating w.r.t. \( \omega_m \):

\[ \frac{dE}{d\omega_m} = 2 \cdot \omega_m \cdot (U_s - \omega_m \cdot \varphi_m)^2 + 2 \cdot \omega_m^2 \cdot (U_s - \omega_m \cdot \varphi_m) \cdot (\omega_m \cdot \varphi_m) = 0 \]

\[ \Rightarrow U_s - \omega_m \cdot \varphi_m = \omega_m \cdot \varphi_m \]

\[ \Rightarrow \omega_{\text{max,E}} = \frac{U_s}{2 \cdot \varphi_m} = \frac{\omega_0}{2}. \]

The maximum error is

\[ \varepsilon(\omega_{\text{max,E}}) = \left( \frac{U_s^2 \cdot L_s}{4 \cdot \varphi_m \cdot R_s} \right)^2. \]

In relative terms

\[ \frac{\varepsilon(\omega_{\text{max,E}})}{U_s^2} = \left( \frac{U_s \cdot L_s}{4 \cdot \varphi_m \cdot R_s} \right)^2. \]

With the numerical data \( \varepsilon(\omega_{\text{max,E}}) = 14.4 \text{ V}^2 \) or in relative terms \( 14.4/30^2 = 1.6\% \).

**Problem 4.16**

A simple thermal model of an electric machine reads

\[ C_{t,m} \cdot \frac{d}{dt} \vartheta_m(t) = P_l(t) - \frac{\vartheta_m(t) - \vartheta_a}{R_{th}}, \]

where \( C_{t,m} \) is an equivalent thermal capacity, \( R_{th} \) is an equivalent thermal resistance, \( \vartheta_m(t) \) is the relevant motor temperature, and \( \vartheta_a \) is the external temperature. Derive the current limitation to \( I_{\text{max}} \) from thermal considerations using the models of Sect. 4.3.3 (DC motor). How would the result change if other losses of the type \( \beta \cdot \omega_m \) (iron losses, mechanical losses) were taken into account?
Fig. 10.7. Visualization of the results of Problem 4.15: (a) Absolute error over \( \omega_{\text{max,E}} \), (b) exact (—) and linear (—−) torque curve.

• Solution

Behind the limitation \( I_a = I_{\text{max}} \) leading to \( T_m = T_{\text{max}} \) (for \( \omega_m < \omega_b \)) there is a temperature limitation \( \vartheta_m = \vartheta_{\text{max}} \). Consider the DC motor model. Here the only loss is due to ohmic losses

\[
P_l = R_a \cdot I_a^2.
\]

The motor (windings) temperature varies according to this power dissipated into heat and according to heat exchange to the ambient, so

\[
C_t \cdot \frac{d\vartheta_m}{dt} = P_l - \frac{\vartheta_m - \vartheta_a}{R_{th}} = R_a \cdot I_a^2 - \frac{\vartheta_m - \vartheta_a}{R_{th}}.
\]  \hspace{1cm} (10.24)

To guarantee that \( \vartheta_m < \vartheta_{\text{max}} \), it should be

\[
R_a \cdot I_a^2 < \alpha \cdot (\vartheta_{\text{max}} - \vartheta_a),
\]

or

\[
I_a < \sqrt{\frac{\vartheta_{\text{max}} - \vartheta_a}{R_{th} R_a}} = \text{const.} = I_{\text{max}}.
\]

If other losses of the type \( \beta \cdot \omega_m \) are considered (the factor \( \beta \) could in turn be dependent on \( \omega_m \)), the condition on temperature reads
\[ R_a \cdot I_a^2 + \beta \cdot \omega_m < \frac{\vartheta_m - \vartheta_a}{R_{th}}, \]

from whence

\[ I_a < \sqrt{\frac{\vartheta_{\text{max}} - \vartheta_a - \beta \cdot \omega_m}{R_{th}R_a}} = I_{\text{max}}(\omega_m). \]

Thus the maximum torque for \( \omega_m < \omega_b \) would decrease with \( \omega_m \).

**Problem 4.17**

Derive a (simplified) rule to express peak torque limits of an electric machine as a function of application time. Use the result of Problem 4.16.

- **Solution**

Relax the condition on stationary temperature. The solution to the ODE (11.1) is then

\[ \vartheta_m(t) = \vartheta_a + (\vartheta_{\text{stat}} - \vartheta_a) \cdot (1 - e^{-t}), \]

where \( \vartheta_{\text{stat}} = \vartheta_a + R_{th}R_a \cdot I_a^2 \) and \( \tau = C_{t,m}R_{th} \). Impose that \( \vartheta(t) = \vartheta_{\text{max}} \) and obtain \( I_{\text{max}} \) as a function of time,

\[ \vartheta_{\text{max}} - \vartheta_a = R_{th}R_a \cdot I_{\text{max}}^2(t) \cdot (1 - e^{-t}) \]

\[ \Rightarrow \quad I_{\text{max}}(t) = \sqrt{\frac{(\vartheta_{\text{max}} - \vartheta_a)}{R_{th}R_a \cdot (1 - e^{-t})}} \]

\[ \Rightarrow \quad T_{\text{max}}(t) = \kappa_a \cdot I_{\text{max}}(t) \]

For \( t \to \infty \), one finds

\[ T_{\text{max}} = \kappa_a \cdot \sqrt{\frac{\vartheta_{\text{max}} - \vartheta_{\text{amb}}}{R_{th}R_a}}, \]

which is the result of Problem 4.16.

**Problem 4.18**

One PMSM has the following design parameter: external diameter \( d_1 = 0.145 \text{ m} \), weight \( m_1 = 14 \text{ kg} \), length \( l_1 = 0.06 \text{ m} \). At 5500rpm it delivers a maximum torque \( T_1 = 12 \text{ Nm} \). Predict the power \( P_2 \) and the weight \( m_2 \) for a similarly designed machine with a diameter \( d_2 = 0.2 \text{ m} \) and a length \( l_2 = 0.2 \text{ m} \). Compare the cases in which the design is made (i) on the basis of constant tangential stress and peripheral speed, or (ii) of constant speed.

- **Solution**
Case (i). The power of machine 1 is

\[ P_1 = \frac{5500 \cdot \pi \cdot 12}{30} = 6.9 \text{ kW}. \]

The mean rotor speed is

\[ c_m = \frac{\omega_1 \cdot d_1}{2} = \frac{5500 \cdot \pi \cdot 0.145}{2} = 41.8 \text{ m/s}. \]

The mean pressure is

\[ p_{me} = \frac{T_1}{2 \cdot V_1} = \frac{12}{2 \cdot (\frac{\pi \cdot 0.145^2}{4} \cdot 0.06)} = 6 \text{ kPa}. \]

Indeed, \( P_1 = \pi \cdot d_1 \cdot l_1 \cdot p_{me} \cdot c_m = \pi \cdot 0.145 \cdot 6 \cdot 10^3 \cdot 41.8 = 6.9 \text{ kW}. \)

For machine 2,

\[ P_2 = \pi \cdot 0.2 \cdot 0.2 \cdot 6 \cdot 10^3 \cdot 41.8 = 31.5 \text{ kW}, \]

\[ m_2 = m_1 \cdot \left( \frac{d_2}{d_1} \right)^2 \cdot \frac{l_2}{l_1} = 14 \cdot \left( \frac{0.2}{0.145} \right)^2 \cdot \frac{0.2}{0.06} = 89 \text{ kg}, \]

assuming constant density.

Case (ii). What changes now is that

\[ P = \omega \cdot p_{me} \cdot \pi \cdot \frac{d^2}{2} \cdot l. \]

For machine 2,

\[ P_2 = \frac{\pi \cdot 5500 \cdot 0.2^2}{30} \cdot 6 \cdot 10^3 \cdot \pi \cdot 0.2^2 = 43 \text{ kW}, \]

while \( m_2 \) is unchanged. The specific power is the same for machine one and two,

\[ \frac{6.9 \cdot 10^3}{14} = \frac{43 \cdot 10^3}{89} = 0.49 \text{ kW/kg}, \]

**Problem 4.43**

Evaluate the specific power of a motor and inverter assembly, knowing that \((\frac{P}{m})_{\text{motor}} = 1.2 \text{ kW/kg}\) and \((\frac{P}{m})_{\text{inverter}} = 11 \text{ kW/kg}\).

**Solution**

The specific power of the motor system is simply

\[ \left( \frac{P}{m} \right) = \frac{1}{(\frac{P}{m})_{\text{motor}}} + \frac{1}{(\frac{P}{m})_{\text{inverter}}} = \frac{1}{1.2} + \frac{1}{11} = 1.08 \text{ kW/kg}. \]
Consider an APU for a series hybrid. Given the engine model

\[ P_f = \frac{P_e + P_0}{e} \]

\[ \frac{1}{e} = 5.07 - 0.0117 \cdot \omega_e + 1.50 \cdot 10^{-5} \cdot \omega_e^2 = a_1 + b_1 \cdot \omega_e + c_1 \cdot \omega_e^2 \]

\[ P_0 = -1.22 \cdot 10^3 + 31.7 \cdot \omega_e + 0.421 \cdot \omega_e^2 = a_2 + b_2 \cdot \omega_e + c_2 \cdot \omega_e^2 \]

\[ T_{max} = 96.9 + 1.35 \cdot \omega_e - 0.0031 \cdot \omega_e^2 = h \cdot \omega_e^2 + g \cdot \omega_e + f, \]

and a generator model with constant efficiency \( \eta_g = 0.92 \), derive an OOL structure \( \hat{\omega}(P_g) \). Then calculate the operating points for (i) \( P_g = 10 \text{ kW} \), (ii) \( P_g = 40 \text{ kW} \), and (iii) \( P_g = 60 \text{ kW} \).

**Solution**

The problem is finding \( \omega_g \) for each \( P_g = \eta_g \cdot P_e \) such that \( P_f \) is minimized. By differentiating \( P_f \) with respect to \( \omega_e = \omega_g \) one obtains

\[ \frac{dP_f}{d\omega_g} = b_2 + 2 \cdot c_2 \cdot \omega_g + b_1 \cdot P_e + 2 \cdot c_1 \cdot \omega_g \cdot P_e = 0, \]

thus

\[ \hat{\omega} = -\frac{b_1 \cdot P_e + b_2}{2 \cdot (c_1 \cdot P_e + c_2)}. \]  

(10.25)

For the case (i), \( \hat{\omega} = 81.7 \text{ rad/s} \), which is below the minimum APU speed, therefore \( \hat{\omega} = 1000 \text{ rpm} = 104.7 \text{ rad/s} \).

For the case (ii), \( \hat{\omega} = 222 \text{ rad/s} \). The torque is \( T_g = 196 \text{ Nm} \), which is below the maximum torque at the speed \( \hat{\omega} \). Thus the operating point is admissible.

For the case (iii), \( \hat{\omega} \) would be equal to 261 rad/s and the torque would be 250 Nm, while the maximum torque at that speed is 239 Nm. Thus a different calculation should be used: find \( \hat{\omega} \) such that

\[ (96.9 + 1.35 \cdot \hat{\omega} - 0.0031 \cdot \hat{\omega}^2) \cdot \hat{\omega} = \frac{60 \cdot 10^3}{0.92}. \]

The solution is \( \hat{\omega} = 279 \text{ rad/s} \).

**Problem 4.21**

For the APU model of Problem 4.20, find a piecewise affine approximation 
\[ P_f = a + b \cdot P_g \]. Evaluate the error with respect to the nonlinear model, for (i) \( P_g = 10 \text{ kW} \), (ii) \( P_g = 40 \text{ kW} \), and (iii) \( P_g = 60 \text{ kW} \).
Fig. 10.8. Visualization of the results of problem 20: (a) true OOL, (b) approximated OOL, (c) blue: true, green: approximated with $a$, $b$ and $c$; red: $a$, $b$ and $c$ approximated with $a_1$, $a_2$ and $c$.

- Solution

Three discontinuity points are identified: $P_e = 0$, $P_e = P_1$ such that $\hat{\omega} = 1000$ rpm, $P_e = P_2$ such that the engine torque limitation is active, and $P_e = P_{\text{max}}$. The value $P_1$ is calculated as the root of the equation

$$\hat{\omega}(P_1) = 1000 \text{ rpm} \quad \Rightarrow \quad b_1 \cdot P_1 + b_2 = -2 \cdot \frac{1000 \cdot \pi}{30} \cdot (c_1 \cdot P_1 + c_2)$$

$$\Rightarrow \quad P_1 = 14 \text{ kW}.$$ 

The value $P_2$ is calculated as the root of the equation
The value of \( P_{\text{max}} \) is given by finding the stationary point of

\[
P(\omega_e) = (h \cdot \omega_e^2 + g \cdot \omega_e + f) \cdot \omega_e \quad \Rightarrow \quad \frac{dP}{d\omega_e} = 3 \cdot h \cdot \omega_e^2 + 2 \cdot g \cdot \omega_e + f = 0,
\]

which gives \( \omega = \omega_e = 324 \text{ rad/s} \) and \( P_{\text{max}} = 68.3 \text{ kW} \).

The value of \( P_f \) at \( P_e = 0 \) is

\[
P_f(0^+) = c_2 \cdot \left( \frac{1000 \cdot \pi}{30} \right)^2 + b_2 \cdot \frac{1000 \cdot \pi}{30} + a_2 = 6.7 \text{ kW}.
\]

The value of \( P_f \) at \( P_e = P_1 \) is

\[
(c_1 \cdot 14 \cdot 10^3 + c_2) \cdot \left( \frac{1000 \cdot \pi}{30} \right)^2 + (b_1 \cdot 14 \cdot 10^3 + b_2) \cdot \frac{1000 \cdot \pi}{30} + a_1 \cdot 14 \cdot 10^3 + a_2 = 62.9 \text{ kW}.
\]

The value of \( P_f \) at \( P_e = P_2 \) is calculated after having calculated the

\[
\omega_e = \frac{b_1 \cdot P_2 + b_2}{2 \cdot (c_1 \cdot P_2 + c_2)} = 256 \text{ rad/s},
\]

with \( b = -694 \) and \( c = 1.35 \) (and \( a = 3.13 \cdot 10^5 \)). Then

\[
P_f = c \cdot \omega_e^2 + b \cdot \omega_e + a = 224.4 \text{ kW}.
\]

The engine power can be replaced by \( P_e = P_g / \eta_g \) to get the affine relationship between fuel power and APU power.

For the operating point (i), \( P_g = 10 \text{ kW} \) and \( P_e = 10.9 \text{ kW} \). The exact \( P_f \) is 50.3 kW. The approximated value is

\[
6.7 \cdot 10^3 + \frac{62.9 - 6.7}{14} \cdot 10.9 \cdot 10^3 = 50.5 \text{ kW},
\]

with a 2% error.
For the operating point (ii), \( P_g = 40 \text{ kW} \) and \( P_e = 43 \text{ kW} \), the exact \( P_f = 166.4 \text{ kW} \). The approximated value is
\[
62.9 \cdot 10^3 + \frac{224.4 - 62.9}{62 - 14} \cdot (43 \cdot 10^3 - 14 \cdot 10^3) = 162.2 \text{ kW},
\]
with a 2.5% error.

For the operating point (iii), \( P_g = 60 \text{ kW} \) and \( P_e = 65.2 \text{ kW} \), the exact \( P_f = 234.7 \text{ kW} \). The approximated value is
\[
224.4 \cdot 10^3 + \frac{248.7 - 224.2}{68.3 - 62} \cdot (65.2 \cdot 10^3 - 62 \cdot 10^3) = 236.8 \text{ kW},
\]
with a 1% error.

**Problem 4.22**

Propose an algorithm to calculate the OOL of an engine for a combined hybrid from the data \( w_4x \) (APU speed breakpoint vector), \( T_4x \) (APU torque breakpoint vector), \( T_{\text{max}}(w_4x) \) (APU maximum torque), and \( \text{mfuel}(w_4x,T_4x) \) (engine consumption map).

- **Solution**

  FOR \( P = 0 \) TO \( \max(w_4x \cdot T_{\text{max}}) \)
  
  WHILE \( w = w_4x \) AND \( P/w < T_{\text{max}}(w) \)
  
  \[ P_f(w) = H_l \cdot \text{mfuel}(w,T); \]
  
  \[ h(w) = P/P_f(w); \]
  
  END
  
  \[ w_{\text{opt}}(P) = \arg \min(h(w)); \]
  
  END

**Problem 4.23**

Find an optimal design for a range extender working at a stationary operating point, i.e., select optimal values for the displacement volume \( V_d \) and the speed \( \omega_e \) of the engine, neglecting stop-and-start effects and making the following approximations. The IC engine is modeled with a Willans line
\[
p_{\text{me}} = e \cdot p_{m,f} - (p_{\text{me}0} + p_{\text{me}2} \cdot \omega_e^2)
\]
with \( e = 0.4 \), \( p_{\text{me}0} = 1.5 \cdot 10^5 \text{ Pa} \), \( p_{\text{me}2} = 1.4 \text{ Pa} \cdot \text{s}^2 \). The generator is a DC machine with constant armature resistance \( R_a = 0.2 \Omega \) and \( \kappa_a = 0.5 \). The battery is modeled as an internal voltage source \( U_{\text{oc}} = 180 \text{ V} \) with an internal resistance \( R_b = 0.3 \Omega \). The overall system efficiency
\[
\eta_{\text{ov}} = \frac{U_{\text{oc}} \cdot I_a}{\dot{m}_f \cdot H_l}
\]
should be optimal. The nominal power $P_e$ of the range extender should be greater than 30 kW and the brake mean effective pressure of the engine smaller than 9 bar. The design parameters can be chosen between the following boundaries: $V_d \in [0.5, 2]$ l, $\omega_e \in [U_{oc}/\kappa_a, 600]$ rad/s.

- Solution

The generator current is

$$ I_a = \frac{T_a}{\kappa_a} = \frac{\kappa_a \cdot \omega_e - U_{oc}}{R}, $$

where $R = R_a + R_b$. The fuel consumption rate is

$$ \dot{m}_f \cdot H_f = \frac{\omega_e \cdot T_e}{e} + \frac{\omega_e \cdot (p_{me0} + p_{me2} \cdot \omega_e^2) \cdot V_d}{4 \cdot \pi}. $$

Thus the overall efficiency is

$$ \eta_{ov} = \frac{e \cdot U_{oc} \cdot (\kappa_a \cdot \omega_e - U_{oc})}{\kappa_a \cdot \omega_e \cdot (\kappa_a \cdot \omega_e - U_{oc}) + \omega_e \cdot (p_{me0} + p_{me2} \cdot \omega_e^2) \cdot \frac{V_d}{R}} = \eta_{ov}(\omega_e, V_d). $$

The power is also expressed as a function of $\omega_e$ and $V_d$ as

$$ P_e = I_a \cdot U_a = I_a \cdot \kappa_a \cdot \omega_e = \frac{(\kappa_a \cdot \omega_e - U_{oc}) \cdot \kappa_a \cdot \omega_e}{R} $$

while the brake m.e.p. is

$$ p_{me} = \frac{4 \cdot \pi \cdot T_e}{V_d} = \frac{4 \cdot \pi \cdot \kappa_a \cdot (\kappa_a \cdot \omega_e - U_{oc})}{V_d \cdot R}. $$

It can be seen by inspection that the efficiency decreases for increasing values of $V_d$. The lower $V_d$ is obtained at the intersection of the two conditions on $P_e$ and $p_{me}$. The former gives

$$ \kappa_a^2 \cdot \omega_e^2 - \kappa_a \cdot U_{oc} \cdot \omega_e - R \cdot P_g = 0 \quad \Rightarrow \quad \omega_e = 484 \text{ rad/s}. $$

The latter condition gives

$$ V_d = \frac{4 \cdot \pi \cdot (\kappa_a^2 \cdot \omega_e - \kappa_a \cdot U_{oc})}{R \cdot p_{me}} = \frac{4 \cdot \pi}{0.5 \cdot 9 \cdot 10^3} \cdot (0.5^2 \cdot 484 - 0.5 \cdot 180) = 0.87 \text{ l}. $$

At this speed, the current is $(0.5 \cdot 484 - 180)/0.5 = 124 \text{ A}$, the torque is $0.5 \cdot 124 = 62 \text{ Nm}$, the engine power is thus $484 \cdot 62 = 30 \text{ kW}$, the fuel power is

$$ \frac{30 \cdot 10^3}{0.4} + 484 \frac{1.5 \cdot 10^5 + 1.4 \cdot 484^2}{4 \cdot \pi} \cdot 0.87 \cdot 10^{-3} = 115 \text{ kW}, $$

(engine efficiency 26%), the armature voltage is $0.5 \cdot 484 - 0.2 \cdot 124 = 217 \text{ V}$, the battery power is $217 \cdot 124 = 26.9 \text{ kW}$ (generator efficiency 26.9/30 = 90%), the battery internal power is $180 \cdot 124 = 22.3 \text{ kW}$ (battery internal efficiency 83%), the overall efficiency is 19%, as it can be verified using the expression calculated for $\eta_{ov}(\omega_e, V_d)$.  


**Problem 4.24**

In Problem 4.23, set the engine displacement volume to \( V_d = 1.6 l \) and find the optimal operating speed that maximizes the overall efficiency while respecting the two constraints on \( P_e \) and \( p_{me} \).

- **Solution**

  The overall efficiency as a function of \( \omega \) is given by

  \[
  \eta_{ov} = \frac{e \cdot U_{ac} \cdot \left( \kappa_a \cdot \omega_e - U_{ac} \right)}{\kappa_a \cdot \omega_e \cdot \left( \kappa_a \cdot \omega_e - U_{ac} \right) + \omega_e \cdot \left( p_{me0} + p_{me2} \cdot \omega_e^2 \right) \cdot \frac{V_e}{2} \cdot R} = \eta_{ov}(\omega_e).
  \]

  To find the maximum efficiency, differentiate with respect to \( \omega_e \),

  \[
  \kappa_a \cdot \left( \kappa_a \cdot \omega_e \cdot \left( \kappa_a \cdot \omega_e - U_{ac} \right) + \omega_e \cdot \left( p_{me0} + p_{me2} \cdot \omega_e^2 \right) \cdot a \right) =
  \]

  \[
  \left( \kappa_a \cdot \omega_e - U_{ac} \right) \cdot \left( 3 \cdot a \cdot p_{me2} \cdot \omega_e^2 + 2 \cdot \kappa_a^2 \cdot \omega_e - \kappa_a \cdot U_{ac} + a \cdot p_{me0} \right) = 0,
  \]

  where \( a = V_d \cdot R/4/\pi \) is a constant. By manipulating the expression above, obtain the third-order equation

  \[
  2 \cdot \kappa_a \cdot a \cdot p_{me2} \cdot \omega_e^3 + (\kappa^3 - 3 \cdot a \cdot p_{me2} \cdot U_{ac}) \cdot \omega_e^2 -
  \]

  \[
  + 2 \cdot \kappa_a^2 \cdot U_{ac} \cdot \omega_e + (\kappa_a \cdot U_{ac}^2 - a \cdot U_{ac} \cdot p_{me0} \cdot \omega_e) = 0,
  \]

  whose solution is \( \omega_e = 503 \text{ rad/s} \) for \( V_d = 1.6 \cdot 10^{-3} \), \( a = 6.37^{-5} \).

  At this speed, the current is \( (0.5 \cdot 503 - 180)/0.5 = 143 \text{ A} \), the torque is \( 0.5 \cdot 143 = 71.5 \text{ Nm} \), the \( p_{me} \) is \( 4 \cdot \pi \cdot 71.5/1.6 \cdot 10^{-3} = 5.6 \text{ bar} \) (thus the constraint is not violated), the engine power is thus \( 503 \cdot 71.5 = 36 \text{ kW} \), the fuel power is

  \[
  \frac{36 \cdot 10^3}{0.4} + \frac{503}{0.4} \cdot (1.5 \cdot 10^5 + 1.4 \cdot 503^2) \cdot \frac{1.6 \cdot 10^{-3}}{4 \cdot \pi} = 171 \text{ kW}
  \]

  (engine efficiency 21%), the armature voltage is \( 0.5 \cdot 503 - 0.2 \cdot 143 = 223 \text{ V} \), the battery power is 223-143 = 31.8 kW and thus also the second constraint is not violated (generator efficiency 31.8/36 = 88%), the battery internal power is \( 180 \cdot 143 = 25.7 \text{ kW} \) (battery internal efficiency 81%), the overall efficiency is 15%, as it can be verified using the expression calculated for \( \eta_{ov}(\omega_e) \). Thus the choice of a non-optimal value for the displacement volume leads to a substantial loss in the overall efficiency.

**Batteries**

**Problem 4.25**

For a battery pack having the following characteristics: \( Q_{cell} = 5 \text{ Ah} \), \( U_{cell} = 3.14 + 1.10 \cdot \xi \) (V), \( R_{cell} = 0.005 - 0.0016 \cdot \xi \) (\( \Omega \)) under discharge and \( R_{cell} = 0.0020 \cdot \xi^2 - 0.0020 \cdot \xi + 0.0041 \) (\( \Omega \)) under charge, \( N = 96 \), calculate the electrochemical power \( P_{ech} \) for an electric power demand of 15 kW in charge and discharge, respectively, and for 20% and 90% state of charge.
Solution

\[ P_{ech} = U_{oc} \cdot I_b \quad \text{with} \quad I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{4 \cdot R_i^2} - \frac{P_b}{R_i}}. \]

For (i) \( P_b = 15 \text{kW} \) and \( q = 0.2 \),

\[ U_{oc} = (3.14 + 1.10 \cdot 0.2) \cdot 96 = 323 \text{V}, \]
\[ R_i = R_d = (0.005 - 0.0016 \cdot 0.2) \cdot 96 = 0.45 \Omega, \]
\[ I_b = \frac{323}{2 \cdot 0.45} - \sqrt{\frac{323^2}{4 \cdot 0.45^2} - \frac{15 \cdot 10^3}{0.45}} = 50 \text{A}, \]
\[ P_{ech} = 323 \cdot 50 = 16.1 \text{kW} \quad \left( \text{efficiency} = \frac{15}{16.1} = 93\% \right). \]

For (ii) \( P_b = 15 \text{kW} \) and \( q = 0.9 \),

\[ U_{oc} = (3.14 + 1.10 \cdot 0.9) \cdot 96 = 396 \text{V}, \]
\[ R_i = R_d = (0.005 - 0.0016 \cdot 0.9) \cdot 96 = 0.34 \Omega, \]
\[ I_b = \frac{396}{2 \cdot 0.34} - \sqrt{\frac{396^2}{4 \cdot 0.34^2} - \frac{15 \cdot 10^3}{0.34}} = 39 \text{A}, \]
\[ P_{ech} = 396 \cdot 39 = 15.4 \text{kW} \quad \left( \text{efficiency} = \frac{15}{15.4} = 97\% \right). \]

For (iii) \( P_b = -15 \text{kW} \) and \( q = 0.2 \),

\[ U_{oc} = 323 \text{V}, \]
\[ R_i = R_c = (0.0020 \cdot 0.04 - 0.0020 \cdot 0.2 + 0.0041) \cdot 96 = 0.36 \Omega, \]
\[ I_b = \frac{323}{2 \cdot 0.36} - \sqrt{\frac{323^2}{4 \cdot 0.36^2} + \frac{15 \cdot 10^3}{0.36}} = -44 \text{A}, \]
\[ P_{ech} = 323 \cdot (-44) = -14.3 \text{kW} \quad \left( \text{efficiency} = \frac{14.3}{15} = 95\% \right). \]

For (iv) \( P_b = -15 \text{kW} \) and \( q = 0.9 \),

\[ U_{oc} = 396 \text{V}, \]
\[ R_i = R_c = (0.0020 \cdot 0.81 - 0.0020 \cdot 0.9 + 0.0041) \cdot 96 = 0.38 \Omega, \]
\[ I_b = \frac{396}{2 \cdot 0.38} - \sqrt{\frac{396^2}{4 \cdot 0.38^2} + \frac{15 \cdot 10^3}{0.38}} = -37 \text{A}, \]
\[ P_{ech} = 396 \cdot 37 = -14.7 \text{kW} \quad \left( \text{efficiency} = \frac{14.7}{15} = 98\% \right). \]

Problem 4.26

Find a quadratic approximation for the relationship between battery power \( P_b \) and electrochemical power \( P_{ech} \). Compare the results with those of Problem 4.25.
• Solution

The relationship between power and current is

\[ I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{2 \cdot R_i^2} - \frac{P_b}{R_i}} \quad \text{or} \quad I_b = c - \sqrt{c^2 - a \cdot P_b}. \]

By expanding this function as a Taylor series, one obtains

\[ I_b(0) = 0, \]

\[ \left. \frac{dI_b}{dP_b} \right|_{P_b=0} = \frac{a}{2 \cdot \sqrt{c^2 - a \cdot P_b}} \bigg|_{P_b=0} = \frac{1}{U_{oc}}, \]

\[ \left. \frac{d^2I_b}{dP_b^2} \right|_{P_b=0} = \frac{a^2}{4 \cdot (c^2 - a \cdot P_b)^{3/2}} \bigg|_{P_b=0} = \frac{2 \cdot R_i}{U_{oc}^3}. \]

Thus

\[ \hat{I}_b = \frac{P_b}{U_{oc}} + 2 \cdot \frac{R_i}{U_{oc}^3} \cdot P_b^2. \]

For the cases of Problem 4.25 and the approximation we get

<table>
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<tr>
<th>( P_b ) (kW)</th>
<th>( q )</th>
<th>( U_{oc} ) (V)</th>
<th>( R_i ) (( \Omega ))</th>
<th>( I_b ) (A)</th>
<th>( \hat{P}_{ech} ) (kW)</th>
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<td>323</td>
<td>0.45</td>
<td>50</td>
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<td>0.9</td>
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</tr>
<tr>
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<td>323</td>
<td>0.36</td>
<td>-44</td>
<td>-14.3</td>
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<tr>
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<td>0.9</td>
<td>369</td>
<td>0.38</td>
<td>-37</td>
<td>-14.7</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>( \hat{I}_b ) (A)</th>
<th>( \hat{P}_{ech} ) (kW)</th>
<th>error (%)</th>
</tr>
</thead>
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<td>5</td>
</tr>
<tr>
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<td>16.0</td>
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</tr>
<tr>
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<td>-13.4</td>
<td>6</td>
</tr>
<tr>
<td>-35.1</td>
<td>-13.9</td>
<td>5</td>
</tr>
</tbody>
</table>

**Problem 4.27**

One couple of electrodes for a lithium cell has the following characteristics: Negative electrode (graphite): capacity \( q_{rev,n} = 340 \text{ mAh/g} \), potential \( U_n = 0.25 \text{ V} \), density \( \rho_n = 2.2 \text{ g/cm}^3 \), thickness \( s_n \leq 80 \mu \text{m} \). Positive electrode (LiCoO\(_2\)): capacity \( q_{rev,p} = 140 \text{ mAh/g} \), potential \( U_p = 3.85 \text{ V} \), density \( \rho_p = 4 \text{ g/cm}^3 \), thickness \( s_p \leq 80 \mu \text{m} \). Separator, collector: surface density 0.047 g/cm\(^2\). Calculate the cell voltage and the cell specific energy.
Solution

\[ U_{\text{cell}} = U_p - U_n = 3.85 - 0.25 = 3.6 \text{ V}. \]

If \( Q = Q_n = Q_p \), then

\[ q_{\text{rev}} = \frac{q_{\text{rev},n} \cdot q_{\text{rev},p}}{q_{\text{rev},n} + q_{\text{rev},p}} = \frac{1}{\frac{340}{3} + \frac{1}{3}} = 99 \text{ mAh/g} \quad \text{(theoretical),} \]

\[ \left( \frac{E}{m} \right)_{\text{rev}} = q_{\text{rev}} \cdot U_{\text{cell}} = 99 \cdot 3.6 = 356 \text{ Wh/kg} \quad \text{(theoretical).} \]

To add the masses of the collector and separator, the surface density \( L_{\text{act}} \) of the active mass must be calculated. To do that, consider the maximum electrode thickness. Since the surface is the same for the anode and the cathode,

\[ \frac{Q_n}{S_n} = \frac{Q_p}{S_p} \rightarrow s_n \cdot q_n \cdot \rho_n \cdot S = s_p \cdot q_p \cdot \rho_p \cdot S \]

\[ \frac{s_n}{s_p} = \frac{q_p \cdot \rho_p}{q_n \cdot \rho_n} = \frac{140 \cdot 4}{340 \cdot 2.2} = 0.75, \quad \text{thus } s_n < s_p \]

Assume \( s_p = 80 \mu\text{m} \), then \( s_n = 60 \mu\text{m} \).

\[ \frac{Q_n}{S_n} = \frac{Q_p}{S_p} = \frac{Q}{S} = 80 \cdot 10^{-4} \cdot 140 \cdot 4 \text{ mAh/cm}^2 = 4.48 \text{ mAh/cm}^2, \]

\[ L_{\text{act}} = \frac{Q}{S} \cdot \frac{1}{q_{\text{rev}}} = \frac{4.48}{99} \text{ g/cm}^2 = 0.045 \text{ g/cm}^2 \quad \text{(active mass).} \]

Add the mass of the separator to obtain

\[ L = L_{\text{act}} + L_{\text{sep}} = 0.045 + 0.047 = 0.092 \text{ g/cm}^2. \]

The practical specific capacity \( q_{\text{cell}} \) is thus

\[ q_{\text{cell}} = \frac{Q}{S} \cdot \frac{1}{L} = \frac{4.48}{0.092} = 49 \text{ mAh/g} \quad \text{(practical),} \]

or approximately 50% of the theoretical capacity. The practical energy density \( e \) is

\[ \left( \frac{E}{m} \right)_{\text{cell}} = 3.6 \cdot 49 = 176 \text{ Wh/kg} \quad \text{(practical).} \]

The pack specific energy will be even lower.

Problem 4.28

Develop an equation for the battery apparent capacity as a function of the current for constant current discharge using the battery modeling equations of Section 4.5.2. Then evaluate the apparent capacity of a battery with nominal capacity \( Q_0 = 72 \text{ Ah}, \kappa_4 = -0.005, \kappa_2 = 1.22, \) at \( C/10, C_1, \) and \( C_{10} \) discharge.
Solution

\[ U_b = U_{oc} - R_i \cdot I_b = (\kappa_1 + \kappa_2 \cdot \xi) - (\kappa_3 + \kappa_4 \cdot \xi) \cdot I_b, \text{ with } \kappa_4 < 0 \]

\[ \dot{\xi} = -\frac{I_b}{Q_0^*}, \]

thus

\[ \dot{U}_b = -\kappa_2 \cdot \frac{I_b}{Q_0^*} + \kappa_4 \cdot \frac{I_b^2}{Q_0^*} = -c, \]

\[ U_b(t) = U_b(0) - c \cdot t, \text{ where } U_b(0) = \kappa_1 + \kappa_2. \]

The discharge ends when \( U_b(t_f) = U_{cut} \), thus for

\[ t_f = \frac{U_b(0) - U_{cut}}{c}. \]

The capacity or Ah rate is

\[ Q_0 = I_b \cdot t_f = Q_0^* \cdot \frac{U_b(0) - U_{cut}}{\kappa_2 - \kappa_4 \cdot I_b}, \]

which is dependent on \( I_b \). To calculate \( Q_0(I_b)/Q_0^*(I_b^*) \), define \( K_c = 1 - \kappa_4/\kappa_2 \cdot I_b^* \). One finds that

\[ \frac{Q_0}{Q_0^*} = \frac{K_c}{1 + (K_c - 1) \cdot \frac{I_b}{I_b^*}}, \]

that is (4.58) with \( n = 2 \).

The nominal capacity must be retrieved for very slow currents, ideally when \( I_b^* = 0 \), as

\[ Q_0^* = Q_0^* \cdot \frac{U_b(0) - U_{cut}}{\kappa_2} = Q_0^* \cdot \frac{\kappa_1 + \kappa_2 - U_{cut}}{\kappa_2}, \]

from whence it must be \( U_{cut} = \kappa_1 \). The dependency \( Q_0(I_b) \) can be rewritten as

\[ Q_0(I_b) = \frac{Q_0^*}{1 + e \cdot I_b}, \text{ where } e = \frac{|\kappa_4|}{\kappa_2}. \]

In the numerical case, \( e = 0.005/1.22 = 0.0041 \) The \( C/10 \) current is \( 72/10 = 7.2 \text{ A} \). For this current, the capacity \( Q_0 \) is

\[ Q_0 = \frac{72}{1 + 0.0041 \cdot 7.2} = 70 \text{ Ah} \quad (97\% \text{ of the nominal capacity}) \]

and the discharge time \( t_f \) is

\[ t_f = \frac{70}{7.2} = 9.7 \text{ h}. \]
For a $C_1$ current $= 72$ A,

$$Q_0 = \frac{72}{1 + 0.0041 \cdot 72} = 56 \text{ Ah} \quad (77\% \text{ of the nominal capacity}),$$

$$t_f = \frac{56}{72} = 0.78 \text{ h} = 46 \text{ min}.$$

For a $C_{10}$ current $= 720$ A,

$$Q_0 = \frac{72}{1 + 0.0041 \cdot 720} = 18 \text{ Ah} \quad (25\% \text{ of the nominal capacity}),$$

$$t_f = \frac{18}{720} = 0.025 \text{ h} = 1.5 \text{ min}.$$

**Problem 4.29**

Verify that the round-trip efficiency of a battery under constant current discharge-charge and for varying parameters $U_{oc}$, $R_i$ as described in Section 4.5.2 has the same form as (4.82) but with $U_{oc}$ and $R_i$ calculated for $\xi = 0.5$.

**Solution**

The energy discharged $E_d$ is

$$E_d = I_b \cdot \int_0^{t_f} U_{oc}(\xi) - R_i(\xi) \cdot I_b \ dt =$$

$$= I_b \cdot \int_0^{t_f} (\kappa_1 + \kappa_2 \cdot \xi) - (\kappa_3 + \kappa_4 \cdot \xi) \cdot I_b \ dt.$$

Since $\xi(t) = 1 - I_b/Q_0 \cdot t$ under constant current discharge,

$$E_d = I_b \cdot \left[ \kappa_1 \cdot t_f + \kappa_2 \cdot \left( t_f - \frac{I_b \cdot t_f^2}{2 \cdot Q_0} \right) - \kappa_3 \cdot t_f \cdot I_b - \kappa_4 \cdot I_b \cdot \left( t_f - \frac{I_b \cdot t_f^2}{2 \cdot Q_0} \right) \right].$$

Since $t_f = Q_0/I_b$,

$$E_d = I_b \cdot \left[ \kappa_1 \cdot t_f + \kappa_2 \cdot \left( t_f - \frac{I_b \cdot t_f^2}{2} \right) - \kappa_3 \cdot t_f \cdot I_b - \kappa_4 \cdot I_b \cdot \left( t_f - \frac{I_b \cdot t_f^2}{2} \right) \right] =$$

$$= I_b \cdot t_f \cdot \left[ \kappa_1 + \frac{\kappa_2}{2} - \kappa_3 \cdot I_b - \kappa_4 \cdot \frac{I_b}{2} \right],$$

which is equal to

$$E_d = I_b \cdot t_f \cdot \left( U_{oc,1} \cdot \frac{1}{2} - R_i,1 \cdot I_b \right).$$

Similarly for $E_c$,

$$E_c = |I_b| \cdot t_f \cdot \left( U_{oc,1} \cdot \frac{1}{2} + R_i,1 \cdot |I_b| \right),$$

and thus

$$\eta_b = \frac{U_{oc,1} \cdot \frac{1}{2} - R_i,1 \cdot I_b}{U_{oc,1} \cdot \frac{1}{2} + R_i,1 \cdot |I_b|}.$$
Supercapacitors

**Problem 4.31**

Derive Equation (4.117).

**Solution**

$P_{sc}$ is considered as a constant. By differentiating (4.116) one obtains

$$2 \cdot U_{sc} \cdot \frac{d}{dt} U_{sc} + \frac{I_{sc}}{C_{sc}} \cdot U_{sc} - \frac{Q_{sc}}{C_{sc}} \cdot \frac{d}{dt} U_{sc} = 0,$$

where the second of (4.115) has been used. By solving (4.116) for $Q_{sc}$, one obtains

$$Q_{sc} = \frac{C_{sc}}{U_{sc}} \cdot (P_{sc} \cdot R_{sc} + U_{sc}^2),$$

thus

$$2 \cdot U_{sc} \cdot \frac{d}{dt} U_{sc} + \frac{P_{sc}}{C_{sc}} - \frac{d}{dt} U_{sc} \cdot \left( P_{sc} \cdot \frac{R_{sc}}{U_{sc}} + U_{sc} \right).$$

Since $2 \cdot U_{sc} \cdot dU_{sc}/dt = d/dt(U_{sc}^2)$, one obtains

$$\frac{d}{dt} U_{sc}^2 + \frac{P_{sc}}{C_{sc}} = \frac{P_{sc} \cdot R_{sc} \cdot \frac{d}{dt} U_{sc}^2}{2} \Rightarrow \frac{d}{dt} U_{sc}^2 + \frac{P_{sc}}{C_{sc}} = \frac{P_{sc} \cdot R_{sc} \cdot \frac{d}{dt} U_{sc}^2}{2} \Rightarrow 0,$$

from whence (4.117) follows. For $R_{sc} = 0$ it is found that $dE_{sc}/dt = d/dt(C_{sc} \cdot U_{sc}^2/2) = -P_{sc}$ ($P_{sc}$ positive during discharge).

**Problem 4.32**

Derive an analytical solution for the discharge of a supercapacitor with maximum power. Then verify the solution with the data of Figure 4.42.

**Solution**

At max power

$$P_{sc} = \frac{Q_{sc}^2}{4 \cdot R_{sc} \cdot C_{sc}^2},$$

then from (4.118)

$$U_{sc} = \frac{Q_{sc}}{2 \cdot C_{sc}},$$

from (4.115)

$$R_{sc} \cdot I_{sc} = \frac{Q_{sc}}{C_{sc}} = \frac{Q_{sc}}{2 \cdot C_{sc}^2},$$

and

$$\frac{dQ_{sc}}{dt} = -\frac{Q_{sc}}{2 \cdot R_{sc} \cdot C_{sc}}.$$
Thus

\[ Q_{sc}(t) = Q_0 \cdot e^{-\frac{t}{\tau}} \cdot R_{sc}, \]
\[ I_{sc}(t) = \frac{Q_0}{2 \cdot C_{sc} \cdot R_{sc}} \cdot e^{-\frac{t}{\tau} \cdot R_{sc}}, \]
\[ U_{sc}(t) = \frac{Q_0}{2 \cdot C_{sc}} \cdot e^{-\frac{t}{\tau} \cdot R_{sc}}, \]
\[ P_{sc}(t) = \frac{Q_0^2}{4 \cdot R_{sc} \cdot C_{sc}^2} \cdot e^{-\frac{t}{\tau} \cdot R_{sc}}. \]

For \( t = 2 \) s, \( C_{sc} = 12.5 \) F, \( R_{sc} = 0.08 \) \( \Omega \), and \( Q_0 = 800 \) C, find

\[ Q_{sc}(t) = 800 \cdot e^{-\frac{2}{12.5 \cdot 0.08}} = 294 \text{ C}, \]
\[ I_{sc}(t) = \frac{800}{2 \cdot 12.5 \cdot 0.08} \cdot e^{-\frac{2}{12.5 \cdot 0.08}} = 147 \text{ A}, \]
\[ U_{sc}(t) = \frac{800}{2 \cdot 12.5} \cdot e^{-\frac{2}{12.5 \cdot 0.08}} = 12 \text{ V}, \]
\[ P_{sc}(t) = \frac{800^2}{4 \cdot 0.08 \cdot 12.5^2} \cdot e^{-\frac{2}{12.5 \cdot 0.08}} = 1.7 \text{ kW}. \]

**Problem 4.33**

Yet another definition of supercapacitor efficiency that is sometimes found (e.g., in [222]) is

\[ \eta_{sc,d} = 1 - \frac{2 \cdot \tau}{t_f}, \]

during discharge at constant current, and

\[ \eta_{sc,c} = \frac{1}{1 + \frac{2 \cdot \tau}{t_f}}, \]

during charge at constant current, where \( \tau = C_{sc} \cdot R_{sc} \) and \( t_f \) has the same meaning as in the text. Explain this definition.

**Solution**

The energy amount that can be obtained from a fully-charged supercapacitor with constant-current discharge in the ideal case of negligible resistance is obtained from (4.119) as

\[ E_{d,id} = \frac{Q_0^2}{2 \cdot C_{sc}}, \]

which coincides with the maximum stored energy defined in (4.129). By defining the efficiency as

\[ \eta_{sc,d} = \frac{E_d}{E_{d,id}}, \]
find the value in the problem statement.

For charge, define the efficiency as

$$\eta_{c,e} = \frac{|E_{c, id}|}{|E_c|},$$

and find the value in the problem statement with $|E_{c, id}| = E_{d, id}$.

**Electric Power Links**

**Problem 4.34**

Consider again Problem 4.9 and account for a battery internal resistance of 0.025 $\Omega$.

- **Solution**

Combine DC motor equations and battery equations with $U_b = U_m$ and $I_b = I_m$. The relationship linking the motor torque, speed, and the DC–DC converter duty cycle is

$$\alpha \cdot U_{oc} - \frac{(R_a + \alpha^2 \cdot R_b) \cdot T_m}{\kappa_a} - \kappa_i \cdot \omega_m = 0.$$

The flux weakening region limit is obtained for $\alpha = 1$ as $0.3 \cdot T_m + 0.25 \cdot \omega_m = 50$ V (axis intercepts at 200 rad/s and 167 Nm). The current limit is obtained by setting $\alpha = 1$ and $\omega_m \cdot T_m = P_{m,max}$. The result is the same as in Problem 4.9 but now $R_a$ should be replaced by $R_a + R_b = 0.075 \Omega$. The new solution is $I_{m,max} = 93$ A (increase).

In the case (i) only the calculation of $\alpha$ changes, since $U_a = 28$ V and $I_a = 60$ A are still admissible values. Knowing that

$$U_b = U_{oc} - R_b \cdot I_b,$$

$$\alpha = \frac{U_a}{U_b} = \frac{I_b}{I_a},$$

one obtains

$$U_a = U_{oc} \cdot \alpha - R_b \cdot I_a \cdot \alpha^2,$$

from whence $\alpha = 57\%$ (increase) and correspondingly $U_b = 49$ V, $I_b = 34$ A (increase). Alternatively, $\alpha$ can be directly calculated from the general equation above.

For the case (ii) again $R_a$ is replaced by the sum of the two resistances. The duty cycle is still fixed at $\alpha = 100\%$ and the flux is now $\kappa_i = \kappa_a = 0.14$ Wb. Thus the flux weakening increases.
Problem 4.35

For an electric drive including a battery, a step-down DC–DC converter (chopper), and a DC motor, calculate the duty cycle that maximizes the regenerated power during braking. Calculate the corresponding motor current, using the data presented in Problem 4.9.

- Solution

The limitation is imposed by the step-down converter for which \( R = U_a/U_b = 1 - \alpha \). Evaluate \( R \) as a function of \( \omega_m \) and \( I_a \):

\[
R \cdot U_{oc} - R_b \cdot R^2 \cdot I_a = R_a \cdot I_a - \kappa_i \cdot \omega = 0.
\]

For \( R = 0 \),

\[
I_a = -\frac{\kappa_i \cdot \omega}{R_a},
\]

but \( I_b = R \cdot I_a = 0 \) (no recuperation). For \( R = \kappa_i \cdot \omega_m/U_{oc} \), \( I_a = 0 \) and \( I_b = 0 \) (again no recuperation). Thus there must be a value \( R > 0 \) that maximizes the recuperated power. The power is

\[
P_a = P_b = I_a \cdot (R_a \cdot I_a + \kappa_i \cdot \omega_m) = R_a \cdot I_a^2 + \kappa_i \cdot \omega_m \cdot I_a,
\]

\[
\frac{dP_a}{dI_a} = 2 \cdot R_a \cdot I_a + \kappa_i \cdot \omega_m = 0 \quad \Rightarrow \quad I_a = -\frac{\kappa_i \cdot \omega_m}{2 \cdot R_a} = I_{a,min}.
\]

For this value of current

\[
R \cdot U_{oc} + \frac{R_b}{2 \cdot R_a} \cdot \kappa_i \cdot \omega_m \cdot R^2 - \frac{\kappa_i \cdot \omega}{2} = 0 \quad \Rightarrow \quad R_{min}.
\]

And inserting the data of Problem 4.9,

\[
I_{a,min} = -\frac{\kappa_i \cdot \omega_m}{2 \cdot R_a} = -\frac{0.25 \cdot 100}{2 \cdot 0.05} = -250 \text{ A},
\]

\[
R^2 \cdot (R_b \cdot I_{a,min}) - u \cdot U_{oc} + R_a \cdot I_{a,min} + \kappa_i \cdot \omega = 0 \quad \Rightarrow \quad R_{min} = 24\%.
\]

Problem 4.36

Consider an electric drive including a battery, a boost DC–DC converter and a motor. Derive a relationship between the maximum torque curve of the motor and the converter ratio, assuming that for \( \omega_m > \omega_b \), \( P_{max} \approx U_{max} \cdot I_{max} \). Conceive a strategy to perform quasistatic simulations in this case.

- Solution
A simplified expression for the motor maximum torque is

\[ T_{\text{max}} = \min\{k \cdot I_{\text{max}}, P_{\text{max}}/\omega_m\} \]

where the constant \( k \) is given by \( \kappa_a \) in DC motors and by \( 3/2 \cdot p \cdot \varphi_m \) in PMSMs. The maximum power is \( P_{\text{max}} = I_{\text{max}} \cdot U_{\text{max}} \) where \( U_{\text{max}} \) is the voltage at the DC side of the motor. One obtains

\[ P_{\text{max}} = U_{\text{max}} \cdot I_{\text{max}} = I_{\text{max}} \cdot R \cdot (U_{\text{oc}} - R_b \cdot R \cdot I_{\text{max}}) = f(R) \]

In backward modeling, the required torque \( T_m \) has to be saturated by the maximum value \( T_{\text{max}} \). However, this value depends on \( R \). Physically, there is one value of \( R \) that realizes the desired speed and torque (see a similar situation in Problem 4.34). For simulation purposes, one could map the conversion ratio as a function of motor speed and torque and then feed the maximum torque map. Alternatively, \( T_{\text{max}} \) could be mapped as a function of speed and voltage \( U_{\text{max}} \) and the latter calculated as \( U_b \cdot I_b/I_{\text{max}} \).

**Problem 4.37**

Consider a semi-active power link with a battery, a supercapacitor, an electric motor, and a DC–DC converter on the supercapacitor branch. Derive an analytical relationship between the control factor \( u \) (4.132) and the DC–DC converter voltage ratio \( R \). Calculate the values of \( R \) to obtain a pure battery supply (\( u = 0 \)) or a pure supercapacitor supply (\( u = 1 \)). Describe the supercapacitor on a quasistatic basis, i.e.,

\[ C_{sc} \cdot \frac{U_{s0} - U_{sc}}{\tau} = -I_{sc}, \]

where \( \tau \) is the time step.

- **Solution**

The battery equation is \( U_b = U_{oc} - R_b \cdot I_b \). The DC link equations are \( U_m = R \cdot U_{sc} = U_b \). Additionally, \( I_b + I_{sc}/R = I_m \). The supercapacitor equation is given in the problem formulation. There are five equations in the six variables \( U_b, I_b, U_{sc}, I_{sc}, U_m, I_m \). However, in dynamic modeling \( I_m \) is given from the downstream powertrain. Thus all the other quantities can be calculated as a function of \( R \). The result is

\[ U_m = U_b = \frac{U_{sc}}{\frac{1}{R_b} + \frac{C}{U_{s0}} R} \cdot I_m \]

from whence \( U_{sc}, I_{sc}, \) and \( I_b \) are calculated as well. The supercapacitor power \( P_{sc} \) is

\[ P_{sc} = U_{sc} I_{sc} = \frac{U_m C}{R} (U_{s0} - U_{sc}) \]
and the control ratio \( u \) is
\[
  u = \frac{P_{sc}}{P_m} = \frac{1}{I_m} \left( \frac{C U_{s0}}{\tau R} - \frac{C}{\tau R^2} \left( \frac{1}{R_s} + \frac{C}{\tau R^2} \right) \right) = u(R, I_m).
\]

To find \( u = 0 \), the DC–DC converter must be regulated such as
\[
  R = \frac{U_{oc} - R_b I_m}{U_{s0}}.
\]

To obtain \( u = 1 \), the DC–DC converter must be regulated such as \( R \) is the solution of the quadratic equation
\[
  \tau I_m R^2 - CU_{s0} R + CU_{oc} = 0.
\]

The condition to have a solution is
\[
  (CU_{s0})^2 - 4 \tau I_m CU_{oc} > 0.
\]

Therefore, pure supercapacitor operation is allowed for
\[
  I_m < \frac{CU_{s0}^2}{4\tau U_{oc}}.
\]

**Torque Couplers**

**Problem 4.38**

Consider a through-the-road parallel hybrid. The torque coupling has the following characteristics: transmission ratio between the rear-axle motor and the wheels \( \gamma_m = 11 \), transmission ratios between the front-axle engine and the wheels \( \gamma_e = \{15.02, 8.09, 5.33, 3.93, 3.13, 2.59\} \), wheel radius \( r_{wh} = 31.7 \text{ cm} \).

Moreover, \( T_{m,\text{max}} = -T_{m,\text{min}} = 140 \text{ Nm} \), \( P_{m,\text{max}} = -P_{m,\text{min}} = 42 \text{ kW} \), engine speed limited to \( \omega_{e,\text{max}} = 4500 \text{ rpm} \), engine maximum torque \( T_{e,\text{max}} = 1.34 \cdot 10^{-5} \cdot \omega_e^3 - 0.0149 \cdot \omega_e^2 + 4.945 \cdot \omega_e - 243 \) (Nm), engine minimum torque \( T_{e,\text{min}} = -20 \text{ Nm} \). For a driving situation with \( V = 30 \text{ m/s} \) and \( T_{wh} = 675 \text{ Nm} \), determine the limits imposed by the motor operation on the engine operation.

- **Solution**

  The engine speed for various gears is
  \[
  \omega_e = \frac{\gamma_e V}{r_{wh}} = \{947, 510, 336, 248, 197, 163\} \text{ rad/s}.
  \]

  Since \( \omega_{e,\text{max}} = 471 \text{ rad/s} \) only the third to sixth gears are admissible. The engine maximum torque \( T_{e,\text{max}} \) for the four admissible speeds is
$T_{e,\text{max}} = \{242, 270, 255, 225\}$ Nm.

However, the torque coupling equation reads

$$T_e \cdot \gamma_e + T_m \cdot \gamma_m = T_{wh}.$$ 

Since $T_m \geq T_{m,\text{min}}$, consequently,

$$T_e \leq \frac{T_{wh} - \gamma_m \cdot T_{m,\text{min}}}{\gamma_e} = \{259, 351, 441, 533\}.$$

In all cases, the motor imposed limits overshadow the engine physical limits during generating operation. For motoring operation,

$$T_e \geq \frac{T_{wh} - \gamma_m \cdot T_{m,\text{max}}}{\gamma_e} = \{1.6, 2.1, 2.7, 3.2\}.$$ 

that prevents, e.g., purely electric operation ($T_e = 0$).

**Problem 4.39**

Consider a power-split combined hybrid powertrain with a planetary gear set linking the engine, the generator, and the output shafts with the following Willis relation

$$\omega_g = 3.6 \cdot \omega_e - 2.6 \cdot \omega_f.$$ 

The second electric machine is mounted directly on the output shaft without any reduction gear. The generator has the following characteristics (both in motor and in generator modes): maximum torque = 160 Nm, maximum power = 25 kW, maximum speed = 1200 rad/s. The motor has the following characteristics (both in motoring and in generating mode): maximum torque = 400 Nm, maximum power = 25 kW, and maximum speed = 700 rad/s. The S.I. engine has the following characteristics: maximum torque curve = \{0 87 110 107\} Nm @ \{0 87 110 107\} rpm. The maximum battery power is 35 kW. Consider a driving situation in which the speed of the output shaft is 343 rad/s and the required torque is 129 Nm. Supposing that the decision variables of the energy management strategy are engine speed and torque, evaluate the admissible range of these variables.

**Solution**

The degrees of freedom are selected as the engine torque and speed. Thus the admissible range is drawn on the engine speed-torque plane. The engine limits themselves are drawn as straight lines (curve $A$).
\[ T_e = 0, \quad \text{for} \quad \omega_e \leq 42 \]
\[ T_e = \frac{87}{94} \cdot (\omega_e - 42), \quad \text{for} \quad 42 < \omega_e \leq 136 \]
\[ T_e = 87 + \frac{23}{251} \cdot (\omega_e - 136), \quad \text{for} \quad 136 < \omega_e \leq 387 \]
\[ T_e = 110 - \frac{3}{136} \cdot (\omega_e - 387), \quad \text{for} \quad 387 < \omega_e \leq 524 \]

The motor speed is fixed, i.e., \( \omega_m = \omega_f = 343 \text{ rad/s} \). The motor base speed is \( 25000/400 = 62 \text{ rad/s} \). Thus the max torque is \( 25000/343 = 73 \text{ Nm} \). The relationship between engine torque, motor torque, and output torque is

\[ T_e \cdot \frac{2.6}{3.6} = T_f - T_m. \]

Thus the engine torque corresponding to the maximum motor torque (curve \( B \)) is

\[ T_e = \frac{129 - 73}{0.72} = 78 \text{ Nm}. \]

Only engine torque values greater than 78 Nm are admissible, since they do not saturate the motor limits.

The relationship between generator torque and engine torque is \( T_g = T_e/3.6 \). The engine torque corresponding to the maximum generator torque is \( 160 \cdot 3.6 = 576 \text{ Nm} \), thus far beyond the engine limits. The generator base speed is \( 25000/160 = 156 \text{ rad/s} \) (in both rotating directions). The engine speed corresponding to the generator base speed is

\[ \frac{\omega_g + 2.6 \cdot \omega_f}{3.6} = \pm 156 + 2.6 \cdot 343 = 291 \text{ and } 204 \text{ rad/s}. \]

Thus outside of this range the max power limit of the generator could limit the engine operation. The max power limit of the generator is \( T_g = 25000/\omega_g \), thus in engine variables (curve \( C \)) is

\[ T_e = \frac{25000}{3.6 \cdot \omega_e - 892}. \]

The intersection of curve \( C \) with the curve \( T_e = 110 \text{ Nm} \) is at 475 rad/s. To be more precise, the intersection should be made with the fourth branch of the curve \( A \), leading to a quadratic equation.

Neglecting in a first approximation the motor and generator losses, the battery power is \( P_b = P_m - P_g = P_f - P_e \). The output power is \( P_f = 129 \cdot 343 = 44247 \text{ W} \). Thus the limit \( P_b = 35 \text{ kW} \) in engine variables becomes \( \omega_e \cdot T_e = 44247 \text{ W} - 35000 \text{ W} = 9247 \text{ W} \) (curve \( D \)). The intersection with curve \( B \) is at 118 rad/s. The curve \( A \) at this engine speed gives 70 Nm, which is below curve \( B \). Thus the battery constraint is not active at these driving conditions. All the other limits (generator minimum torque, motor minimum torque) are not active as well.
The engine admissible range is thus between curve $B$ ($T_e = 78$ Nm), curve $A$ (between $\omega_e = 126$ rad/s and 475 rad/s), and curve $C$ (between $\omega_e = 475$ rad/s and 524 rad/s).

![Fig. 10.9. Admissible operating range of the engine and limiting curves.](image)

**Problem 4.40**

Derive the coupling matrix for the four torque levels of a PSD from the elements of the kinematic matrix in the case of quasistatic modeling. Do the same in the case of forward modeling, i.e., derive (4.166).

- **Solution**

In backward modeling the kinematic matrix generally reads

$$
\omega_e = A \cdot \omega_f + B \cdot \omega_g,
\omega_m = C \cdot \omega_f + D \cdot \omega_g.
$$

Moreover, the power balance of the PSD reads

$$
T_e \cdot \omega_e + T_m \cdot \omega_m = T_g \cdot \omega_g + T_f \cdot \omega_f.
$$

In backward modeling one wants to calculate $T_e$ and $T_m$ as a function of $T_f$ and $T_g$. To do so, use the kinematic relationships

$$
T_e \cdot A \cdot \omega_f + T_e \cdot B \cdot \omega_g + T_m \cdot C \cdot \omega_f + T_m \cdot D \cdot \omega_g = T_g \cdot \omega_g + T_f \cdot \omega_f.
$$
By equating the factors of $\omega_g$ and, respectively, $\omega_f$, one obtains

\begin{align*}
T_e \cdot A + T_m \cdot C &= T_f, \\
T_e \cdot B + T_m \cdot D &= T_g,
\end{align*}

thus,

\begin{align*}
T_e &= \frac{1}{A \cdot D - B \cdot C} \cdot (D \cdot T_f - C \cdot T_g), \quad \text{and} \\
T_m &= \frac{1}{A \cdot D - B \cdot C} \cdot (-B \cdot T_f + A \cdot T_g).
\end{align*}

In the case of forward modeling, the input variables are $\omega_f$ and $\omega_g$, while the output variables are $T_g$ and $T_f$. As a result, Equation 4.141 is obtained.

**Problem 4.41**

For the compound power split device architecture shown in Fig. 11.4, derive the kinematic matrix $M$ and the values of the two kinematic nodes.

![Diagram](image)

**Fig. 10.10.** Compound power-split configuration for Problem 4.41

• **Solution**

Let the first electric machine be the motor and the second the generator. The general relationship of a PGS is

$$\omega_r + z \cdot \omega_s = (1 + z) \cdot \omega_c.$$ 

For the first PGS,

$$\omega_c + z_1 \cdot \omega_m = (1 + z_1) \cdot \omega_f.$$ 

For the second PGS,

$$\omega_f + z_2 \cdot \omega_g = (1 + z_2) \cdot \omega_e.$$ 

Thus the kinematic matrix is $A = 1/(1 + z_2)$, $B = z_2/(1 + z_2)$, $C = (z_1 + z_2 + z_1 z_2)/(1 + z_2)$, $D = -1/z_1(1 + z_2)$. The two nodes are calculated (see Problem 4.42) as $K_1 = 1/(1 + z_1)$ and $K_2 = 1 + z_2$.
Problem 4.42

Derive equation (4.167) for $K_v$ as a function of $K$ and equation (4.168) for $r$ as a function of $K$, including the definitions of $K_1$ and $K_2$.

- Solution

We use the definition of $M$ as in Problem 4.40. By defining $K = \omega_f / \omega_e$, we have

$$\omega_g / \omega_e = A \cdot K + B$$

and

$$\omega_m / \omega_e = C \cdot K + D.
$$

Thus,

$$K_v = \frac{\omega_m}{\omega_g} = \frac{C \cdot K + D}{A \cdot K + B} = \frac{D}{B \left( \frac{1}{K} - \frac{K_1}{K_2} \right)}$$

from whence the definition of $K_1 = -D/C$ and $K_2 = -B/A$. The power split ratio is

$$r = \frac{\omega_g}{\omega_e} \cdot \frac{T_g}{T_e} = \frac{\omega_m}{\omega_e} \cdot \frac{T_m}{T_e}.$$ 

By using the torque matrix calculated in Problem 10,

$$r = (A \cdot K + B) \cdot \frac{D \cdot \frac{T_f}{T_e} + C}{B \cdot C - A \cdot D} = (C \cdot K + D) \cdot \frac{B \cdot \frac{T_f}{T_e} + A}{B \cdot C - A \cdot D}.$$

Equating the last two equations and defining $T_f/T_e = X$, derive that

$$(D \cdot X + C) \cdot (A \cdot K + B) = (C \cdot K + D) \cdot (B \cdot X + A)$$

from whence

$$X \cdot (D \cdot (A \cdot K + B) - B \cdot (C \cdot K + D)) = A \cdot (C \cdot K + D) - C \cdot (A \cdot K + B)$$

$$\Rightarrow X \cdot (A \cdot D \cdot K - B \cdot C \cdot K) = A \cdot D - B \cdot C,$$

thus $X = 1/K$. Hence,

$$r = \frac{1}{K} \cdot \frac{(C \cdot K + D) \cdot (B + A \cdot K)}{B \cdot C - A \cdot D} = \frac{1}{K} \cdot D \cdot B \cdot \frac{(1 + \frac{K_1}{K_2}) \cdot (1 + \frac{K_2}{K_1})}{B \cdot C - A \cdot D}.$$

As $B \cdot C - A \cdot D$ can be written as

$$D \cdot B \left( \frac{C}{D} - \frac{A}{B} \right) = D \cdot B \left( \frac{1}{K_2} - \frac{1}{K_1} \right),$$

equation (4.143) is obtained.
Problem 4.43

Derive (4.163)-(4.164).

Solution

Speed balance:

\[ \omega_e = a \cdot \omega_g + (1 - a) \cdot \omega_f, \]

where \( a = z/(1 + z) \), \( 1 - a = 1/(1 + z) \).

Steady-state torque balance:

\[ T_e = T_g + T_r, \]
\[ T_r = T_f - T_m. \]

Dynamic torque balance (note the sign of inertia torque in the right-hand side):

\[ T_e - \Theta_e \cdot \frac{d\omega_e}{dt} = T_g + \Theta_s \cdot \frac{d\omega_g}{dt} + T_r + \Theta_r \cdot \frac{d\omega_f}{dt}. \]

Power balance:

\[
\begin{aligned}
&\left( T_e - \Theta_e \cdot \frac{d\omega_e}{dt} \right) \cdot \omega_e = \left( T_g + \Theta_s \cdot \frac{d\omega_g}{dt} \right) \cdot \omega_g + \left( T_r + \Theta_r \cdot \frac{d\omega_f}{dt} \right) \cdot \omega_f = \\
&= \left( T_e - \Theta_e \cdot \frac{d\omega_e}{dt} \right) \cdot a \cdot \omega_g + \left( T_e - \Theta_e \cdot \frac{d\omega_e}{dt} \right) \cdot (1 - a) \cdot \omega_f = \\
&= \left( T_g + \Theta_s \cdot \frac{d\omega_g}{dt} \right) \cdot \omega_g + \left( T_r + \Theta_r \cdot \frac{d\omega_f}{dt} \right) \cdot \omega_f = \\
&= \left( T_e - \Theta_e \cdot a \cdot \frac{d\omega_g}{dt} - \Theta_e \cdot (1 - a) \cdot \frac{d\omega_f}{dt} \right) \cdot a \cdot \omega_g + \\
&+ \left( T_e - \Theta_e \cdot a \cdot \frac{d\omega_g}{dt} - \Theta_e \cdot (1 - a) \cdot \frac{d\omega_f}{dt} \right) \cdot (1 - a) \cdot \omega_f
\end{aligned}
\]

from whence, by equalizing the terms multiplying \( \omega_g \) and those multiplying \( \omega_f \), one obtains

\[
\begin{aligned}
&\quad a \cdot T_e - \Theta_e \cdot a^2 \cdot \frac{d\omega_g}{dt} - \Theta_e \cdot (1 - a) \cdot a \cdot \frac{d\omega_f}{dt} = T_g + \Theta_s \cdot \frac{d\omega_g}{dt} \\
&(1 - a) \cdot T_e - \Theta_e \cdot a \cdot (1 - a) \cdot \frac{d\omega_g}{dt} - \Theta_e \cdot (1 - a)^2 \cdot \frac{d\omega_f}{dt} = T_r + \Theta_r \cdot \frac{d\omega_f}{dt}
\end{aligned}
\]

which are the equations sought, since

\[
\begin{aligned}
a \cdot (1 - a) &= \frac{z}{(1 + z)^2}, & a^2 &= \frac{z^2}{(1 + z)^2}, & (1 - a)^2 &= \frac{1}{(1 + z)^2}.
\end{aligned}
\]
Problem 4.44

Extend equations (4.149)-(4.150) to the case where there are losses in the planetary gearset.

Solution

Assume that \( e_s \) and \( e_r \) are the efficiencies of the contacts carrier-sun and carrier-ring, respectively, the power balance neglecting the inertia terms is simply written as

\[
\begin{align*}
P_e &= \frac{1}{e_s} \cdot P_g + \frac{1}{e_r} \cdot P_r, \\
T_e \cdot \omega_e &= \frac{T_g}{e_s} \cdot \omega_g + \frac{T_r}{e_r} \cdot \omega_f, \\
T_g &= e_s \cdot a \cdot T_e, \\
T_r &= e_r \cdot (1 - a) \cdot T_e,
\end{align*}
\]

from whence \( T_g = e_s \cdot a \cdot T_e \), and \( T_r = e_r \cdot (1 - a) \cdot T_e \), where \( a = \frac{z}{1 + z} \).

The lost power \( P_{\text{lost}} \) is

\[
P_{\text{lost}} = T_e \cdot \omega_e - T_g \cdot \omega_g - T_r \cdot \omega_f =
\]

\[
= a \cdot T_e \cdot \omega_g \cdot (1 - e_s) + (1 - a) \cdot T_e \cdot \omega_f \cdot (1 - e_r).
\]

Non-electric Hybrid Propulsion Systems

Hybrid-inertial Powertrains

Problem 5.1

Derive a Ragone curve similar to (5.1) and (5.2) for a flywheel battery. Then evaluate the maximum energy and power. Use a simplified expression for the loss power of the type \( P_l = R \cdot \omega_f^2 \).

Solution

For constant power \( P_f \), the dynamic equation \( \Theta_f \cdot \omega_f \cdot \dot{\omega}_f = -P_f - R \cdot \omega_f^2 \) can be rewritten as

\[
\frac{\Theta_f \cdot d\omega_f}{P_f + R \cdot \omega_f^2} = -dt
\]

By integrating both terms from \( t = 0, \omega_f = \omega_0 \) to \( t = t_\infty, \omega_f = 0 \) yields

\[
t_\infty = \tau \cdot \ln(1 + R \cdot \omega_0^2 / P_f),
\]

where \( \tau \triangleq \Theta_f / (2R) \). The energy delivered is \( E_f = P_f \cdot t_\infty \), while the initial energy stored in the flywheel is \( E_0 = 1/2 \cdot \Theta_f \cdot \omega_0^2 \).

The Ragone curve \( E_f = E_f(P_f) \) has therefore a monotonously increasing trend, contrarily to battery and supercapacitors. That is, the larger is the power, the larger is the energy that can be extracted from the flywheel and thus its efficiency.
**Problem 5.2**

Dimension a flywheel for the following application (F1 KERS): $P_{\text{max}} = 60\, \text{kW}$, $E_{\text{max}} = 500\, \text{kJ}$, $\omega_{\text{max}} = 60000\, \text{rpm}$. Assume $\beta = 0.5$, $\rho = 1658\, \text{kg/m}^3$ (carbon fiber), $\rho_0^{\frac{1}{3}} n_0^{0.2} = 0.48$ (sealed flywheel).

- **Solution**

The maximum speed is $\omega_{\text{max}} = 60000 \cdot 2\pi / 60 = 6283\, \text{rad/s}$. The moment of inertia is given by

$$\Theta_f = \frac{2 \cdot E_{\text{max}}}{\omega_{\text{max}}} = 2 \cdot 510 \cdot 10^3 / (6283)^2 = 0.026 \, \text{kg} \cdot \text{m}^2.$$

The diameter is found from (5.5) by setting $P_{\alpha} = P_{\text{max}}$, i.e.,

$$0.04 \cdot 0.48 \cdot d^{4.6} \cdot (6283/2)^2 \cdot 0.83 = 60 \cdot 10^3,$$

from whence $d = 0.2\, \text{m}$. Thus $b = \beta \cdot d = 0.5 \cdot 0.2 = 0.1\, \text{m}$.

Using (5.7), find then

$$0.026 = 3.14 \cdot 1658 / 32 \cdot (1 - q^4) \cdot 0.1 \cdot 0.2^4,$$

from whence $q = 0.2$.

The mass is obtained from (5.8) as

$$m_f = 0.1 \cdot 0.2^2 \cdot 3.14 \cdot 1658 \cdot (1 - 0.2^2) / 4 = 5\, \text{kg}.$$

**Problem 5.3**

Evaluate the charging efficiency of the flywheel of Problem 5.2 for a braking at maximum power for 2s. Evaluate the round-trip efficiency.

- **Solution**

During charging, the flywheel dynamics reads $\Theta_f \cdot \omega_f \cdot \dot{\omega}_f = P_f - R \cdot \omega_f^2$. Using the data of Problem 5.2, an approximation for $R$ could be

$$R = P_{\text{max}} / \omega_{\text{max}}^2 = 60 \cdot 10^3 / (60000 \cdot 2\pi / 60)^2 = 0.0015.$$

Now, the speed trajectory starting from rest is

$$t = -\tau \cdot \ln \left( \frac{P_f - R \cdot \omega_f^2}{P_f - R \cdot \omega_0^2} \right),$$

thus $\omega_f^2 = P / R \cdot (1 - e^{-t/\tau})$.

The available energy is $E_{\text{in}} = P_f \cdot t$. The stored energy is $E = 1/2 \cdot \Theta_f \omega_f^2$.

The efficiency is therefore
\[ \eta_c = \frac{E}{E_{in}} = \frac{\tau}{t} \cdot (1 - e^{-t/\tau}). \]

With the data of the problem,

\[ \tau = \Theta_f/(2 \cdot R) = 0.26/2/0.015 = 8.55 \text{ s}, \]

\[ \eta_c = 8.55/2 \cdot (1 - e^{-2/8.55}) = 0.89. \]

The final speed is

\[ \omega_f^2 = 60 \cdot 10^3/0.0015 \cdot (1 - e^{-2/8.55}) = 8231 \text{ rad/s}. \]

During discharge, use the result of Problem 5.1 with \( \omega_0 = 8231 \text{ rad/s}. \) Define \( x \triangleq R \cdot \omega_0^2/P_f = 1 - e^{-t/\tau} = 0.2085, \) from which obtain

\[ \eta_d = \frac{E_f}{E_0} \frac{\ln(1 + x)}{x} = \ln(1 + 0.208)/0.208 = 0.91. \]

The round-trip efficiency is \( \eta = \eta_c \cdot \eta_d = \ln(1 + x) \cdot (t/\tau) = 0.81. \)

**Problem 5.4**

Evaluate the CVT ratio during a deceleration of a vehicle equipped with a flywheel-based KERS and the opening time of the clutch. Use the flywheel data of Problem 5.3. The flywheel is connected to the input stage of the CVT through a fixed-reduction gear with ratio 8.33. Final drive and wheel ratio \( \left( \frac{\gamma_{fd}}{\gamma_{rw}} \right) = 13. \) Initial conditions: \( v(0) = 80 \text{ km/h}, \) \( \omega_f(0) = 10000 \text{ rpm}, \) \( m_v = 600 \text{ kg}, \) braking time 2 s (assume a constant braking power), CVT range \( \nu_{max}/\nu_{min} = 6. \)

- **Solution**

The initial ratio is

\[ \nu(0) = \frac{\omega_f(0)}{\tau_f \cdot v(0) \cdot \left( \frac{2\pi}{\nu_{max}} \right)} = (10000 \cdot 2\pi/60)/(8.33/(80/3.6))/13 = 0.43. \]

The average braking power is

\[ P_b = \frac{m_v \cdot \omega^2}{t_b} = (80/3.6)^2 \cdot 600/2/2.5 = 59 \text{ kW}. \]

Vehicle speed varies as

\[ v^2(t) = v(0)^2 - 2 \cdot P_b \cdot t/m_v. \]

Flywheel speed varies as
\[ \omega_f(t)^2 = \frac{P_f}{R} - \left( \frac{P_f}{R} - \omega_f(0)^2 \right) \cdot e^{-t/\tau}, \]

where \( R = 0.015 \) and \( \tau = 8.55 \) have been calculated in Problem 5.3.

At \( t = t_b \), \( v = 0 \) and \( \omega_f = 34.6 \) krpm. The maximum CVT ratio is \( \nu_{\text{max}} = 6 \cdot 0.43 = 2.58 \). The clutch opening time \( t_{\text{oc}} \) is such that \( \nu(t_{\text{oc}}) = \nu_{\text{max}} \), thus \( 2.58 = \frac{\omega_f(t_{\text{oc}})/8.33}{131x734} \cdot v(t_{\text{oc}}) \). Find \( t_{\text{oc}} = 1.9 \) s. For \( t > t_{\text{oc}} \), the clutch is slipping and the vehicle can further decelerate until rest.

**Hybrid-hydraulic Powertrains**

*Problem 5.5*

Derive a Ragone curve similar to (5.1) and (5.2) for a hydraulic accumulator. Show that for high power this definition is equivalent to that adopted in the text.

- **Solution**

The differential equation for the accumulator energy reads

\[ \dot{E} = -P_h - h \cdot A_w \cdot (\vartheta - \vartheta_w) = -\frac{E}{\tau} + h \cdot A_w \cdot \vartheta_w. \]

Integrate between \( E(0) = E_0 = m_g \cdot c_{v,g} \cdot \vartheta_B \) and \( E(t_{\infty}) = E_w = m_g \cdot c_{v,g} \cdot \vartheta_w \) to obtain

\[ t_{\infty} = \tau \cdot \ln \left( 1 + \frac{E_0 - E_w}{P_h} \right). \]

Now, observe that \( W_{AB} = m_g \cdot c_{v,g} \cdot (\vartheta_B - \vartheta_A) = E_0 - E_w \), from whence

\[ E_{ha}(P_h) = P_h \cdot t_{\infty} = P_h \cdot \tau \cdot \ln \left( 1 + \frac{W_{AB}}{P_h} \right). \]

Observing that, for \( P_h \to \infty \), also \( E_{ha} \to W_{AB} \), the validity of definition (5.36) is higher as higher is the power.

*Problem 5.6*

Derive (5.48) and (5.50).

- **Solution**

**Fuel-Cell Propulsion Systems**

*Fuel Cells*

*Problem 6.1*

For high pressures, the thermodynamic properties of gas have to be calculated using the Redlich–Kwong equation of state instead of the ideal gas law. The Redlich–Kwong equation reads
\[ p = \frac{\hat{R} \cdot \vartheta}{V - b} - \frac{a}{\sqrt{\vartheta \cdot V \cdot (V + b)}}, \]  
(10.26)

where \( p \) is pressure, \( \hat{R} \) is the universal gas constant, \( \vartheta \) is temperature, \( \hat{V} \) is the molar volume. The constants \( a \) and \( b \) are defined as

\[ a = \frac{0.4275 \cdot \hat{R}^2 \cdot \vartheta_c^{2/5}}{p_c}, \quad b = \frac{0.08664 \cdot \hat{R} \cdot \vartheta_c}{p_c}, \]  
(10.27)

where \( \vartheta_c \) is the temperature at the critical point, and \( p_c \) is the pressure at the critical point. Using this equation of state, evaluate the gaseous density of hydrogen at 350 bar, 700 bar, and 300 K.

\textbf{Solution}

From thermodynamic tables, find \( \hat{R} = 8.3141 \text{ J/mol/K}, \vartheta_c = 32.97 \text{ K} \) and \( p_c = 1.293 \text{ MPa} \). Then find

\[ a = \frac{0.4275 \cdot 8.314 \cdot 32.97^{2/5}}{1.293 \cdot 10^6} = 0.143, \]
\[ b = \frac{0.08664 \cdot 8.314 \cdot 32.97}{1.293 \cdot 10^6} = 1.837 \cdot 10^{-5}. \]

Now, for \( p = 700 \cdot 10^5 \text{ Pa} \) and \( \vartheta = 300 \text{ K} \) solve the equation of state for \( \hat{V} \) (a third-order algebraic equation) to find \( \hat{V} = 5.29 \text{ m}^3/\text{mol} \). The density is \( \rho_h = M_b/\hat{V} \) where \( M_b = 2 \text{ g/mol} \) is the molar mass of hydrogen. Thus \( \rho_h = 2 \cdot 10^{-3}/5.28 \cdot 10^{-5} = 37.8 \text{ kg/m}^3 \). Its estimation with the ideal gas equation would be

\[ \rho_h = \frac{700 \cdot 10^5}{300 \cdot 8.3144/2 \cdot 10^{-3}} = 56.1 \text{ kg/m}^3, \]

a substantially higher value.

The \textit{compressibility factor} defined as \( Z = \frac{p \cdot \hat{V}}{\hat{R} \cdot \vartheta} \), describes the deviation of a real gas from the ideal gas law (for which \( Z = 1 \)). In our case, \( Z \) is calculated as \( Z = 56.1/37.8 = 1.48 \).

For \( p = 350 \text{ bar}, \hat{V} = 8.78 \cdot 10^{-5} \text{ m}^3/\text{mol} \), thus \( \rho_h = 22.8 \text{ kg/m}^3 \). With the ideal gas equation, \( \rho_h = 28.1 \text{ kg/m}^3 \), thus \( Z = 1.23 \).

\textbf{Problem 6.2}

A good approximation of the compressibility factor of hydrogen between pressures \( p \) and \( p_0 \) is

\[ Z = 1 + 0.00063 \cdot \left( \frac{p}{p_0} \right) \]  
(10.28)
(verify it with the results of Problem 6.1). With this assumption evaluate the energy required to compress 1 kg of hydrogen (from 1 bar) to 350 bar and 700 bar, respectively, at 300 K, under the further assumptions of (i) isothermal compression, (ii) adiabatic compression. Evaluate the result as a percentage of the energy content of hydrogen.

- Solution

The elementary compression work is

\[ dW = v dp, \]

where \( v \) is the specific volume (\( v = 1/\rho \)). From the definition of compressibility factor, \( p \cdot v = Z \cdot R \cdot \vartheta \). Therefore the compression work is obtained as

\[ W = \int_{p_0}^{p} \frac{Z \cdot R \cdot \vartheta}{p \cdot v} dp. \]

Calculating the integral in the isothermal case (\( \vartheta = \text{const.} \)), obtain

\[ W_c = R \cdot \vartheta \cdot \left( \ln \left( \frac{p}{p_0} \right) + 0.00063 \cdot (p - p_0) \right). \]

In the adiabatic case, a simple expression for the compression work is obtained using an average compressibility factor, yielding

\[ W_c = \frac{Z \cdot R \cdot \vartheta}{\gamma - 1} \cdot \left( \left( \frac{p}{p_0} \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right), \]

where \( \gamma \) is the specific heat ratio (1.4 for hydrogen).

For 700 bar, \( \bar{Z} = (1 + 1.48)/2 = 1.24 \). The compression work is \( W_c = 8.7 \text{ MJ/kg} \) in the isothermal case and \( W_c = 21.2 \text{ MJ/kg} \) in the adiabatic case. The ratio to the lower heating value (120 MJ/kg) is 7.2% in the case (i), 17.7% in the case (ii).

For 350 bar, \( \bar{Z} = (1 + 1.23)/2 = 1.11 \). The compression work is \( W_c = 7.6 \text{ MJ/kg} \) in the isothermal case and \( W_c = 15.0 \text{ MJ/kg} \) in the adiabatic case. The ratio to the lower heating value (120 MJ/kg) is 6.3% in the case (i), 12.5% in the case (ii).

**Problem 6.3**

Typical characteristics of various metal-hydride materials (1–4) for hydrogen storage are listed in the following table [350]. Evaluate the energy density for these storage systems.

- Solution
<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material density (g/cm³)</td>
<td>6.2</td>
<td>1.25</td>
<td>1.26</td>
<td>0.66</td>
</tr>
<tr>
<td>Porosity (%)</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>Mass storage capacity (%)</td>
<td>1.8</td>
<td>5.55</td>
<td>6.5</td>
<td>11.5</td>
</tr>
</tbody>
</table>

The energy density is given by

$$\frac{E_{ht}}{V_{ht}} = \frac{H_h \cdot \xi_{ht}}{\gamma_{ht}}$$

where $\gamma_{ht}$ is the reciprocal of the material density. For material (1),

$$\frac{E_{ht}}{V_{ht}} = 6.2 \left( \frac{kg_{mat}}{l} \right) \cdot 1.8 \cdot 10^{-2} \left( \frac{kg_h}{kg_{mat}} \right) \cdot 33.3 \left( \frac{kWh}{kg_h} \right) = 1.86 \left( \frac{kWh}{l} \right).$$

For materials (2)–(4), the results are \{1.15, 1.36, 1.26\} kWh/kg, respectively. Despite an increase in the gravimetric storage capacity, the energy density decreases when changing from material (1) to (4). Even the specific energy may decrease if the necessary system becomes significantly larger and thus heavier (ancillaries, etc.).

**Problem 6.4**

Evaluate the increase of energy density obtained with the cryo-compressed tank (CcH2) concept operated at 77 K with respect to conventional, ambient-temperature pressurized tanks.

- **Solution**

Using the method of Problem 6.1, find

- For $p = 700$ bar and $\vartheta = 77$ K, $\rho_h = 77$ kg/m³ (twice the density at 300 K).
- For $p = 350$ bar and $\vartheta = 77$ K, $\rho_h = 61$ kg/m³ (more than twice the density at 300 K).
- For liquid cryogenic storage, $\rho_h = 71$ kg/m³ (comparable with the CcH2 but at the cost of a higher liquefaction energy).

**Problem 6.5**

Explain the values of $\gamma_{ht}$ in Table 6.2.1, for storage tanks pressurized at 350 bar.

- **Solution**

Use the equation (5.33)

$$w = \frac{p \cdot d}{4 \cdot \sigma},$$
to estimate the width $w$ of a spherical vessel as a function of gas pressure $p$ and tensile strength $\sigma$ of the vessel material. The vessel mass is therefore

$$m_{ht} = \rho \cdot \frac{\pi}{2} \cdot d^2 \cdot w = \frac{3}{2} \cdot V \cdot p \cdot \frac{\rho}{\sigma},$$

where $\rho$ is the density of the vessel material and $V$ the contained volume. Find $\gamma_{ht}$ as $V/m_{ht}$.

For steel, use $\rho = 8 \text{ kg/l}$ and $\sigma = 460 \text{ MPa}$. Find $\gamma_{ht} = \frac{2}{3} \cdot 460/35/8 = 1.11/\text{kg}$. For an aluminium alloy, use $\rho = 2.71/\text{kg}$ and $\sigma = 210 \text{ MPa}$ and find $\gamma_{ht} = \frac{2}{3} \cdot 210/35/2.7 = 1.51/\text{kg}$. For a magnesium-based composite, $\rho = 1.91/\text{kg}$ and $\sigma = 1000 \text{ MPa}$ and find $\gamma_{ht} = \frac{2}{3} \cdot 1000/35/1.9 = 101/\text{kg}$. Of course at 700 bar these values are halved.

Currently materials with tensile strength of 6000 MPa and more are being introduced.

**Problem 6.6**

Evaluate the storage pressure and the specific strength (ratio of tensile strength to density) of the tank material that would be necessary to meet the 2015 DOE targets of Table 6.2.1 with gaseous hydrogen.

- **Solution**

The target on energy density fixes the required hydrogen density, $\rho_h = (E_{ht}/V_{ht})/H_h = 2.7/33.3 = 0.081 \text{ kg/l}= 81 \text{ kg/m}^3$. Using the results of Problem 6.1, the required pressure is

$$p = \frac{\tilde{R} \cdot \vartheta}{V - b} - \frac{a}{\sqrt{\vartheta} \cdot V \cdot (V + b)},$$

where $\tilde{V} = M_h/\rho_h$. Find $\tilde{V} = 2.47 \cdot 10^{-5} \text{ m}^3/\text{mol}$. Find the pressure as $p = 385 \text{ MPa}$, a very large value compared with current technology.

The storage capacity is fixed by the specific energy target,

$$\gamma_{ht} = (E_{ht}/m_{ht})/(H_h \cdot \rho_h) = 3/2.7 = 1.1 \text{ l/kg}.$$  

The specific strength is calculated using the results of Problem 6.5, i.e., $p = 3/2 \cdot \gamma_{ht} \cdot p = 1.5 \cdot 1.385 = 635 \text{ kNm/kg}$. Compare with $460/8=57.5 \text{ kNm/kg}$ for steel, $1000/1.88=532 \text{ kNm/kg}$ for a composite material etc. However, carbon fiber can have a specific strength of a few thousands kNm/kg.

**Problem 6.7**

Explain the explicitness of the number of cells $N$ in (6.72).

- **Solution**
Problem 6.8

For the fuel cell stack of Fig. 6.11, find (i) the maximum output power \( P_{fcs,max} \), (ii) the current \( I_{fc,P} \) at which this power is yielded, and (iii) the current \( I_{fc,\eta} \) that maximizes the overall efficiency. Compare the result with the curves shown in the figure.

**Solution**

Evaluate \( P_{fcs,max} \) from (6.79) in such a way that the term under the square root is zero,

\[
P_{fcs,max} = N^2 \cdot \frac{(U_{oc} - k_{aux})^2}{4 \cdot N \cdot R_{fc}} - P_0.
\]

With the data given in the caption of Fig. 6.11, \( P_{fcs,max} = 250^2 \cdot \frac{(0.82 - 0.05)^2}{(4 \cdot 250 \cdot 0.024)} - 100 = 15.34 \text{ kW} \).

The current at which this power is yielded is found as

\[
I_{fc,P} = N \cdot \frac{U_{oc} - k_{aux}}{2 \cdot N \cdot R_{fc}}.
\]

With the data \( I_{fc,P} = 250 \cdot \frac{(0.82 - 0.05)}{(2 \cdot 250 \cdot 0.0024)} = 160.4 \text{ A} \).

As for the stack current that maximizes the efficiency, use \( I_{fc,\eta} = \sqrt{P_0/(N \cdot R_{fc})} \). With the data given in Fig. 6.11, \( I_{fc,\eta} = \sqrt{100/(250 \cdot 0.0024)} = 12.9 \text{ A} \).

Problem 6.9

Calculate the same quantities as in Problem 6.8 for a small fuel cell stack powering a racing FCHEV (see Sect. 8.6). Use the quadratic expression (6.71) for \( P_{aux} \) and the following data: \( N \cdot U_{oc} = 16.8 \), \( N \cdot R_{fc} = 0.137 \), \( P_0 = 19.89 \), \( \kappa_1 = 6.6 \), \( \kappa_2 = -0.024 \).

**Solution**

Find the maximum power

\[
P_{fcs,max} = \frac{(N \cdot U_{oc} - \kappa_1)^2}{4 \cdot (N \cdot R_{fc} + \kappa_2)} - P_0 = \frac{(16.8 - 6.6)^2}{4 \cdot (0.137 - 0.024)} = 19.89 = 210 \text{ W},
\]

for the current at maximum power

\[
I_{fc,P} = \frac{N \cdot U_{oc} - \kappa_1}{2 \cdot (N \cdot R_{fc} + \kappa_2)} = \frac{16.8 - 6.6}{2 \cdot (0.137 - 0.024)} = 45 \text{ A},
\]

and for the current at maximum efficiency

\[
I_{fc,\eta} = \sqrt{\frac{P_0}{N \cdot R_{fc} + \kappa_2}} = \sqrt{\frac{19.89}{0.137 - 0.024}} = 13.25 \text{ A},
\]
Reformers

Problem 6.10

Derive (6.95).

Solution

The methanol conversion ratio is defined as

\[ x = \frac{n_m(0) - n_m}{n_m(0)}, \]

so that \( n_m = n_m(0) \cdot (1 - x) \). Since one mole of methanol reacts with one of water (steam), the moles of water at a certain \( x \) are \( n_s = n_s(0) - \Delta n_m = n_s(0) - n_m(0) \cdot x \). Introducing the ratio \( \sigma = n_s(0)/n_m(0) \) of the feed gas, \( n_s = n_m(0) \cdot (\sigma - x) \). As for the hydrogen, there is an increase of 3 moles per each mole of methanol. Therefore, the moles of hydrogen are \( n_h = 3 \cdot x \cdot n_m(0) \). Similarly, there is an increase of \( x \cdot n_m(0) \) moles of CO\(_2\). The molar concentration of methanol is thus

\[
C_m(x) = \frac{n_m}{n_m + n_s + n_h + n_{CO_2}} = \frac{n_m(0)}{n_m(0) \cdot (1 - x) + n_m(0) \cdot (\sigma - x) + 3 \cdot x \cdot n_m(0) + x \cdot n_m(0)} = \frac{1 - x}{1 + 2 \cdot x + \sigma}.
\]

Since the initial value of the concentration is \( C_m(0) = 1/(1 + \sigma) \), find (6.95).

Supervisory Control Algorithms

Driver’s Interpretation

Problem 7.1

An ICE-based powertrain has the following characteristics: \( \gamma = \{15.0, 8.1, 5.3, 3.9, 3.1, 2.6\} \), wheel radius \( r_w = 0.32 \text{ m} \), rated engine power \( P_{e,\text{max}} = 92 \text{ kW} \) at a speed \( \omega_{e,\text{max}} = 524 \text{ rad/s} \), engine braking torque \( T_{e,\text{min}} = 20 \text{ Nm} \). Build a driver’s interpretation map. Then, follow a torque control structure to generate an engine torque setpoint for a driver pedal request of 50% at a vehicle speed of 100 km/h and fourth gear.

Solution
The curve describing the maximum force available at the wheels consists of a first part that reproduces the engine maximum torque curve (not known in this exercise) for the 1st gear, and a second part that is the envelop of the maximum-power engine points at different gears. The maximum-power engine point is at $\omega_e = 524 \text{ rad/s}$ and $T_e = 92 \cdot 10^3 / 524 = 176 \text{ Nm}$.

The vehicle speed at gear $i$ is related to the engine speed by the equation $v_i = r_w / \gamma_i \cdot \omega_e$, for $i = 1, \ldots, 6$. The force at the wheels is $F_{t,i} = \gamma_i / r_w \cdot T_e$. Since $v = 100 / 3.6 = 27.8 \text{ m/s}$ is greater than $v_1 = 524 \cdot 0.32 / 15 = 11.2 \text{ m/s}$, the maximum-power range is active. Therefore, $F_t = 92 \cdot 10^3 / V$ corresponds to 100% accelerator pedal. The maximum brake power is calculated from the engine data as $20 \cdot 524 = 10.5 \text{ kW}$. Thus $F_t = 10.5 \cdot 10^3 / V$ corresponds to 0% accelerator pedal. At the current speed, the maximum force is 3312 N, the minimum force is 378 N. For a pedal depression of 50%, assuming linear interpolation, we have a force request $F_t = -378 + (3312 + 378) \cdot 0.5 = 1467 \text{ N}$.

Assuming 4th gear, the engine torque is $T_e = 0.32 / 3.9 \cdot 1467 = 120 \text{ Nm}$ at a speed $\omega_e = 339 \text{ rad/s} = 3235 \text{ rpm}$.

**Problem 7.2**

Add an electric machine to the powertrain of Problem 7.1, having the following characteristics: maximum torque $T_{m,max} = 140 \text{ Nm}$, base speed $\omega_b = 300 \text{ rad/s}$, maximum power $P_{m,max} = 42 \text{ kW}$. Calculate the total torque demand for the same driving situation as in Problem 7.1 if power assist is authorized at each vehicle speed. Assume coupled regenerative braking.

* Solution

The vehicle speed corresponding to the base speed of the motor is $300 \cdot 0.32 / 11 = 8.7 \text{ m/s} = 31.4 \text{ km/h}$, thus lower than $v_1$ (see Problem 7.1). Therefore, the actual vehicle speed corresponds to the maximum-power range of the motor. Summing the two powers yields $92 + 42 = 134 \text{ kW}$. The minimum force at the wheels does not change with respect to the ICE case. The maximum force is now $F_t = 134 \cdot 10^3 / 27.8 = 4820 \text{ N}$. For a 50% pedal position, the force demand is $F_t = -378 + (4820 + 378) \cdot 0.5 = 2221 \text{ N}$, which corresponds to a total powertrain torque of $T_t = 2221 \cdot 0.32 = 711 \text{ N}$. In the 4th gear, that would correspond to an engine torque $T_e = 711 / 3.9 = 182 \text{ Nm}$, which is very close to its maximum torque.

**Problem 7.3**

Propose a driver’s interpretation function for a BEV whose motor and battery have the same data as in Problem 4.9. Assume a coupled braking circuit. Calculate the torque setpoint for (i) $\omega_m = 0 \text{ rad/s}$ and 0% pedal depression, (ii) $\omega_m = 100 \text{ rad/s}$ and 0% pedal depression, (iii) $\omega_m = 250 \text{ rad/s}$ and 50% pedal depression.
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• Solution

The 0% pedal position corresponds to the minimum between \( I_{a,\text{min}} \) as calculated in Problem 4.35 and a negative torque for which the driver has a similar feeling than with an ICE-based powertrain. Fix this torque to, say, 1/10 of the maximum torque. Thus

\[
I_{a,\text{min}} = \begin{cases} 
-\kappa_a \cdot \frac{\omega_m}{2 \cdot R_a} & \text{for } \omega_m < \frac{2 \cdot R_a \cdot (0.1 \cdot I_{\text{max}})}{\kappa_a}, \\
-0.1 \cdot I_{\text{max}} & \text{for } \omega_m < \frac{P_{\text{max}}}{\kappa_a \cdot 0.1 \cdot I_{\text{max}}}, \\
-\frac{P_{\text{max}}}{\kappa_a \cdot \omega_m} & \text{else},
\end{cases}
\]

while for 100% acceleration

\[
I_{a,\text{max}} = \begin{cases} 
I_{\text{max}} & \text{for } \omega_m < \frac{P_{\text{max}}}{\kappa_a \cdot I_{\text{max}}}, \\
\frac{P_{\text{max}}}{\kappa_a \cdot \omega_m} & \text{else}.
\end{cases}
\]

The curve \( I_a = f(\alpha, \omega_m) \) is obtained by interpolation between these two limits, \( I_a = I_{a,\text{min}} + \alpha \cdot (I_{a,\text{max}} - I_{a,\text{min}}) \).

For the case (i), \( I_{a,\text{min}} \) is 0 and \( I_a = T_m = 0 \).

For the case (ii), \( \omega_m = 100 \text{ rad/s} \) and \( \omega_m > \frac{2 \cdot R_a \cdot (0.1 \cdot I_{\text{max}})}{\kappa_a} = 2 \cdot 0.05 \cdot (0.1 \cdot 88)/0.25 = 3.52 \text{ rad/s} \). Thus \( I_a = I_{a,\text{min}} = -0.1 \cdot 88 = -8.8 \text{ A} \), \( T_m = -2.2 \text{ Nm} \).

For the case (iii), \( \omega_m = 250 \text{ rad/s} \) and \( \omega_m < 4 \cdot 10^3/(0.25 \cdot 0.1 \cdot 88) = 1818 \text{ rad/s} \). Thus \( T_m = -2.2 + 0.5 \cdot (22 + 2.2) = 9.9 \text{ Nm} \).

Problem 7.4

Derive a PI model for a human driver of an electric vehicle that tries to follow a prescribed drive cycle acting on the acceleration pedal. Derive a gain-scheduling tuning of the PI parameters. Vehicle data: mass \( m_v = 1360 \text{ kg} \), \( \frac{1}{2} \cdot \rho_b \cdot A_f \cdot c_d = 0.25 \text{ N/(m/s)}^2 \). Evaluate the PI coefficients for a vehicle speed of 20 m/s. Assume perfect recuperation (decoupled braking).

• Solution

Vehicle dynamics can be generally written as

\[
\frac{dv}{dt} = \frac{F_t - c_0 - c_2 \cdot v^2}{m_v}.
\]

The driver is sensitive to the difference between \( v(t) \) and \( v_s(t) \). Its action is on the acceleration and brake pedals. Define the general driver’s output as the force \( F_t \) (positive for traction, negative for braking). Thus

\[
\frac{dv}{dt} = \frac{u - c_0 - c_2 \cdot v^2}{m_v}.
\]
\[ u = K_p \cdot (v - v_s) + K_i \cdot \int (v - v_s) \, dt. \]

Linearize around an operating point \( v_s \), then define \( u_s = c_0 + c_2 \cdot v_s^2 \), \( z = v - v_s \), \( w = u - u_s \). Then evaluate

\[
\frac{dz}{dt} = w - c_2 \cdot 2 \cdot v_s \cdot (v - v_s) = \frac{w - 2 \cdot c_2 \cdot v_s \cdot z}{m_v} = K \cdot w - a \cdot z,
\]

where \( K = 1/m_v \) and \( a = 2 \cdot c_2 \cdot v_s / m_v \).

Now close the loop with the driver regulator

\[ w = K_p \cdot z + K_i \cdot \int z \, dt \]

Obtain

\[ s^2 \cdot z = K \cdot K_p \cdot s \cdot z + K \cdot K_i \cdot z - a \cdot s \cdot z \rightarrow z(s) \cdot (s^2 + (a - K \cdot K_p) \cdot s - K \cdot K_i) = 0 \]

By pole-placement, target at

\[ \omega_n = \sqrt{-K \cdot K_i} = 1 \text{ rad/s}, \]

and

\[ \zeta = (a - K \cdot K_p) / (2 \cdot \omega_n) = 0.7 \]

With the numerical values, \( K = 1/1360 = 7.35 \cdot 10^{-4}, a = 2 \cdot 0.25 \cdot v_s / 1360 = 3.7 \cdot 10^{-4}, v_s, K_i = -1/K = -1360, K_p = (a - 0.7 \cdot 2) / K = (3.7 \cdot 10^{-4} \cdot v_s - 0.7 \cdot 2) / 7.35 \cdot 10^{-4} \). For \( v_s = 20 \text{ m/s} \) obtain \( K_p = -1898 \text{ N} \).

**Regenerative Braking Control**

**Problem 7.5**

Derive an ideal law to split the braking effort between the two axles under the assumption that the adherence is the same at each wheel.

- **Solution**

Write force and torque balance equations for a two-wheel equivalent vehicle. We have four equations in the four unknowns \( N_1, N_2 \) (normal forces), \( F_1, F_2 \) (longitudinal forces), where subscript 1 is for front wheels and subscript 2 for rear wheels, while the total required force \( F_t \) is known:

\[
\begin{align*}
F_1 + F_2 &= F_t, \quad (\text{balance of longitudinal forces}) \\
N_1 + N_2 &= N, \quad (\text{balance of normal forces}) \\
N \cdot b &= F_1 \cdot h + N_1 \cdot (a + b), \quad (\text{balance of momenta}) \\
\frac{F_1}{N_1} &= \frac{F_2}{N_2} \quad (\text{equal adherence})
\end{align*}
\]
where $N = m_v \cdot g$ is the vehicle weight, $a$ and $b$ are the horizontal distances of the wheel axles from the center of gravity (CoG), and $h$ the height of the CoG.

By combining these four equations, obtain

$$N_1 = \frac{N \cdot b}{a + b} - \frac{F_t \cdot h}{a + b}$$

$$N_2 = N - N_1 = \frac{N \cdot a}{a + b} + \frac{F_t \cdot h}{a + b}$$

During braking, $F_t < 0$ and the front weight increases. Moreover, the equal adherence condition reads

$$F_1 \cdot N_2 = F_2 \cdot N_1.$$  \hspace{1cm} (10.29)

For a given $F_t$, obtain the ideal split

$$F_1 \cdot (N \cdot a - F_t \cdot h) = (F - F_1) \cdot (N \cdot b - F_t \cdot h) \rightarrow F_1 \cdot N(a+b) = F_t \cdot (N \cdot b - F_t \cdot h),$$

or

$$F_1 = \frac{F_t \cdot b}{a + b} - \frac{F_t^2 \cdot h}{N \cdot (a + b)}. $$

In terms of torques

$$T_1 = \frac{T_1 \cdot b}{a + b} - \frac{T_2^2 \cdot h}{N \cdot r_w \cdot (a + b)},$$

$$T_2 = \frac{T_1 \cdot a}{a + b} + \frac{T_2^2 \cdot h}{N \cdot r_w \cdot (a + b)}.$$

The equal adherence curve $T_2 = T_2(T_1)$ is obtained by manipulating (11.6) as

$$F_1 \cdot (N \cdot a + F_t \cdot h) = F_2 \cdot (N \cdot b - F_t \cdot h)$$

or

$$T_1^2 \cdot h + 2 \cdot T_1 \cdot T_2 \cdot h + T_1 \cdot N \cdot a \cdot r_w - T_2 \cdot N \cdot b \cdot r_w + T_2^2 \cdot h = 0 \hspace{1cm} (10.30)$$

**Problem 7.6**

Consider a vehicle having an electric powertrain on the rear axle, with $T_{m, \text{max}} = 1540$ Nm (at the wheels), $P_{m, \text{max}} = 42$ kW, and the following vehicle characteristics (see Problem 7.5): static weight distribution fraction $s = 0.40$, height of CG $h = 55$ cm, wheelbase $l = 2.685$ m, wheel radius $r_w = 0.32$ m, vehicle mass $m_v = 1932$ kg. Evaluate the regenerative braking torque and the frictional braking torque on the front and rear axles for a total
braking torque $T_t = -1200 \text{Nm}$, vehicle speed $v = 90 \text{km/h}$, under (i) a maximum regeneration strategy, (ii) a constant braking distribution between the axles of 70%–30%, (iii) ideal braking as in the result of Problem 7.5, and (iv) a modified brake pedal that induces regenerative braking up to a deceleration of 0.05g and then frictional braking with a constant braking distribution of 70%–30%.

- Solution

At $v = 90 \text{km/h}$ the maximum regenerative capability of the electric powertrain is

$$\frac{42 \cdot 10^3}{3.6 \cdot 0.32} = 537 \text{Nm}.$$  

The braking power is

$$\frac{90 \cdot 400}{3.6 \cdot 0.32} = 31.2 \text{kW}.$$  

In the case (i), $T_{rec} = -537 \text{Nm}$; $T_1 = -1200 + 537 = -663 \text{Nm}$, $T_2 = 0 \text{Nm}$. The braking split ratio is 55%/45%. However, this value is above the equal-adherence curve, thus it is not admissible. The quantity $T_{rec}$ should be limited as in the case (iii).

In the case (ii), $T_{rec} = -0.30 \cdot 1200 = -360 \text{Nm}$, $T_1 = -1200 + 360 = -840 \text{Nm}$, $T_2 = 0 \text{Nm}$.

In the case (iii), using the formula of Problem 7.5,

$$T_{rec} = 0.40 \cdot (-1200) + \frac{1200^2 \cdot 0.55}{1932 \cdot 9.81 \cdot 0.32 \cdot 2.685} = -431 \text{Nm},$$

d and $T_1 = -1200 + 431 = -769 \text{Nm}$, $T_2 = 0 \text{Nm}$ (braking split ratio 64%/36%).

In the case (iv), the threshold torque is $m_v \cdot a \cdot r_w = 1932 \cdot (-0.05) \cdot 9.81 \cdot 0.32 = -300 \text{Nm}$. Thus $T_{rec} = -300 \text{Nm}$, $T_1 = 0.7 \cdot (-1200 + 300) = -630 \text{Nm}$, $T_2 = -270 \text{Nm}$.

**Problem 7.7**

Consider a conventional (coupled) braking system where $T_2 = k \cdot T_1$ ($T_1 < 0, T_2 < 0$). Calculate the maximum value of the adherence that can be obtained under the assumption of equal adherence between the axles (ideal distribution curve) and the corresponding total braking torque. Check what happens for higher braking torques. Then, calculate the limit value of $k$ that can be achieved. Use the numerical values of Problem 7.6.

- Solution

The limit of conventional braking systems occurs when the practical braking split curve meets the ideal split curve. The latter is given by (11.7), while the former reads $T_2 = k \cdot T_1$. By combining these two equations, obtain...
\[ T_1 = \frac{-N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1 + k)^2} \]

and

\[ T_2 = \frac{-k \cdot N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1 + k)^2} \]

Since both \( T_1 \) and \( T_2 \) are negative quantities, a condition on \( k \) is that \( a - k \cdot b > 0 \), or

\[ k < \frac{a}{b} \]

For \( k = a/b \), the braking distribution is such that

\[ \frac{T_1}{T_t} = 1 \quad \frac{T_2}{T_t} = \frac{k}{1 + k} \]

The common value of the adherence factor (\( \mu_1 = \mu_2 = \mu \)) is obtained by calculating \( N_1 \) and \( N_2 \). First evaluate

\[ T_t = T_1 + T_2 = (1 + k) \cdot T_1 = \frac{-N \cdot r_w \cdot (a - k \cdot b)}{h \cdot (1 + k)} \]

Then, find

\[ N_1 = \frac{N \cdot b - T_1 \cdot h \cdot r_w}{a + b} = \frac{N \cdot b + N \cdot (a \cdot k - b)}{(1 + k) \cdot (a + b)}, \quad N_2 = \frac{N \cdot k}{1 + k} \]

and the adherence factor as

\[ \mu = \frac{-T_1}{N_1 \cdot r_w} = \frac{-T_2}{N_2 \cdot r_w} = \frac{a - k \cdot b}{h \cdot (1 + k)} \]

With the data of Problem 7.6, the limit \( k = 0.4/0.6 = 0.667 \). For a value \( k = 0.3/0.7 = 0.429 \), the maximum adherence is 0.488. The limit values of \( T_1 \) and \( T_2 \) are -2055 Nm and -881 Nm. The maximum braking torque is \( T_t = -2935 \) Nm.

For higher braking torques, e.g., \( T_t - 3500 \) Nm, the braking split ratio would be 2450/1050 Nm. The vertical forces would be \( N_1 = \frac{13632}{0.32} = 13632 \) N, \( N_2 = 1932 \cdot 9.81 - 13632 = 5321 \) N. Thus \( \mu_1 = 2450/13632/0.32 = 0.56 \) while \( \mu_2 = 1050/0.32/5321 = 0.62 \). This circumstance \( \mu_2 > \mu_1 \) is potentially dangerous and should be avoided.

**Problem 7.8**

Derive the ideal braking distribution law as in Problem 7.5 when one axle is motoring while the other is braking (for instance, battery recharge mode in an HEV with an engine on the front axle and an electric machine on the rear axle). For simplicity, assume \( a = b \).
Solution

As in Problem 7.5,

\[ F_1 + F_2 = F_t, \quad \text{(balance of logitudinal forces)} \]
\[ N_1 + N_2 = N, \quad \text{(balance of normal forces)} \]
\[ N \cdot b = F_t \cdot h + N_1 \cdot (a + b), \quad \text{(balance of momenta)} \]

but now

\[ \frac{F_1}{N_1} = -\frac{F_2}{N_2}, \]

the latter being the condition of equal adherence \( \mu_1 = -\mu_2 \). By combining these four equations, obtain

\[
N_1 = \frac{N \cdot b}{a + b} - \frac{F_t \cdot h}{a + b}, \\
N_2 = N - N_1 = \frac{N \cdot a}{a + b} + \frac{F_t \cdot h}{a + b}
\]

\[ F_1 \cdot N_2 = -F_2 \cdot N_1, \quad (10.31) \]

from whence

\[ F_1 \cdot (N \cdot a + F_1 \cdot h + F_2 \cdot h) = -F_2 \cdot (N \cdot b - F_1 \cdot h - F_2 \cdot h) \rightarrow F_1 \cdot N \cdot a + h \cdot F_t^2 = -N \cdot b \cdot F_2 + h \cdot F_2^2. \]

There are two solutions to this equation for \( a = b \). One is \( F_2 = F_1 + \frac{N \cdot a}{a + b} \). The other is \( F_2 = -F_1 \), which does not imply the satisfaction of the total force \( F_t \) but constitutes a limit in the 2nd and 4th quadrant of the plane \( F_1 - F_2 \).

From the first solution, find

\[ T_1 = \frac{T_t}{2} - \frac{N \cdot a \cdot r_{\text{w}}}{2 \cdot h}, \quad T_2 = \frac{T_t}{2} + \frac{N \cdot a \cdot r_{\text{w}}}{2 \cdot h}. \]

Dynamic Coordination

Problem 7.9

In a series HEV the supervisory control yields engine torque and speed set-points \( T_e \) and \( \omega_e \). Derive a simple generator controller in order to achieve the desired speed of the APU.

Solution

The dynamics of the APU can be described by the simplified equation

\[ \Theta_{\text{apu}} \frac{d\omega_e(t)}{dt} = T_e(t) - T_g(t). \]
The generator torque open-loop setpoint is $T_{g,sp} = T_{e,sp}$. However, in order to let the generator speed converge toward the value $\omega_e$, at least a proportional correction should be added. Assuming $T_e(t) = T_{e,sp}(t)$,

$$T_g(t) = T_e(t) + k_p \cdot (\omega_g(t) - \omega_e(t)).$$

The closed-loop dynamics therefore reads

$$\Theta_{apu} \cdot s \cdot \ddot{\omega} = -K_p \cdot \ddot{\omega},$$

which converges to $\ddot{\omega} = 0$ with a time constant $\Theta_{apu}/k_p$.

**Problem 7.10**

Derive the dynamic equations to control the generator torque in a simple PSD-based system like that of the Toyota Prius. Neglect the generator inertia.

- **Solution**

Manipulate the dynamic equations (4.163)–(4.164) with $\Theta_{sun} = 0$ to obtain

$$\Theta_{carrier} \cdot \frac{d\omega_e(t)}{dt} = T_e(t) - \frac{1 + z}{z} \cdot T_g(t),$$

$$\Theta_{ring} \cdot \frac{d\omega_f(t)}{dt} = \frac{1}{z} \cdot T_g(t) + T_m(t) - T_f(t).$$

where $\Theta_{carrier}$ is represented by $\Theta_e$ and $\Theta_{ring}$ by the vehicle inertia.

The resulting dynamics for the engine speed is rather similar to that of Problem 7.9, except for the $\frac{1 + z}{z}$ factor now multiplying the generator torque. It is laborious but straightforward to show that the same dynamic equation for $\omega_e$ (but not for $\omega_f$) applies also to the case where $\Theta_{sun}$ is not negligible.

**Problem 7.11**

In a post-transmission parallel HEV, in principle it is possible to compensate the torque gap at the wheels during a gear shift. Evaluate the time lapse after which the vehicle speed before the shift is recovered ($t_r$, recovery time) and the necessary electric energy for a downshift from 4th to 3rd gear occurring during a constant vehicle acceleration. Data: gear ratios including final gear $\gamma = 5.4$, motor gear ratio = $\gamma_m = 11$, transmission efficiency $\eta_t = 0.97$, motor efficiency $\eta_m = 0.85$, wheel radius $r_w = 0.29 \text{ m}$, engine shift speed $\omega_e = 4500 \text{ rpm}$, shift duration $t_s = 1 \text{ s}$, acceleration $a = 0.5 \text{ m/s}^2$, vehicle mass $m_v = 1360 \text{ kg}$, $c_r = 0.009$, $c_d \cdot A_f = 0.5 \text{ m}^2$.

- **Solution**
Define the two time points \( t_1 \) and \( t_2 \) as the beginning and the end of the gear shift \((t_2 - t_1 = t_s)\). Evaluate

\[
v_1 \triangleq v(t_1) = \frac{\dot{\omega}_{e,shift}}{r_e} = \frac{4500 \pi}{20 \cdot 0.29} = 27 \text{ m/s} = 98 \text{ km/h}.
\]

Without compensation, the vehicle speed decreases during the shift according to the law

\[
\frac{dv(t)}{dt} = -g \cdot c_r - \frac{\rho_a \cdot A_f \cdot c_d \cdot v^2}{2 \cdot m_w},
\]

or (2.17) with \( \alpha = \sqrt{\frac{0.5 \cdot 1.2}{1360 \cdot 0.8}} = 0.015 \), \( \beta = \sqrt{9.8 \cdot 0.009} = 0.3 \). The coasting velocity at \( t_2 \) is

\[
\begin{align*}
v_2 &= \frac{\beta}{\alpha} \cdot \tan \left( \arctan \left( \frac{\alpha}{\beta} \cdot v_1 \right) - \alpha \cdot \beta \cdot (t_2 - t_1) \right) \\
&= \frac{0.3}{0.015} \tan \left( \arctan \left( \frac{0.015}{0.3} \cdot 27 \right) - 0.015 \cdot 0.3 \cdot 1 \right) \\
&= 26.7 \text{ m/s} = 96 \text{ km/h}.
\end{align*}
\]

After the engine is engaged again, the speed increases according to the linear law \( v(t) = v_2 + a \cdot t \). The recovery time is

\[
t_3 = \frac{v_1 - v_2}{a} = \frac{0.3}{0.5} = 0.6 \text{ s}.
\]

Thus the total time lost is \( t_r = t_s + t_3 = 1 + 0.6 = 1.6 \text{ s} \). This time can be recuperated if the motor provides the missing torque during the shift. This torque is

\[
T_m(t) = \frac{F_i(t)}{r_w} = \frac{m_w \cdot a + m_v \cdot c_r + \rho_a \cdot c_d \cdot A_f \cdot v(t)^2}{2 \cdot r_w}.
\]

The energy provided by the motor is calculated from

\[
\Delta E_m = \int \omega_m(t) \cdot T_m(t) dt = \frac{1}{\eta} \cdot \int F_i(t) \cdot v(t) dt = \frac{1}{\eta} \cdot \left( \left( m_w \cdot a + m_v \cdot c_r \right) \cdot \int v(t) dt + \int 0.5 \cdot \rho_a \cdot c_d \cdot A_f \cdot v(t)^3 dt \right)
\]

\[
= \frac{1}{\eta} \cdot \left( \frac{m_w \cdot v_2^2 - v_1^2}{2} + \frac{m_v \cdot c_r}{2 \cdot a} \cdot (v_2^2 - v_1^2) + 1.2 \cdot c_d \cdot A_f \cdot (v_2^4 - v_1^4)/(8 \cdot a) \right),
\]

where now \( v_2 = v_1 + a \cdot t_s = 27 + 0.5 \cdot 1 = 27.5 \text{ m/s} \). Thus the energy is

\[
\Delta E_m = \frac{1}{0.97} \cdot \left( 1360 \cdot \frac{27.5^2 - 27^2}{2} + 1360 \cdot 0.009 \cdot \frac{(27.5^2 - 27^2)}{2 \cdot 0.5} + 0.5 \cdot 1.2 \cdot 0.5 \cdot \frac{27.5^4 - 27^4}{4 \cdot 0.5} \right) = 25.7 \text{ kJ},
\]

with an average power \( \dot{P}_m = 25.7 \text{ kW} \).
Consider a parallel HEV with an electric machine mounted on the primary shaft of the gearbox with a reduction gear ratio $\gamma_m$. During a gear shift, the inertia of the motor sums up to the inertia of the primary shaft. To reduce the synchronization lag, the motor in principle could yield a torque to compensate its own inertia. Model this situation with simple equations. Then calculate the motor energy consumption for the following data: $\gamma_m = 3.3$, downshift from 4th to 3rd gear with $\gamma_3 = 5.5$, $\gamma_4 = 3.9$, vehicle speed $v = 60 \text{ km/h}$, motor inertia $\Theta_m = 0.07 \text{ kg/m}^2$.

**Solution**

During synchronization without motor assist, the dynamics of the primary shaft reads

$$\Theta_p \cdot \frac{d\omega_p(t)}{dt} = T_{m,i}(t) \cdot \gamma_m + k \cdot \left( \frac{v(t) \cdot \gamma_3}{r_w} - \omega_p(t) \right),$$

where the second right-hand term simulates the action of the synchronizer which is proportional to the difference between the secondary speed with the new gear ratio and the primary speed. The term $T_{m,i}$ is transmitted from the motor inertia

$$T_{m,i} = -\Theta_m \cdot \frac{d\omega_m}{dt}.$$

Since $\omega_m = \gamma_m \cdot \omega_p$, the equation becomes

$$\left( \Theta_p + \gamma_m^2 \cdot \Theta_m \right) \cdot \frac{d\omega_p(t)}{dt} = k \cdot \left( \frac{v(t) \cdot \gamma_3}{r_w} - \omega_p(t) \right),$$

and the primary shaft speed increases from the initial value $\omega_{p,4} = \frac{v_4}{r_w}$ up to the new value $\omega_{p,3} = \frac{v_3}{r_w}$. The variation law is (ideally) asymptotic,

$$\omega_p(t) = \frac{\gamma_3 \cdot v(t)}{r_w} + \left( \frac{\gamma_4 \cdot v(t)}{r_w} - \frac{\gamma_3 \cdot v(t)}{r_w} \right) \exp \left( -\frac{k \cdot t}{\Theta} \right),$$

where $\Theta \triangleq \Theta_p + \Theta_m \cdot \gamma_m^2$. To decrease the motor inertia, one should apply a torque $T_m$ such that

$$\Theta_m \cdot \frac{d\omega_m(t)}{dt} = T_m - T_{m,i},$$

such that $T_{m,i} = 0$, thus

$$T_m = \Theta_m \cdot \frac{d\omega_m(t)}{dt} = \Theta_m \cdot \gamma_m \cdot \frac{d\omega_p(t)}{dt} = \frac{\Theta_m \cdot \gamma_m \cdot k \cdot \left( v(t) \cdot \frac{\gamma_4}{r_w} - \omega_p(t) \right)}{\Theta_p}.$$
where $\omega_p(t)$ is still calculated with the equation above but with $\Theta_p$ instead of $\Theta$.

The motor power is

$$P_m = T_m \cdot \omega_m = \Theta_m \cdot \gamma_m \cdot \frac{d\omega_p}{dt} \cdot \gamma_m \cdot \omega_p = \Theta_m \cdot \gamma_m^2 \cdot \omega_p \cdot \frac{d\omega_p}{dt}.$$ 

The energy consumed results from the integral of $P_m$ or

$$E_m = \Theta_m \cdot \gamma_m^2 \cdot \frac{\omega_p^2}{2} - \frac{\omega_p^4}{2}.$$ 

Numerically,

$$\omega_{p,3} = \frac{70 \cdot 5.5}{3.6 \cdot 0.32} = 334 \text{ rad/s}$$

$$\omega_{p,4} = \frac{70 \cdot 3.9}{3.6 \cdot 0.32} = 237 \text{ rad/s}$$

$$E_m = 0.07 \cdot 3.3^2 \cdot \frac{334^2 - 237^2}{2} = 21 \text{ kJ}$$

### Heuristic Energy Management Strategies

#### Problem 7.13

Consider a pre-transmission, single-shaft parallel HEV with fixed gear reduction. System data: gear ratio including final gear ratio $\gamma = 4$, engine maximum torque curve $T_{e,\text{max}}(\omega_e) = 50 + 0.7 \cdot \omega_e - 1 \cdot 10^{-3} \cdot \omega_e^2$, motor maximum torque $T_{m,\text{max}} = 150 \text{ Nm}$, motor maximum power $P_{m,\text{max}} = 25 \text{ kW}$, vehicle data $c_D = 0.33$, $A_f = 2.5 \text{ m}^2$, $c_r = 0.013$, $m_v = 1500 \text{ kg}$, $\Theta_w = 0.25 \text{ kg m}^2$, $r_w = 0.25 \text{ m}$. Consider the simple, SOC-independent heuristic energy-management strategy:

- EV mode if $\omega_e < 1000 \text{ rpm}$ or if $T_e < 40 \text{ Nm}$,
- power assist mode if $T_e > T_{e,\text{max}}$,
- else, recharge mode if $T_e > 0$
- regenerative braking if $T_e < 0$.

Evaluate the scheduled mode, the engine torque, and the motor torque for the following driving situations: (i) $v = 17 \text{ km/h}$, $a = 1.37 \text{ m/s}^2$; (ii) $v = 38.76 \text{ km/h}$, $a = 0.0944 \text{ m/s}^2$; (iii) $v = 28.8 \text{ km/h}$, $a = 1.56 \text{ m/s}^2$; (iv) $v = 95 \text{ km/h}$, $a = 0.19 \text{ m/s}^2$.

- Solution
For the case (i) the required propulsion force is
\[
F_t = \left( m_v + \frac{\Theta_w}{r_w \cdot \gamma^2} \right) \cdot a + m_v \cdot 9.81 \cdot c_v + \frac{1}{2} \cdot \rho_a \cdot c_D \cdot A_f \cdot v^2 = 1503 \cdot 1.369 + 191.3 + 0.47 \cdot 4.75^2 = 2257 \text{ N.}
\]
The engine torque would be \( T_e = F_t \cdot \frac{r_w}{\gamma} \) = 2257 \cdot 0.25/4 = 141 \text{ Nm.} The engine speed would be \( \omega_e = v \cdot \frac{1}{r_w} = 75.9 \text{ rad/s} = 725 \text{ rpm.} \) The mode selected would be the ZEV (\( \omega_e < 1000 \text{ rpm} \) and \( T_e > 0 \)). The base speed is \( 25 \cdot 10^3/150 = 167 \text{ rad/s.} \) The motor torque \( T_m = 141 \text{ Nm} \) is lower than the motor maximum torque at 725 rpm, which is 150 Nm.

For the case (ii), the required force is \( 1503 \cdot 0.094 + 191.3 + 0.47 \cdot (38.76/3.6)^2 = 387 \text{ Nm.} \) The engine torque would be of 387 \cdot 0.25/4 = 24.2 Nm. The engine speed would be \( 38.76/0.25 \cdot 4 = 172 \text{ rad/s} = 1645 \text{ rpm.} \) The mode selected would be again the ZEV \( (T_e < 40 \text{ Nm} \) with \( \omega_e > 1000 \text{ rpm}). \) At 1645 rpm (higher than the motor base speed) the motor maximum torque is \( 25 \cdot 10^3/172 = 145 \text{ Nm}. \) Thus the ZEV mode is feasible.

For the case (iii), the required force is \( 1503 \cdot 1.56 + 191.3 + 0.47 \cdot (28.8/3.6)^2 = 2535 \text{ N.} \) The engine torque would be of 160 Nm. The engine speed would be of 1230 rpm. The engine max torque at that speed would be \( T_{e,\text{max}} = 50 + 0.7 \cdot 128 - 1 \cdot 10^{-3} \cdot 128^2 = 123 \text{ Nm.} \) Thus the mode selected would be the boost \( (T_e > T_{e,\text{max}} \) and \( \omega_e > 1000 \text{ rpm}). \) The motor torque would be 160 – 123 = 37 Nm, which is lower than the motor max torque.

For the case (iv), the required force is 805 N. The engine torque would be of 50 Nm. The engine speed would be 4033 rpm = 422 rad/s. Thus the selected mode would be the battery recharge \( (T_e > 40 \text{ Nm} \) and \( \omega_e > 1000 \text{ rpm}). \) The maximum generating torque at 422 rad/s is \( -25 \cdot 10^3/422 = -59 \text{ Nm.} \) Thus the maximum engine torque could be 50 + 59 = 119 Nm (feasible because the maximum engine torque is 167 Nm).

**Problem 7.14**

Consider the following SOC-dependent energy-management heuristic strategy:

- engine on with \( P_e = P_t - P_{t,b}(\xi) \) if \( P_t > P_{e,\text{start}}(\xi) \).
- else, engine off,

with the definitions \( P_{t,b} = -P_{m,\text{max}} + 2 \cdot P_{m,\text{max}}/(\xi_{hi} - \xi_{lo}) \cdot (\xi - \xi_{lo}) \), \( P_{e,\text{start}} = P_{m,\text{max}}/(\xi_{hi} - \xi_{lo}) \cdot (\xi - \xi_{lo}) \) and the numerical values \( \xi_{hi} = 80\%, \xi_{lo} = 40\% \).

Assume a unit-efficiency motor operation. Perform again the calculations of Problem 7.13, for \( \xi = \{55, 70\}\% \).

- Solution

For the case (ii), engine possible speed and torque \( \omega_e = 172 \text{ rad/s, } T_e = 24.2 \text{ Nm} \) would lead to \( P_e = 4.16 \text{ kW.} \) Since \( P_{m,\text{max}} = 25 \text{ kW} \), \( P_{e,\text{max}} = 24.2 \text{ kW} \), evaluate
\[ P_{e,\text{start}} = \frac{\xi - 40}{80 - 40} \cdot P_{m,\text{max}} = 9.4 \text{ kW} \quad \text{for} \quad \xi = 55\%, \]
\[ = 18.75 \text{ kW} \quad \text{for} \quad \xi = 70\%. \]

Now, \( P_e < P_{e,\text{start}} \), thus the selected mode is ZEV.

For the case (iii), \( \omega_e = 129 \text{ rad/s}, T_e = 160 \text{ Nm} \) would lead to \( P_e = 20.6 \text{ kW} \). In this case \( P_{m,\text{max}} = 150 \cdot 129 = 19.3 \text{ kW}, P_{e,\text{max}} = 15.95 \text{ kW} \), thus \( P_{e,\text{start}} = \{7.2; 14.5\} \text{ kW} \) for the two SOC values. In both cases, \( P_e > P_{e,\text{start}} \). Evaluate \( P_{t,b} = -19.3 + (\xi - 40)/40 \cdot (19.3 \cdot 2) = \{-4.8; 9.7\} \text{ kW} \). Therefore, \( P_e \) would be \( \{25.4; 10.9\} \text{ kW} \). After saturation, \( P_e = \{15.95; 10.9\} \text{ kW} \) and obtain as a difference \( P_m = \{4.65; 9.7\} \text{ kW} \) (boost mode).

For the case (iv), \( \omega_e = 422 \text{ rad/s}, T_e = 50 \text{ Nm} \) would lead to \( P_e = 21 \text{ kW} \). Since \( P_{m,\text{max}} = 25 \text{ kW} \) and \( P_{e,\text{max}} = 70.6 \text{ kW} \), evaluate \( P_{e,\text{start}} = \{9.4; 18.75\} \text{ kW} \). In both cases, \( P_e > P_{e,\text{start}} \). Evaluate \( P_{t,b} = \{-0.55; 18.2\} \text{ kW} \) and find \( P_e = \{21.55; 2.8\} \text{ kW} \). No saturation is needed and \( P_m = \{-0.55; 18.2\} \text{ kW} \) thus the selected mode is recharge, resp., boost.

**Problem 7.15**

Give an interpretation of the heuristic energy-management strategy of Problem 7.14 in terms of equivalent “cost” of the battery power with respect to the fuel power. Assume a Willans-type engine model with constant parameters \( e \) and \( P_0 \) and a unit-efficiency electric machine.

- **Solution**

The heuristic rule reads (\( P_t \) is the demand power)

\[ P_e = 0 \quad \text{if} \quad P_t > P_{e,\text{start}} + P_{t,b} \]
\[ P_e = P_t - P_{t,b} \quad \text{if} \quad P_t > P_{e,\text{start}} + P_{t,b}. \]

Define an “equivalent” power consumption as \( H = P_e + s \cdot P_m \), where \( s \) is the equivalence factor. Using a Willans engine model,

\[ H = \frac{P_0}{e} \cdot (P_{mt} > 0) + \frac{P_e}{e} + s \cdot (P_t - P_e). \]

Note that:

- For \( P_e = 0 \), \( H = s \cdot P_t \).
- For \( P_e = P_t - P_{t,b} \), \( H = \frac{P_0}{e} + \frac{P_t - P_{t,b}}{e} + s \cdot P_{t,b} \).

The switching condition for which \( P_e = 0 \) is preferable is that \( s \cdot P_t < \frac{P_0}{e} + \frac{P_t - P_{t,b}}{e} + s \cdot P_{t,b} \), that is, \( P_t \cdot (s - \frac{1}{e}) < \frac{P_0}{e} + P_{t,b} \cdot (s - \frac{1}{e}) \), or \( P_t < \frac{P_0}{e \cdot s - 1} + P_{t,b} \).

By comparing this switching condition with the heuristic rule, derive \( P_{e,\text{start}} \) as

\[ P_{e,\text{start}} = \frac{P_0}{e \cdot s - 1} + P_{t,b}. \]
from whence derive
\[ s \cdot e = \frac{1 + P_0}{P_{e,start} - P_{t,b}} \]
as the equivalence rule between the two strategies (i.e., between \( s \) and \( P_{e,start} \)).

**Optimal Energy Management Strategies**

*Problem 7.16*

Derive the exact formulation of the Euler-Lagrange equation (7.14) if the equivalent-circuit parameters of the battery are affine functions of SoC as described by (4.64) and (4.66). Consider the following system and operating point: battery capacity \( Q_b = 6.5 \text{ Ah} \), nominal open-circuit voltage \( U_{oc} = 250 \text{ V} \), nominal internal resistance \( R_i = 0.3 \Omega \), electric power \( P_b = 15 \text{ kW} \), variation of the open-circuit voltage with respect to SOC \( \kappa_2 = 20 \text{ V} \), and variation of the internal resistance \( \kappa_4 = -0.1 \Omega \). Evaluate the characteristic time constant associated with the variation of the Lagrange multiplier and assess the constant-\( \mu \) approximation.

- **Solution**

The Hamiltonian function is
\[ H = \dot{m}_f + \mu \cdot \dot{x}, \]
where \( \dot{x} = -\frac{I_b}{Q_0} \). The Euler–Lagrange equation is written as
\[ \dot{\mu} = -\frac{\partial H}{\partial x} = \frac{\mu}{Q_0} \frac{\partial I_b}{\partial x} \]

For an equivalent circuit model,
\[ I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\frac{U_{oc}^2}{4 \cdot R_i^2} - \frac{P_b}{2 \cdot R_i}} = \frac{U_{oc}}{2 \cdot R_i} - A \]
\[ \frac{\partial I_b}{\partial U_{oc}} = \frac{1}{2 \cdot R_i} - \frac{1}{2 \cdot A} \cdot \frac{U_{oc}}{2 \cdot R_i^2} \]
\[ \frac{\partial I_b}{\partial R_i} = -\frac{U_{oc}}{2 \cdot R_i^2} - \frac{1}{2 \cdot A} \cdot \left( -\frac{U_{oc}^2}{2 \cdot R_i^3} + \frac{P_b}{R_i^2} \right) \]

Numerically,
\[ A = \sqrt{\frac{250^2}{(2 \cdot 0.3)^2}} = \frac{15 \cdot 10^3}{0.3} = 351.58 \text{ A} \]
\[
\frac{\partial I_b}{\partial U_{oc}} = \frac{1}{2 \cdot 0.3} - \frac{1}{2 \cdot 351.58} \cdot \frac{250}{2 \cdot 0.3^2} = -0.31 \, \text{A/V}
\]
\[
\frac{\partial I_b}{\partial R_i} = -\frac{250}{2 \cdot 0.3^2} - \frac{1}{2 \cdot 351.58} \left( \frac{-250^2}{2 \cdot 0.3^4} + \frac{15 \cdot 10^3}{0.3^2} \right) = 20 \, \text{A/Ω}
\]
\[
\frac{\partial I_b}{\partial x} = \frac{\partial I_b}{\partial U_{oc}} \cdot \frac{\partial U_{oc}}{\partial x} + \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial R_i}{\partial x} = -0.31 \cdot 20 + 20 \cdot (-0.1) = -8.2 \, \text{A/}.
\]
and finally obtain
\[
\dot{\mu} = \frac{\partial I_b}{\partial x} \cdot \frac{1}{Q_0} = -8.1 \cdot \frac{6.5 \cdot 3600}{2854} = -\frac{1}{2854} \, \text{s}.
\]

**Problem 4.36**

Starting from the results of Problems 7.16, 4.26, find an approximated expression for the variation of the Lagrange multiplier. Evaluate the error with respect to the exact solution.

• Solution

Using the result of Problem 4.26,
\[
I_b \approx \hat{I} = \frac{P_b}{U_{oc}} + 2 \cdot \frac{R_i}{U_{oc}^3} \cdot P_b^2,
\]
thus
\[
\frac{\partial \hat{I}}{\partial U_{oc}} = -\frac{P_b}{U_{oc}^2} - \frac{R_i}{U_{oc}^3} \cdot P_b^2
\]
\[
\frac{\partial \hat{I}}{\partial R_i} = 2 \cdot \frac{R_i}{U_{oc}^3} \cdot P_b^2
\]
With the numerical values of Problem 7.16 compared to the exact solution,
\[
\frac{\partial \hat{I}}{\partial U_{oc}} = \frac{-15 \cdot 10^3}{250^2} - \frac{0.3}{250^3} \cdot (15 \cdot 10^3)^2 = -0.34 \, \text{A/V} \quad (10\% \, \text{error})
\]
\[
\frac{\partial \hat{I}}{\partial R_i} = \frac{2}{250^3} \cdot (15 \cdot 10^3)^2 = 28.8 \, \text{A/Ω} \quad (50\% \, \text{error})
\]
\[
\frac{\partial \hat{I}}{\partial x} = -0.34 \cdot 20 + 29 \cdot (-0.1) = -9.7 \, \text{A} \quad (20\% \, \text{error})
\]

**Problem 7.18**

At low temperature operation, the variation of the internal parameters of a battery can be significant. Develop a version of the ECMS where variations of internal resistance, via the parameter \( \kappa_3 \) of (4.66), with temperature are accounted for. Cell data: \( \kappa_1 = 3.4 \, \text{V}, \kappa_2 = 0.5 \, \text{V}, \kappa_4 = 0, \) and
\[ \kappa_3 = 0.015 - \vartheta_b \cdot \frac{0.01}{40}, \]

with \( \vartheta_b \) in °C. The nominal SOC is \( \xi = 0.5 \), temperature \( \vartheta = 25 \) °C, power \( P_b = 0.1 \cdot P_{b,max} \), thermal capacitance \( C_{t,b} = 300 \) J/K, thermal conductance \( 1/R_{th} = 0.5 \) W/K, and capacity \( Q_b = 6 \) Ah. Evaluate the time constant of the adjoint states.

- Solution

The quantity to be minimized is still the fuel consumption rate. The SOC variation is still proportional to the current. However the latter varies as a function of SOC and temperature. Thus the state equation for the temperature must be taken into account. The Hamiltonian is

\[ H = m_f + \mu \cdot \frac{\partial \xi}{\partial t} + \nu \cdot \frac{\partial \vartheta}{\partial t}, \]

where \( \frac{\partial \xi}{\partial t} = -\frac{I_b}{Q_0} \) and

\[ \frac{\partial \vartheta}{\partial t} = \frac{\partial \tilde{\vartheta}}{\partial t} = \left( R_i \cdot I_b^2 - \alpha \cdot \tilde{\vartheta} \right) \cdot \frac{1}{C_{t,b}}, \]

where \( \tilde{\vartheta} = \vartheta - \vartheta_{amb} \). The Euler–Lagrange equations read

\[ \dot{\mu} = -\frac{\partial H}{\partial \xi} = \mu \cdot \frac{1}{Q_0} \cdot \frac{\partial I_b}{\partial \xi} \]
\[ \dot{\nu} = -\frac{\partial H}{\partial \vartheta} = -\nu \cdot \left( 2 \cdot R_i \cdot I_b \cdot \frac{\partial I_b}{\partial \vartheta} - \alpha \right) \cdot \frac{1}{C_{t,b}} \]

The quantity \( \frac{\partial I_b}{\partial \xi} \) is calculated as

\[ \frac{\partial I_b}{\partial \xi} = \frac{\partial I_b}{\partial U_{oc}} \cdot \frac{\partial U_{oc}}{\partial \xi} + \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial R_i}{\partial \xi} = \frac{\partial I_b}{\partial U_{oc}} \cdot \kappa_2 + \frac{\partial I_b}{\partial R_i} \cdot \kappa_4 \]

while

\[ \frac{\partial I_b}{\partial \vartheta} = \frac{\partial}{\partial R_i} \cdot \frac{\partial R_i}{\partial \vartheta} = \frac{\partial I_b}{\partial R_i} \cdot \frac{\partial \kappa_3}{\partial \vartheta} \]

With the numerical values for the open-circuit voltage and the internal resistance are calculated as

\[ U_{oc} = \kappa_1 + \kappa_2 \cdot q = N \cdot (3.4 + 0.5 \cdot 0.5) = 3.65 \cdot N \text{ V} \]
\[ R_i = \kappa_3 + \kappa_4 \cdot q = N \cdot \left( 0.015 - 0.01 \cdot \frac{25}{40} \right) = 0.009 \cdot N \text{ Ω}, \]

while the maximum power can be calculated with equation (4.74)

\[ P_{b,max} = \frac{U_{oc}^2}{4 \cdot R_i} = \frac{3.65^2 \cdot N^2}{4 \cdot 0.09 \cdot N} = 370 \cdot N \text{ W} \]
\[ P_e = 0.1 \cdot P_{b,max} = 37 \cdot N \text{ W}. \]
This results in the numerical value for the current

\[ A = \sqrt{\frac{U_{sc}^2}{4 \cdot R_i^2}} - \frac{P_b}{R_i} = \sqrt{\frac{3.65^2}{4 \cdot 0.009^2} - \frac{37}{0.009}} = 192 \, \text{A} \]

\[ I_b = \frac{U_{oc}}{2 \cdot R_i} - \sqrt{\left( \frac{U_{oc}}{2 \cdot R_i} \right)^2 - \frac{P_b}{R_i}} = \frac{3.65}{2 \cdot 0.009} - \sqrt{\left( \frac{3.65}{2 \cdot 0.009} \right)^2 - \frac{37}{0.009}} = 10.5 \, \text{A} \]

The variations in the current relative to variations in the states \( \{\xi, \theta\} \) are now

\[
\frac{\partial I_b}{\partial U_{oc}} = \frac{1}{2 \cdot R_i} \cdot \left( 1 - \frac{U_{oc}}{2 \cdot R_i} \cdot A \right) = \frac{1}{2 \cdot 0.009} \cdot N \cdot \left( 1 - \frac{3.65}{2 \cdot 0.009} \cdot 192 \right) = -\frac{3.12}{N} \, \text{A/V} \\
\frac{\partial I_b}{\partial R_i} = -\frac{U_{oc}}{2 \cdot R_i^2} \cdot \frac{1}{2} \cdot A \cdot \left( \frac{P_b}{R_i} - \frac{U_{sc}^2}{2 \cdot R_i^2} \right) = -\frac{3.65}{2 \cdot 0.009^2} \cdot \left( 1 - \frac{37}{0.009^2} - \frac{3.65^2}{2 \cdot 0.009^2} \right) = \frac{75.2}{N} \, \text{A/} \Omega \\
\frac{\partial I_b}{\partial \xi} = -3.12 \cdot 0.5 + 75.2 \cdot 0 = -1.56 \, \text{A} \\
\frac{\partial I_b}{\partial \theta} = 75.2 \cdot \frac{1}{40} = -0.019 \, \text{A/K} 
\]

Following time constants of the lagrange multipliers are obtained

\[
\dot{\mu} = \frac{1}{Q_0} \frac{\partial I_b}{\partial \xi} = -\frac{1.56}{6 \cdot 3600} = -7.2 \cdot 10^{-5} \, /s \\
\dot{\nu} = -2 \cdot R_i \cdot I_b \cdot \frac{\partial I_b}{\partial \theta} - \alpha = -2 \cdot 0.009 \cdot 10.5 \cdot 0.019 - 5 \cdot 10^{-1} \cdot \frac{300}{300} = 0.0017 \, /s 
\]

**Problem 7.19**

Find the optimal-control formulation (Hamiltonian function and Euler-Lagrange equation) of the energy management of a hybrid powertrain with an ICE and a supercapacitor. Find under which approximation the costate is time-invariant.

- **Solution**

The state equation (4.117) of the supercapacitor reads

\[
\frac{d}{dt} U_{sc}^2 \cdot \left( 1 - \frac{R_{sc} \cdot P_{sc}}{U_{sc}^2} \right) = -\frac{2 \cdot P_{sc}}{C_{sc}} 
\]
Define \( x \triangleq C_{sc} \cdot U_{sc}^2 = 2 \cdot E_{sc} \) as the state variable, then
\[
\frac{dx}{dt} = -\frac{2 \cdot P_{sc}}{1 - R_{sc} \cdot P_{sc} \cdot \frac{C_{sc}}{x}}.
\]
In this way the Hamiltonian can be built in power terms as
\[
H = P_f + s \cdot P_{ech},
\]
where \( P_{ech} = \frac{dx}{dt} \). The Euler–Lagrange equation reads
\[
\frac{ds}{dt} = -\frac{\partial H}{\partial x} = -s \cdot \frac{\partial P_{ech}}{\partial x},
\]
where
\[
\frac{\partial P_{ech}}{\partial x} = -\frac{2 \cdot P_{sc}^2 \cdot R_{sc} \cdot C_{sc}}{(x - P_{sc} \cdot R_{sc} \cdot C_{sc})^2},
\]
which depends on \( x \), thus is not constant. Only if one neglects the resistance \( R_{sc} \), then \( \partial P_{ech}/\partial x \) is zero.

**Problem 7.20**

Formulate the energy-optimal energy management in the case of a double-source electric powertrain, with a battery and a supercapacitor.

- **Solution**

In this case the optimization criterion is the minimization of the battery consumption, i.e.,
\[
L = P_{ech} = U_{oc} \cdot I_b,
\]
while the global constraint is over the supercapacitor SOC or voltage,
\[
x(t_f) = \frac{1}{2} \cdot C_{sc} \cdot U_{sc}^2(t_f) = x(0).
\]
Therefore the Hamiltonian reads
\[
H = U_{oc} \cdot I_b(P_b) + s \cdot P_{sc},
\]
where \( P_{sc} = \frac{dx}{dt} \), and the Euler–Lagrange equation is
\[
\frac{ds}{dt} = -\frac{\partial H}{\partial x} = -s \cdot \frac{\partial P_{sc}}{\partial x},
\]
for whose development see Problem 7.19. The global constraint over the state \( x \) can be used to find the unknown initial value of \( s \). However, the constraints locally applied to the state \( x \) are even more critical in this case.
Problem 7.21

Formulate the optimal energy management for a parallel HEV that includes engine temperature variations. Assume that the cold-engine fuel consumption is given by an equation of the type

$$m_f(T_e, \omega_e, \vartheta_e) = m_{f,w}(T_e, \omega_e) \cdot f(T_e, \omega_e, \vartheta_e),$$

where $\vartheta_e$ is one engine relevant temperature and $m_{f,w}$ is the warm-engine fuel consumption. Moreover, assume an engine temperature dynamic of the type

$$C_{t,e} \cdot \dot{\vartheta_e} = P_{\text{heat}}(T_e, \omega_e, \vartheta_e) - \alpha \cdot (\vartheta_e - \vartheta_{\text{amb}}).$$

• Solution

The cost function is $L = \star m_f$. There exist two state variables, namely, the SOC of the battery $\xi$ and the engine temperature $\vartheta_e$. Thus the Hamiltonian is

$$H = \star m_f(T_e, \omega_e, \vartheta_e) + \mu \cdot \frac{d\xi}{dt} + \nu \cdot \frac{d\vartheta_e}{dt},$$

where $\dot{\xi} = g(T_e, t, \xi)$ and $\dot{\vartheta_e}$ is given in the problem text.

The Euler–Lagrange equations read

$$\dot{\mu} = -\frac{\partial H}{\partial \xi} = -\mu \cdot \frac{\partial \dot{\xi}}{\partial \xi} \quad \text{as usual (see Problem 7.16)}$$

$$\dot{\nu} = -\frac{\partial H}{\partial \vartheta_e} = \frac{\partial \star m_f}{\partial \vartheta_e} - \nu \cdot \frac{\partial \dot{\vartheta_e}}{\partial \vartheta_e} = - \star m_{f,w} \frac{\partial f}{\partial \vartheta_e} - \nu \cdot \frac{P_{\text{heat}}}{C_{t,e}} \left( \frac{\partial P_{\text{heat}}}{\partial \vartheta_e} - \alpha \right).$$

Define $\nu = -\nu \cdot C_{t,e}$ to have a third term in the Hamiltonian that has the units of a power. In that case, the state variable would be the thermal energy accumulated $E_{th}$. Since there is no constraint over the state $\vartheta_e$ (or $E_{th}$), the terminal value of the second Lagrange multiplier $\nu$ must be $\nu(T) = 0$.

Problem 7.22

Evaluate the optimal gear ratio profile during an ICE-based vehicle acceleration from rest to $v_f$ on a flat road. Use (i) the acceleration time $t_f$ and (ii) the fuel consumption $m_f$ as the performance index. Make the following simplifying assumptions: constant engine parameters $T_e = T_{e,max}$, $e$, $P_0$, continuously variable gear ratio, linearized vehicle dynamics

$$\dot{v} = \frac{F_t}{m_v} - b \cdot v,$$

where $F_t = u \cdot T_e$ and $u = \gamma/r_w$. Verify the solution given by optimal control theory by analyzing the dependency of the criterion on the gear ratio.

Numerical data: $b = 10^{-2}$, $u = \gamma/r_w \in [u_{\text{min}}, u_{\text{max}}] = [2, 12]$, $m_v = 1000 \text{ kg}$, $v_f = 100 \text{ km/h}$, $T_e = 150 \text{ Nm}$, $e = 0.4$, $P_0 = 2 \text{ kW}$. 
Fuel power consumption:

\[ P_f = \frac{T_e \cdot \omega_e + P_0}{e} = \frac{T_e \cdot u \cdot v + P_0}{e}. \]

**Case (i)**

\[ J = \int_0^{t_f} dt = t_f \]
\[ L = 1 \]
\[ H = 1 + \mu \cdot \left( \frac{u \cdot T_e}{m_v} - b \cdot v \right) \]
\[ s = \frac{\partial H}{\partial u} = \mu \cdot \frac{T_e}{m_v} \]
\[ u^o = \begin{cases} u_{min} & \text{if } s < 0 \\ u_{max} & \text{if } s > 0 \end{cases} \]

Euler–Lagrange equation:

\[ \dot{\mu} = -\frac{\partial H}{\partial v} = \mu \cdot b, \]

thus

\[ \mu(t) = \mu(0) \cdot e^{bt} \]

Use the condition that \( H \equiv 0 \) for “free final time” problems, to find that \( \mu(0) \) must be negative,

\[ \mu(0) = -\frac{1}{m_v} \]

since the quantity \( u \cdot T_e/m_v \) is positive. Thus \( \mu(t) \) is always negative and \( s(t) \) is negative as well. Consequently,

\[ u^o(t) = u_{max}. \]

Intuitively, the longest gear is to accelerate in the least time possible. Of course, speed limits of the engine force the gear shift as soon as the upper limit is reached. The criterion \( J = t_f \) is obtained after having calculated

\[ \mu(t_f) = -\frac{1}{m_v} \frac{u_{max} \cdot T_e}{m_v} - b \cdot v_f. \]

Then

\[ t_f = \frac{1}{b} \cdot \ln \frac{\mu(t_f)}{\mu(0)} = \frac{1}{b} \cdot \ln \left( \frac{1}{1 - x} \right). \]
where \( x = b \cdot v_f \cdot m_v/(u_{\text{max}} \cdot T_e) \). It is easy to verify that \( t_f \) is a decreasing function of \( u \). Also verify that

\[
v(t) = \frac{u \cdot T_e}{m_v \cdot b} \cdot (1 - e^{-b \cdot t})
\]

and

\[
v(t_f) = v_f \Rightarrow e^{-b \cdot t_f} = 1 - \frac{b \cdot v_f \cdot m_v}{u_{\text{max}} \cdot T_e} = 1 - x \Rightarrow e^{b \cdot t_f} = \frac{1}{1 - x}
\]
as in the previous equation.

**Case (ii)**

\[
J = \int_0^{t_f} T_e \cdot u \cdot v + P_0 \, dt
\]

\[
L = \frac{T_e \cdot u \cdot v}{e}
\]

\[
H = \frac{T_e \cdot u \cdot v}{e} + \mu \cdot \left( \frac{u \cdot T_e}{m_v} - b \cdot v \right)
\]

\[
s = \frac{\partial H}{\partial u} = \frac{T_e \cdot v}{e} + \mu \cdot \frac{T_e}{m_v} \Rightarrow s = \frac{v}{e} + \frac{\mu}{m_v}
\]

Euler–Lagrange equation:

\[
\dot{\mu} = -\frac{T_e \cdot u}{e} + \mu \cdot b
\]

\[
\mu(t) = \frac{u \cdot T_e}{b \cdot e} \cdot (\mu(0) - \frac{u \cdot T_e}{b \cdot e}) \cdot e^{b \cdot t}
\]

Again, the Hamiltonian must be constantly zero for the optimal solution:

\[
H(0) = \mu(0) \cdot \frac{u \cdot T_e}{m_v} = 0 \Rightarrow \mu(0) = 0
\]

Thus

\[
\mu(t) = \frac{u \cdot T_e}{e \cdot b} \cdot (1 - e^{b \cdot t})
\]

Moreover, from the state equation for \( v \) and the constancy of the control \( u \) (either \( u_{\text{min}} \) or \( u_{\text{max}} \)), obtain

\[
v(t) = \frac{u \cdot T_e}{b \cdot m_v} \cdot (1 - e^{-b \cdot t})
\]

\[
s(t) = \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot (1 - e^{-b \cdot t}) + \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot (1 - e^{b \cdot t})
\]
Analyze $s(t)$:

\[
\begin{align*}
    s(0) &= 0 \\
    s(\infty) &= -\infty \\
    \dot{s}(0) &= \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot b + \frac{u \cdot T_e}{b \cdot m_v \cdot e} \cdot (-b) = 0
\end{align*}
\]

Thus $s(t)$ is always negative (except for $t = 0$). Consequently,

\[
u^0(t) = u_{max}.
\]

Verify the criterion

\[
J = \int L\, dt = \frac{u \cdot T_e}{e} \int \frac{u \cdot T_e}{b \cdot m_v} \left(1 - e^{-b \cdot t}\right)\, dt = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \left(t_f + \frac{1}{b} \cdot e^{-b \cdot t_f} - \frac{1}{b}\right).
\]

However,

\[
e^{-b \cdot t_f} = 1 - \frac{v_f \cdot b \cdot m_v}{u \cdot T_e}
\]

and thus

\[
J = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \cdot \left(t_f + \frac{1}{b} - \frac{v_f \cdot m_v}{u \cdot T_e} \cdot \frac{1}{b}\right) = \frac{u^2 \cdot T_e^2}{e \cdot b \cdot m_v} \cdot t_f - \frac{u \cdot T_e \cdot v_f}{e \cdot b}
\]

After inserting the expression for $t_f$, it is easy to see that $J$ is a decreasing function of $u$.

**ECMS Problem 7.23**

Consider a parallel HEV. The engine is a Willans machine with $e = 0.3$ and $P_{e,0} = 2 \text{ kW}$. The electric drivetrain has a constant efficiency $\eta_{el} = 0.8$ and a maximum/minimum power $P_{m,max/min} = \pm 20 \text{ kW}$. Calculate for which values of the equivalence factor $s$ a purely electric drive and a full recharge, respectively, are optimal for a power demand $P_t = 20 \text{ kW}$.

* Solution

If $u = P_e / P_d$, i.e., the ratio between the power delivered by the IC engine and the power demand, then

\[
H(P_t, s, u) = \frac{u \cdot P_t + P_{e,0} \cdot h(u)}{e} + s \cdot (1 - u) \cdot P_t \cdot \eta_{el}^{\text{sign}(u-1)},
\]

where $h(.)$ is the unit step function. Since

\[
\frac{\partial H}{\partial u} = \frac{P_t}{e} - s \cdot P_t \cdot \eta_{el}^{\text{sign}(u-1)}
\]
is piecewise constant, the optimal \( u^o \) is either at \( u = 0 \), \( u = 1 \), or at \( u = u_{\text{max}} \) (discontinuities). The three values of the Hamiltonian are

\[
H(0) = s \frac{P_t}{\eta_{\text{el}}},
\]
\[
H(1) = \frac{P_t}{e} + \frac{P_{e,0}}{e},
\]
\[
H(u_{\text{max}}) = \frac{P_t}{e} \cdot u_{\text{max}} + s \cdot \frac{\eta_{\text{el}}}{e} \cdot (1 - u_{\text{max}}) \cdot P_t + \frac{P_{e,0}}{e}.
\]

The value of \( u_{\text{max}} \) is such that \((1 - u_{\text{max}}) \cdot P_t = P_{m,\text{min}} \), which leads to \( u_{\text{max}} = 2 \). After inspection, the purely electric drive \( (u = 0) \) is selected when \( H(0) < H(1) \) ⇒ \( s < \frac{\eta_{\text{el}}}{e} \cdot \frac{P_t + P_{e,0}}{P_t} = 0.8 \frac{0.3}{1.1} = 2.9 \)

and

\[
H(0) < H(2) \quad \Rightarrow \quad s < \frac{2 \cdot P_t}{e} + \frac{1}{\eta_{\text{el}} + \frac{1}{\eta_{\text{el}}}} = \frac{2.1}{0.3 \cdot (0.8 + 0.8)} = 3.4.
\]

Thus, \( s \) must be lower than 2.9 for the purely electric drive to be optimal.

The full recharge is optimal when

\[
H(2) < H(1) \quad \Rightarrow \quad \frac{2 \cdot P_t}{e} - s \cdot \frac{P_t}{\eta_{\text{el}}} < \frac{P_t}{e} \quad \Rightarrow \quad s > \frac{1}{e \cdot \eta_{\text{el}}} = 4.2.
\]

and

\[
H(2) < H(0) \quad \Rightarrow \quad s > 3.4.
\]

Thus full recharge is optimal when \( s > 4.2 \). For \( 2.9 < s < 4.2 \), the purely ICE operation is optimal.

**Problem 7.24**

Derive a look-up table yielding the optimal engine torque \( T_e \) of a post-transmission parallel hybrid as a function of \( \omega_w \), \( T_t \) and \( s \). Use the following engine model,

\[
P_f = \begin{cases} 
\frac{P_{e,0} + P_e}{e}, & \text{for } T_e > 0 \\
0, & \text{for } T_e > 0
\end{cases}
\]

with the following parameters:

\[
\frac{1}{e} = \begin{cases} 
1.21 \cdot 10^{-5} \cdot \omega_e^2 - 0.0053 \cdot \omega_e + 2.94, & T_e < \min(T_{e,\text{max}}, T_{e,\text{turbo}}) \\
-1.63 \cdot 1^{-4} \cdot \omega_e^2 - 0.0876 \cdot \omega_e - 6.80, & T_e > T_{e,\text{turbo}}
\end{cases}
\]
\[
\frac{P_{e,0}}{e} = \begin{cases} 
0.166 \cdot \omega_e^2 + 1.174 \cdot \omega_e + 4.59 \cdot 10^3, & T_e < \min(T_{e,\text{max}}, T_{e,\text{turbo}}) \\
5.19 \cdot \omega_e^2 - 2.83 \cdot 10^3 \cdot \omega_e + 2.27 \cdot 10^5, & T_e > T_{e,\text{turbo}}
\end{cases}
\]

with \(T_{e,\text{turbo}} = 200\ \text{Nm}\) and \(T_{e,\text{max}} = -0.0038 \cdot \omega_e^2 + 2.32 \cdot \omega_e - 79\ \text{Nm}\). Use the motor model

\[
P_m = \omega_m \cdot T_m + (0.0012 \cdot \omega_m + 0.0179) \cdot T_m^2 + (-0.0002 \cdot \omega_m^2 + 0.789 \cdot \omega_m + 384) = \omega_m \cdot T_m + a(\omega_m) \cdot T_m^2 + c(\omega_m)
\]

with \(P_{m,\text{max}} = 42\ \text{kW}\) and \(T_{m,\text{max}} = 140\ \text{Nm}\), and the battery model of Problem 4.26 with \(R_i = 0\). Find the optimal \(T_e\) for \(T_t = 1000\ \text{Nm}, \omega_w = 39\ \text{rad/s}, \gamma_m = 11, \gamma = 8.1\) (including the final gear), and \(s = 2.8\).

- Solution

The unconstrained optimum of Problem 7.25 must fulfil the constraints

\[
T_m > T_{m,\text{min}} = -T_{m,\text{max}} \quad \text{and} \quad T_e < T_{e,\text{max}}
\]

Moreover, the coefficient \(1/e\) changes across \(T_e = 200\ \text{Nm}\).

The motor limits at \(\omega_m = \omega_r \cdot \gamma_m = 429\ \text{rad/s}\) are ±98 Nm (both in motoring and generating). These two values correspond to \(T_e = -10\ \text{Nm}\) and \(T_e = 256\ \text{Nm}\), respectively. At \(\omega_e = \omega_w \cdot \gamma = 315\ \text{rad/s}\), the engine maximum torque is

\[
T_{e,\text{max}} = -0.0038 \cdot 315^2 + 2.32 \cdot 315 - 79 = 274\ \text{Nm}.
\]

Summarizing, the admissible \(T_e\) range is between -10 Nm and 256 Nm, with a discontinuity at 200 Nm.

For the assigned operating point, assume first that the optimal solution is below the turbocharging limit. Thus \(1/e = 2.47\) (\(e = 0.40\)), \(a = 0.53\). Consequently, from (11) of Problem 7.25 find

\[
T_m = \frac{315 \cdot 2.47 - \frac{8.1}{11} \cdot 2.8 \cdot 429}{2 \cdot 0.53} = -48.7\ \text{Nm}
\]

and

\[
T_e = \frac{T_t - \gamma_m \cdot T_m}{\gamma} = 189.6\ \text{Nm}
\]

which is a point below the turbocharging limit. To confirm this result, test the other set of Willans parameters. In particular, \(1/e = 4.65\) (\(e = 0.21\)). Consequently,

\[
T_m = \frac{315 \cdot 4.65 - \frac{8.1}{11} \cdot 2.8 \cdot 429}{2 \cdot 0.53} = 547\ \text{Nm}
\]

which is clearly beyond the motor limit. Thus the optimal point is \(T_e = 190\ \text{Nm}, T_m = -49\ \text{Nm}\).
For a further verification, calculate the Hamiltonian for \( T_e = -10 \) Nm, \( T_e = 0 \) Nm, \( T_e = 190 \) Nm, \( T_e = 200 \) Nm and \( T_e = 256 \) Nm (\( a = 0.53, c = 686 \)):

For \( T_e = -10 \), \( \frac{1}{e} = 2.47, \frac{P_{e,0}}{e} = 21.43 \cdot 10^3, T_m = 98 \) \( \Rightarrow H = 1.34 \cdot 10^5 \)

For \( T_e = 0 \), \( \frac{1}{e} = 2.47, \frac{P_{e,0}}{e} = 21.43 \cdot 10^3, T_m = 91 \) \( \Rightarrow H = 1.23 \cdot 10^5 \)

For \( T_e = 190 \), \( \frac{1}{e} = 2.47, \frac{P_{e,0}}{e} = 21.43 \cdot 10^3, T_m = -49 \) \( \Rightarrow H = 1.159 \cdot 10^5 \)

For \( T_e = 200 \), \( \frac{1}{e} = 2.47, \frac{P_{e,0}}{e} = 21.43 \cdot 10^3, T_m = -56 \) \( \Rightarrow H = 1.163 \cdot 10^5 \)

For \( T_e = 256 \), \( \frac{1}{e} = 4.65, \frac{P_{e,0}}{e} = -1.49 \cdot 10^5, T_m = -98 \) \( \Rightarrow H = 1.25 \cdot 10^5 \)

which confirms the bounded optimum at 190 Nm.

**Problem 7.25**

Find the unconstrained optimal engine torque for a post-transmission parallel hybrid with an engine model of the type

\[
P_f = \frac{P_{e,0}}{e} + P_b,
\]

an electric machine model of the type

\[
P_m = \omega_m \cdot T_m + a \cdot T_m^2 + c,
\]

and the battery model of Problem 4.26,

\[
P_{ech} = P_b + P_b^2 \frac{2 \cdot R_i \cdot U_{oc}^2}{U_{oc}^2},
\]

where \( P_b = P_m \). Neglect the SOC influence.

- **Solution**

The optimal operating point is the pair \((T_e, \gamma)\). The optimal \( T_e \) is calculated for each transmission ratio \( \gamma \). For each \( \gamma \) the engine speed \( \omega_e = \gamma \cdot \omega_w \) and the motor speed \( \omega_m = \gamma_m \cdot \omega_w \) are fixed (\( \gamma_m \) is usually a constant). Thus the coefficients \( 1/e, P_{e,0}/e, a, \) and \( c \) are also fixed.

The Hamiltonian is

\[
H = P_f + s \cdot P_{ech} = \\
= \frac{1}{e} \cdot \omega_e \cdot T_e + \frac{P_{e,0}}{e} + \\
+ s \cdot \left( \left( \omega_m \cdot T_m + a \cdot T_m^2 + c \right) + 2 \cdot \frac{R_i}{U_{oc}^2} \cdot \left( \omega_m \cdot T_m + a \cdot T_m^2 + c \right)^2 \right).
\]
To find the optimal $T_e$, differentiate the Hamiltonian

$$\frac{\partial H}{\partial T_e} = \frac{\omega_e}{e} - s \cdot \left(1 + \frac{2 \cdot R_i}{\frac{U_{oc}^2}{2}} \cdot 2 \cdot (\omega_m \cdot T_m + a \cdot T_m^2 + c)\right) \cdot (\omega_m + 2 \cdot a \cdot T_m) \cdot \gamma = 0$$

since $\gamma_m \cdot T_m = T_t - \gamma \cdot T_e$ and $\partial T_m / \partial T_e = -\gamma / \gamma_m$. If one neglects the loss term in the battery model, the resulting equation is

$$\frac{\omega_e}{e} = \frac{\gamma}{\gamma_m} \cdot s \cdot (\omega_m + 2 \cdot a \cdot T_m)$$

and the optimal solution would be

$$T_m = \frac{\omega_m - \frac{s \cdot \omega_m}{2 \cdot a} \cdot \frac{1}{s \cdot \gamma}}{1 \cdot \gamma}$$

from whence

$$T_e = \frac{T_t - \frac{\gamma_m \cdot T_m}{\gamma}}{\gamma} = \frac{T_t}{\gamma} - \left(\frac{\gamma_m^2}{2 \cdot a \cdot e \cdot s \cdot \gamma} - \frac{\gamma_m^2}{2 \cdot a \cdot \gamma}\right) \cdot \omega_w.$$

**Problem 7.26**

Use the result of Problem 4.21 and a simplified battery model $P_{ech} = P_b$ to derive an analytical solution of the optimal energy management of a series hybrid. Following Problem 4.21, consider the engine Willans parameter varying as

$$\frac{1}{e} = \begin{cases} 
0 & \text{for } P_g = 0 \\
4.01 & \text{for } 0 < P_g \leq 14 \cdot 0.92 \cdot 10^3 \\
3.36 & \text{for } 14 \cdot 0.92 \cdot 10^3 < P_g \leq 62 \cdot 0.92 \cdot 10^3 \\
3.89 & \text{for } 62 \cdot 0.92 \cdot 10^3 < P_g \leq 68 \cdot 0.92 \cdot 10^3 
\end{cases}$$

and $\eta_g = 0.92$.

- **Solution**

The Hamiltonian is

$$H = P_f + s \cdot P_{ech} = \frac{P_0}{e} + \frac{P_g}{\eta_g \cdot e} + s \cdot (P_m - P_g).$$

This function is affine in $P_g$, and

$$\frac{\partial H}{\partial P_g} = \frac{1}{\eta_g \cdot e} - s$$

However, the coefficient $1/e$ changes with $P_g$; now, the possible solutions are at the discontinuity points. For low $s$, the optimum is at $P_g = 0^-$ and $H(0^-) =$
For increasing $s$, the optimum switches toward $P_g = P_1 \cdot \eta_g = 57$ kW, where $P_1$ is the engine power at $\omega_e = 1000$ rpm. The switching value of $s$ is calculated by equating

$$s \cdot P_m = 214.8 \cdot 10^3 + s \cdot (P_m - 62 \cdot 10^3 \cdot \eta_g) \Rightarrow s = 3.76$$

For $s > 3.95$, the optimum shifts to $P_g = P_{\text{max}} \cdot \eta_g = 62.8$ kW.

**Problem 7.27**

Derive equations (7.24) – (7.25) from PMP.

- **Solution**

For a parallel hybrid with constant efficiencies $\eta_f$ and $\eta_e$, the Hamiltonian is

$$H(s, P_e) = \begin{cases} \frac{P_e}{\eta_f} + s \cdot \eta_e \cdot (P_t - P_e), & \text{for } 0 < P_e < P_t \\ \frac{P_e}{\eta_f} + s \cdot \eta_e \cdot (P_t - P_e), & \text{for } P_t < P_e < P_{\text{max}} \end{cases}$$

that is, the dependency $H(P_e)$ is piecewise affine and consists of two segments. For small values of $s$, $\partial H/\partial P_e$ is always positive, thus the optimal value is $P_e = 0$.

The value of $s$ for which $\partial H/\partial P_e = 0$ in the first segment is $s_1 = \eta_e / \eta_f$. Beyond this value, the first segment is increasing and the second is decreasing, thus the optimal control is $P_e = P_t$.

The value of $s$ for which $\partial H/\partial P_e = 0$ in the second segment is $s_2 = 1 / (\eta_e \cdot \eta_f)$. For higher values of $s$, the second segment is decreasing, thus the optimal control is $P_e = P_{\text{max}}$.

Summarizing, one recovers that $s = s_1$ or lower values lead to a battery discharge; $s = s_2$ or higher values lead to a battery recharge, while any value between $s_1$ and $s_2$ leads to pure ICE operation, thus charge-sustained operation (but no hybrid operation).

**Problem 7.28**

For the simple parallel HEV model of Problem 7.23, with $e = 0.4$, $P_{e,0} = 3$ kW, $\eta_{el} = 0.9$, find the conditions on $P_t$ for which the ZEV mode, the ICE mode or the battery recharge with $P_b = -2 \cdot P_t$ are optimal, respectively.

- **Solution**

As opposed to the results of Problem 7.23 the solution is now to be found as a function of $s$. After inspection, three possibilities arise:

- If $s < s_1 = \frac{\eta_{el}}{\eta_f} = 3$, then only the purely electric mode ($P_e = 0$) could be optimal.
• If \( s_1 < s < s_2 = \frac{1}{\eta_{el}} = 3.7 \), then the optimum is either the purely electric mode or the purely ICE operation. The switch is when

\[
P_t = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - \eta_{el}} = \frac{27 \text{ kW}}{4 \cdot s - 9},
\]

thus below which ZEV mode, above which ICE mode are optimal.

• If \( s > s_2 \), then again two possibilities arise, namely, the ZEV or the recharge mode. The switch is for

\[
P_t = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - 2 \cdot \eta_{el} + s \cdot \eta_{el}^2 \cdot e} = \frac{2700 \text{ kW}}{724 \cdot s - 1800}.
\]

**Problem 7.29**

Use the result of Problem 7.28 to evaluate \( s \) over a drive cycle with the following characteristics: \( \bar{E}_{trac} - \bar{E}_{rec} = \Delta \bar{E} = 0.183 \text{ MJ} \), \( \bar{E}_{trac} = 0.670 \text{ MJ} \), \( P_{max} = 18.9 \text{ kW} \). Assume a linear relationship between cumulative energy and power demand. Then perform again the calculations for the data of Problem 7.23.

• Solution

The condition for \( s \) to be optimal is that the electrical energy is balanced over the cycle, thus

\[
\eta_{el} \cdot (\Delta \bar{E} + \bar{E}_{chg}) = \frac{\bar{E}_{zev}}{\eta_{el}}
\]

where \( \bar{E}_{chg} \) is the mechanical energy demand during the recharge phase (the same quantity is sent from the engine to the generator because \( u = 2 \)) and \( \bar{E}_{zev} \) is the mechanical energy demand during the ZEV phase.

Assume first that \( s > s_2 \). Then two phases exist, ZEV or recharge. The switch power is

\[
P_{lim} = \frac{\eta_{el} \cdot P_{e,0}}{s \cdot e - 2 \cdot \eta_{el} + s \cdot \eta_{el}^2 \cdot e}
\]

and this relationship will be used to calculate \( s \) once \( P_{lim} \) has been found.

To calculate \( P_{lim} \), observe that \( \bar{E}_{zev} \) is the energy for power demand lower than \( P_{lim} \). Since a linear relationship between energy and power demand is assumed,

\[
E(P_t) = \frac{\bar{E}_{trac}}{P_{max}} \cdot P_t,
\]

then

\[
\bar{E}_{zev} = \frac{\bar{E}_{trac}}{P_{max}} \cdot P_{lim}.
\]

Consequently, \( \bar{E}_{chg} = \bar{E}_{trac} - \bar{E}_{zev} \). Using these equations, obtain
\[ \eta_{el} \cdot \Delta \bar{E} + \eta_{el} \cdot \bar{E}_{trac} - \eta_{el} \cdot \frac{\bar{E}_{trac}}{P_{max}} \cdot P_{lim} = \frac{\bar{E}_{trac}}{P_{max} \cdot \eta_{el}} \cdot P_{lim} \]

\[ \Rightarrow P_{lim} = \frac{\eta_{el}^2}{1 + \eta_{el}} \cdot \frac{\bar{E}_{trac} + \Delta \bar{E}}{E_{trac}} \cdot P_{max} = 10.8 \text{ kW}, \]

from whence, calculate the charge-sustaining and optimal value of \( s \) as

\[ s = \frac{2 \cdot P_{lim} + P_{e,0}}{e \cdot (\frac{P_{lim}}{\eta_{el}} + \eta_{el} \cdot P_{lim})} = 2.83. \]

To verify the initial assumption,

\[ s_2 = \frac{1}{\eta_{el}} = 2.78 < s. \]

So the assumption was correct and the result is valid.

For the data of Problem 7.23, \( e = 0.3, P_{e,0} = 2 \text{ kW}, \eta_{el} = 0.8, \) one would obtain \( s_1 = 2.67, s_2 = 4.17. \) Assuming \( s > s_2 \) would lead to \( P_{lim,a} = 9.41 \text{ kW} \) and \( s_a = 3.60 \) which is not greater than \( s_2. \)

Assuming instead that \( s_1 < s < s_2, \) there is a switch between ZEV and ICE modes. Thus the energy balance is

\[ \eta_{el} \cdot \Delta \bar{E} = \frac{\bar{E}_{zev}}{\eta_{el}}, \text{ or } P_{lim,b} = \frac{\eta_{el}^2 \cdot \Delta \bar{E} \cdot P_{max}}{E_{trac}} = 3.3 \text{ kW}. \]

This switch power corresponds to

\[ s_b = \frac{P_{lim,b} + P_{e,0}}{P_{lim,b}} \cdot \frac{\eta_{el}}{e} = 4.28, \]

which is not lower than \( s_2 \) as assumed.

Finally the only possible result could be \( s = s_2, \) so that the recharge and the ICE mode are equally optimal. A switch will be added in order to balance the battery energy. For \( s = s_2, \) the limit power for the ZEV mode is

\[ \frac{\eta_{el}^2}{1 - \eta_{el}} \cdot P_{e,0} = 3.56 \text{ kW}. \]

The corresponding ZEV energy is 0.126 MJ. To be balanced, a recharge energy

\[ \bar{E}_{chg} = \frac{\bar{E}_{zev}}{\eta_{el}} - \Delta \bar{E} = 13.3 \text{ kJ} \]

is needed. Thus a further power limit

\[ P_{max} \cdot \left(1 - \frac{\bar{E}_{chg}}{E_{trac}}\right) = 15.4 \text{ kW} \]

can be taken, above which the recharge mode is selected and below which the ZEV mode is selected.
Problem 7.30

Consider a post-transmission parallel HEV with the following simplified data: motor transmission ratio $\gamma_m = 11$, wheel radius $r_w = 0.317\,\text{m}$, engine transmission ratio $\gamma = \{15.02, 8.09, 5.33, 3.93, 3.13, 2.59\}$, transmission efficiency $\eta_t = 0.95$. Consider the following driving situation: torque demand at the wheels $T_t = 378\,\text{Nm}$, vehicle speed $v = 69.25\,\text{km/h}$, engine on, electric consumers off, 4th gear. In this situation, $T_{m,max} = 140\,\text{Nm}$, $P_{m,max} = 42\,\text{kW}$, $U_{b,min} = 300\,\text{V}$, $U_{b,max} = 420\,\text{V}$, $P_{b,start} = 3\,\text{kW}$, $P_m = 0.9345 \cdot T_{m,max}^2 + 673.97 \cdot T_{m,max} + 127.44$, $U_{oc} = 381.12\,\text{V}$, $R_i = 0.3648\,\Omega$ (discharge), $R_i = 0.3264\,\Omega$ (charge), Coulombic efficiency $\eta_c = 0.95$, $T_{e,max} = 269.7\,\text{Nm}$, $T_{e,min} = -20\,\text{Nm}$, fuel consumption $m_f = (T_e - T_{e,min}) \cdot \left(2.9 \cdot 10^{-8} \cdot T_e + 1.112 \cdot 10^{-5}\right) = 2.9 \cdot 10^{-8} \cdot T_e^2 + 1.17 \cdot 10^{-5} \cdot T_e + 2.225 \cdot 10^{-4}$.

Find the engine and motor torque calculated by the ECMS. The current estimation of the equivalence factor is $s = 3$.

- Solution

Calculate first the electric-mode Hamiltonian $H_{ev}$:

\[
\omega_m = \frac{v_t}{r_w} \cdot \gamma_m = 667\,\text{rad/s}, \quad \omega_b = \frac{42 \cdot 10^3}{140} = 300\,\text{rad/s}
\]

\[
T_m(0) = T_t = 34.4\,\text{Nm}, \quad T_{m,max} = \frac{P_{m,max}}{\omega_m} = 63\,\text{Nm}
\]

The motor speed $\omega_m$ is above the base speed $\omega_b$, further the motor torque is smaller than the maximum possible motor torque. The electrical power of the motor is calculated with the given relationship

\[
P_m(0) = 0.9345 \cdot 34.4^2 + 673.97 \cdot 34.4 + 127.44 = 24.4\,\text{kW} = P_b(0).
\]

The battery power is limited by

\[
P_{b,max} = \min \left( \frac{U_{oc}^2}{4 \cdot R_i}, \frac{U_{oc} \cdot U_{b,min} - U_{b,min}^2}{R_i} \right) = 66.7\,\text{kW},
\]

\[
P_{b,min} = -\frac{U_{b,max}^2 - U_{oc} \cdot U_{b,max}}{R_i} = -50.0\,\text{kW}.
\]

Thus the condition $P_{b,min} < P_b < P_{b,max}$ is fulfilled. Evaluate

\[
I_b(0) = \frac{U_{oc}}{2 \cdot R_i} - \frac{\sqrt{U_{oc}^2 - R_i \cdot 4 \cdot P_b}}{4 \cdot R_i} = 68.5\,\text{A},
\]

\[
P_{ech}(0) = U_{oc} \cdot I_b(0) = 26.1\,\text{kW}.
\]
Thus the Hamiltonian for the ZEV case is
\[ H_{ev} = s \cdot P_{ech} = 3 \cdot 26.1 = 79.2 \text{ kW}. \]

Now calculate hybrid Hamiltonians. For simplicity take only three candidate values:
\[ T_e(1) = T_{e,\text{max}}, \quad T_e(2) = \frac{T_t}{\gamma_e \cdot \eta_t} = 101.2 \text{ Nm}, \quad T_e(3) = T_{e,\text{min}} \]

The corresponding fuel consumptions are
\[ \hat{m}_f(1) = (269.7 + 20) \cdot (2.9 \cdot 10^{-8} \cdot 269.7 + 1.112 \cdot 10^{-5}) = 5.49 \text{ g/s} \]
\[ \hat{m}_f(2) = (101.2 + 20) \cdot (2.9 \cdot 10^{-8} \cdot 101.2 + 1.112 \cdot 10^{-5}) = 1.70 \text{ g/s} \]
\[ \hat{m}_f(3) = 0 \text{ g/s} \]

The electric motor has to provide the following torque (note the role of the transmission efficiency):
\[ T_m(1) = \frac{T_t - \gamma_e \cdot T_{e,\text{max}} \cdot \eta_t}{\gamma_m} = -57.2 \text{ Nm}, \]
\[ T_m(2) = 0 \text{ Nm}, \]
\[ T_m(3) = \frac{T_t - \gamma_e \cdot T_{e,\text{min}} \cdot \eta_t}{\gamma_m} = 41.9 \text{ Nm}. \]

All three absolute values are lower than 63 Nm, so the constraints are not violated. Calculate the electric power
\[ P_m(1) = 0.9345 \cdot 57.2^2 - 673.97 \cdot 57.2 + 127.44 = -35.37 \text{ kW} = P_b(1), \]
\[ P_m(2) = 0.13 \text{ kW} = P_b(2), \]
\[ P_m(3) = 0.9345 \cdot 41.9^2 + 673.97 \cdot 41.9 + 127.44 = 30 \text{ kW} = P_b(3). \]

All three values are between -50 kW and 66.7 kW, so the battery limit is not overstepped. Again, the battery current is derived with the same formula as above as
\[ I_b(1) = \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 + 0.3648 \cdot 4 \cdot 35.37 \cdot 10^3}{4 \cdot 0.3648^2}} = -86.4 \text{ A}, \]
\[ I_b(2) = \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 127}{4 \cdot 0.3648^2}} = 0.33 \text{ A}, \]
\[ I_b(3) = \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 30 \cdot 10^3}{4 \cdot 0.3648^2}} = 85.8 \text{ A}, \]

which results in electrochemical power consumptions.
\[ P_{ech}(1) = -381.12 \cdot 86.4 \cdot 0.95 = -31.28 \text{ kW}, \]
\[ P_{ech}(2) = 381.12 \cdot 0.33 = 0.13 \text{ kW}, \]
\[ P_{ech}(3) = 381.12 \cdot 85.8 = 32.7 \text{ kW}. \]

Combining these results leads to
\[ H_{hyb} = 42.6 \cdot 10^6 \cdot \{5.49, 1.705, 0\} \cdot 10^{-3} + 3 \cdot \{-31.28, 0.13, 32.7\} \cdot 10^3 = \]
\[ = \{140, 73, 98\} \text{ kW}. \]

Finally, the chosen operating point will be the pure ICE operation. Thus the engine will continue to stay on.

**Problem 7.31**

Solve again Problem 7.30 for the situation in which the engine is turned off.

- **Solution**

Nothing changes up to the calculation of \( P_b(1, \ldots, 3) \). In order to account for the engine turning on phases, add \( P_{e,\ \text{start}} = 3 \text{ kW} \) to the battery power:

\[ P_b(1) = -35.37 + 3 = -32.37 \text{ kW}, \]
\[ P_b(2) = 0.13 + 3 = 3.13 \text{ kW}, \]
\[ P_b(3) = 30 + 3 = 33 \text{ kW}. \]

All three values are still admissible. Again the battery currents are calculated as

\[ I_b(1) = \frac{381.12}{2 \cdot 0.3264} - \sqrt{\frac{381.12^2 + 0.3264 \cdot 4 \cdot 32.37 \cdot 10^3}{4 \cdot 0.3264^2}} = -79.5 \text{ A}, \]
\[ I_b(2) = \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 3.13}{4 \cdot 0.3648^2}} = 8.27 \text{ A}, \]
\[ I_b(3) = \frac{381.12}{2 \cdot 0.3648} - \sqrt{\frac{381.12^2 - 0.3648 \cdot 4 \cdot 33 \cdot 10^3}{4 \cdot 0.3648^2}} = 95.3 \text{ A}, \]

which results in electrochemical power consumptions

\[ P_{ech}(1) = -381.12 \cdot 79.5 \cdot 0.95 = -28.79 \text{ kW}, \]
\[ P_{ech}(2) = 381.12 \cdot 8.27 = 3.15 \text{ kW}, \]
\[ P_{ech}(3) = 381.12 \cdot 94.2 = 36.3 \text{ kW}. \]

Combining these results leads to
\[ H_{hyb} = 42.6 \cdot 10^6 \cdot \{5.49, 1.705, 0\} \cdot 10^{-3} + 3 \cdot \{-28.79, 3.15, 36.3\} \cdot 10^3 = \]
\[ = \{147.5, 82, 109\} \text{ kW}. \]

In this case \( H_{ev} \leq \min(H_{hyb}) \) and it is more convenient to keep the engine off and use the ZEV mode.


**Problem 7.32**

Consider an ECMS with a stop-start strategy implementation based on hysteresis thresholds. In order to start the engine, $H_{hev}$ (see Problem 7.30) must fulfill the condition

$$H_{hev} < x_{on} \cdot H_{ev}.$$ 

At the previous calculation step, the lower Hamiltonian value was $H_{ev}$, thus the engine is off. At the current time step, the power demand is $P_t = 13.28$ kW. The equivalence factor estimation is $s = 3.2813$. Calculate the mode selected for a hysteresis threshold $x_{on}$ of (i) 95% and (ii) 90%, respectively. Use the following data and models: post-transmission parallel HEV architecture, transmission efficiency, $\eta_t = 0.95$, fuel consumption $P_f = 2.5446 \cdot P_e + 9.6525 \cdot 10^3$ if $P_e > 0$, electrochemical power $P_{ech} = \begin{cases} 1.2707 \cdot P_m + 2.7703 \cdot 10^3 - 2.014 \cdot 10^3, & \text{if } P_m > -595 \text{ W} \\ 0.7397 \cdot P_m + 2.4544 \cdot 10^3 - 2.014 \cdot 10^3, & \text{if } P_m < -595 \text{ W}. \end{cases}$

The cost of engine start is $P_{e,start} = 2.014$ kW (in electrochemical power units).

**Solution**

The Hamiltonian function is bilinear. Thus the optimum combination can be either at $P_e = 0$ W (ZEV mode), at $P_m = -595$ W (discontinuity), or at $P_e = P_{e,max}$. The engine power at the discontinuity is

$$P_{e,dis} = \frac{13.282 \cdot 10^3 + 595}{0.95} = 14.607 \text{ kW}.$$ 

However, a simple inspection of equations above shows that

$$\frac{\partial H}{\partial P_e} = \begin{cases} -1.1465, & \text{for } 0 < P_e < 14.607 \cdot 10^3 \text{ W} \\ 0.2388, & \text{for } P_e > 14.607 \cdot 10^3 \end{cases}$$ 

Thus the minimum of $H$ is either at the discontinuity point or at the ZEV mode:

- for $P_e = 0$, $P_m = 13.282$ kW and $P_{ech} = 19.648 - 2.014$ kW, thus $H_{ev} = 57.86$ kW;
- for $P_e = 14.607$ kW, $P_f = 46.821$ kW, $P_m = -595$ W, $P_{ech} = 2.014$ kW, the engine must be started and $H_{hyb} = 53.4295$ kW.

In order to choose between $H_{ev}$ and $H_{hyb}$ the hysteresis threshold $x_{on}$ must be considered:
• for (i) $x_{on} = 95\%$, the engine will be turned on if $53.43 \leq 0.95 \cdot 57.86 = 54.97$, which is true.

• for (ii) $x_{on} = 90\%$, the engine should be turned on if $53.43 \leq 0.90 \cdot 57.86 = 52.07$, which is not true.

Thus in this case the engine should be kept off.

**Problem 7.33**

Consider an HEV under several repetitions of an elementary driving cycle. An ECMS has a PI adaptation of $s$ as a function of SoC that yields a new estimation every cycle repetition. Assume that the overall behavior of the system on a cycle-by-cycle basis depends on $s$ as follows:

$$
\Delta \xi(n) = \xi(n) - \xi(n-1) = K_s \cdot (s(n) - s_0),
$$

where $\xi(n)$ is the SoC at the end of the $n$-th repetition, $s_0$ is the optimal value of $s$, $s(n)$ is the value adopted during the $n$-th cycle, and $K_s > 0$ is a constant depending on the particular system. Evaluate the stability and the dynamic characteristics of the controlled system on a cycle-by-cycle basis. Evaluate the influence of the integral term in the PI controller.

**Solution**

Let $\Delta \xi(n)$ be the variation of SOC on the $n$-th cycle, i.e., in a first approximation, a quantity proportional to the electrochemical energy consumption of the cycle. The cycle-by-cycle dependency between SOC and $s$ can be linearized as

$$
\Delta \xi(n) = \xi(n) - \xi(n-1) = K_s \cdot (s(n) - s_0),
$$

where $s_0$ is the optimal theoretical value of $s$, $s(n)$ is the value adopted during the $n$-th cycle, and $K_s > 0$ is a constant depending on the particular system. Correspondingly, the adaptation rule for $s(n)$ reads

$$
s(n) = s_p - k_p \cdot \xi(n-1) - k_i \cdot \sum_{i=0}^{n-1} \xi(i),
$$

where $\Delta \xi(0) = 0$ by definition, $k_p > 0$, $k_i \geq 0$ and $s_p$ initial value of $s$. By shifting from the discrete time to the continuous time, and by combining the two equations, one obtains

$$
\frac{d^2 s}{dt^2} = -k_p \cdot K_s \cdot \frac{ds}{dt} - k_i \cdot K_s \cdot (s - s_0).
$$

This dynamics is stable. The factor $s$ converges to $s_0$ (unknown) as prescribed. Thus the quantity $\Delta x$ converges to $0$. The true SOC error, the integral of $\Delta \xi$, converges to zero as well (the prescribed value). The integral of the SOC error converges to the value $(s_p - s_0)/k_i$.

If $k_i = 0$, the SOC error does not vanish but it tends to $(s_p - s_0)/k_p$. 
**Problem 7.34**

Compare (7.22) with (7.27) – (7.31). Under which assumptions are they equivalent?

- Solution

Combining (7.27) – (7.31), obtain

\[ s(t) = s_{chg} + (s_{dis} - s_{chg}) \cdot p(t) = s_{chg} + (s_{dis} - s_{chg}) \cdot (p_0 + p_1(t) \cdot E_{ech}(t)). \]

In the assumption that the difference \( E_h - E_h(t) \triangleq \Delta E_h \) is kept constant (sliding horizon), the coefficient \( p_1 \) is constant as well and

\[ s(t) = s_{chg} + (s_{dis} - s_{chg}) \cdot p_0 + (s_{dis} - s_{chg}) \cdot p_1 \cdot E_e(t). \]

The term \( E_{ech}(t) \) is the electrochemical energy consumed. In terms of SOC,

\[ E_{ech}(t) = Q_0 \cdot (\xi(0) - \xi(t)) \cdot U_{oc}(t) \]

and, assuming an averagely constant open-circuit voltage,

\[ E_{ech}(t) = K \cdot (\xi(0) - \xi(t)). \]

If \( \xi(0) = \xi_t \), obtain (7.22) with

\[ s_t = s_{chg} + (s_{dis} - s_{chg}) \cdot p_0, \]

\[ k_p = (s_{dis} - s_{chg}) \cdot p_1 \cdot K. \]

Now give a closer look to \( p_0 \) and \( p_1 \). To simplify the analysis, assume \( u_r = u_l = 1 \) and neglect \( \lambda \). Thus,

\[ p_0 = \frac{1}{\eta_e + \eta_f} \quad \text{and} \quad p_1 = \frac{1}{(\frac{1}{\eta_e} + \eta_f) \cdot \Delta E_h}. \]

Further impose (7.26)

\[ \frac{1}{\eta_e} = s_{dis} \cdot \eta_f, \quad \eta_e = s_{chg} \cdot \eta_f \]

to find

\[ p_0 = \frac{s_{dis}}{s_{dis} + s_{chg}} \]

\[ p_1 = \frac{1}{(s_{dis} + s_{chg}) \cdot \eta_f \cdot \Delta E_m} \]

\[ s_t = s_{chg} + (s_{dis} - s_{chg}) \cdot \frac{s_{dis}}{s_{dis} + s_{chg}} = \frac{s_{chg}^2 + s_{dis}^2}{s_{dis} + s_{chg}}. \]

It is easy to show that under the aforementioned assumptions \( s_{chg} < s_t < s_{dis} \), thus \( s_t \) plays the role of the constant optimal equivalence factor \( s_0 \).
Problem 7.35

Express \( s_0 \) as a function of \( u_r, \) \( u_l, s_{dis}, \) and \( s_{chg}. \) Evaluate \( s_0 \) for \( s_{max} = 5, \) \( s_{min} = 2, \) knowing from a cycle analysis that \( u_r/\eta_c = 1.2 \) and \( u_l \cdot \eta_c = 1.8. \)

- Solution

Using the method of Problem 7.34, but with \( u_r \neq u_l, \) obtain

\[
\begin{align*}
s_0 &= \frac{u_l \cdot s_c^2 + u_r \cdot s_d^2}{u_r \cdot s_d + u_l \cdot s_c}.
\end{align*}
\]

For the numerical case \( s_{max} = s_d \) and \( s_{min} = s_c. \)

\[
\begin{align*}
\eta_c &= \sqrt{\frac{s_c}{s_d}} = 0.63 \\
u_r &= 0.76 \\
u_l &= 2.85 \\
s_0 &= \frac{2.85 \cdot 2^2 + 0.76 \cdot 5^2}{0.76 \cdot 5 + 2.85 \cdot 2} = 3.2.
\end{align*}
\]

Remark: the value of \( u_r \) found shows that pure ZEV would not be allowed in this case.