Solutions

Exam Duration: 120 minutes examination time + 15 minutes reading time at the beginning of the exam

Number of Questions: 5 (differently weighted, in total 35 points)

Rating: It is not required to answer all questions to achieve the maximum grade. The number of points is indicated in front of each question separately.

Permitted aids: 20 sheets A4 (40 pages)

A pocket calculator will be provided.

The assistants are not allowed to answer technical questions and you are not permitted to use any electronic devices (exception is the pocket calculator provided officially).

Important: Use only these prepared sheets for your solutions. You may ask for extra sheets if you need more space.
**Question 1 (General Knowledge) 5 Points**

You have to develop a controller for an instable plant, which will run on a microprocessor allowing for a sampling time of $T = 0.3$ s. The spectra of the plant, of the disturbance and of the measurement noise are shown in Figure 1. The plant has one unstable pole at $\pi_+ = 1$. The reference for the control is $r(t) = 0$.

![Figure 1: Spectra of the plant (P), the disturbance (D) and the measurement noise (N)](image)

a) (1 point) Draw the block diagram of a generic, complete discrete-time control system, including interfaces, filters, and all relevant signals.

b) (2 points) Can you do controller emulation in the case at hand? Justify your answer.

c) (1 points) Is an anti-aliasing filter necessary? Justify your answer.

d) (1 points) Based on your answers to the previous questions, what plant model do you use to design your controller?
Solution 1

a) The diagram is shown in Figure 2.

![Figure 2: Discrete-time control system](image)

b) It is generally advisable to have the crossover frequency of the loop gain centered between the crossover frequencies of the disturbance and the noise, which is at approximately $8 \text{rad/s}$. Also, the control system should be at least one octave (better: one decade) faster than the fastest unstable pole of the plant. This requirement also forces a crossover frequency of at least $8 \text{rad/s}$.

The sampling frequency is $\omega_s = \frac{2\pi}{T} \approx 21 \text{rad/s}$, and thus only 2.6 times faster than the required crossover frequency. For emulation to possibly work well, a factor of approximately 10 is required. Therefore, we cannot do emulation.

c) Yes. Even if the control system has a steep descent after the crossover frequency (and thus the sampled signal does not have substantial energy at frequencies higher than the Nyquist frequency of roughly $10.5 \text{rad/s}$), an anti-aliasing filter (AAF) is still needed to attenuate the noise.

d) The four combinations of emulation/discrete controller design and with/without AAF are possible. For emulation, the continuous representation of the plant is used when designing the (continuous) controller. In the case of ‘real’ discrete-time controller design, the plant (and possibly the AAF) have to be discretized first. All four cases are listed here, the correct one based on the answers to b) and c) is the discrete-time design with AAF.

<table>
<thead>
<tr>
<th></th>
<th>discrete-time design</th>
<th>emulation</th>
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<tbody>
<tr>
<td>with AAF</td>
<td>$(1 - z^{-1}) \mathcal{Z} \left{ \mathcal{L}^{-1} \left{ \frac{\text{AAF}(s) \cdot P(s)}{s} \right} \right</td>
<td>_{t=kT} \right} \right</td>
</tr>
<tr>
<td>no AAF</td>
<td>$(1 - z^{-1}) \mathcal{Z} \left{ \mathcal{L}^{-1} \left{ \frac{P(s)}{s} \right} \right</td>
<td>_{t=kT} \right} \right</td>
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Question 2 (Controller Emulation) 6 Points

For a continuous-time plant model, an engineer obtained the following continuous-time controller:

\[ C(s) = \frac{2s + 1}{s + \alpha}, \]

whereas \( \alpha \in \mathbb{R} \) is a tuning factor.

a) (2 points) The engineer implements the controller on a microprocessor using the Euler backward emulation approach. What is the resulting transfer function \( C(z) \) for a generic sampling time \( T \) (\( T \in \mathbb{R}^{+} \))?

b) (3 points) What is the range of the tuning factor \( \alpha \) to produce an asymptotically stable discrete-time controller using the Euler backward emulation approach?

c) (1 points) What is the condition on \( \alpha \) to have an asymptotically stable continuous-time controller \( C(s) \) and an asymptotically stable discrete-time controller using the Euler backward emulation approach?

Solution 2

a) The Euler backward emulation approach is given by

\[ s = \frac{z - 1}{Tz}. \]

Inserting this equation into the controller \( C(s) \) yields

\[ C(z) = \frac{2z - 1 + 1}{z - 1 + \alpha} = \frac{z(2 + T) - 2}{z(1 + \alpha T) - 1}. \]

b) The controller \( C(z) \) is asymptotically stable if its pole \( z_p \) fulfills the condition \( |z_p| < 1 \).

The pole \( z_p \) is obtained from the equation \( z(1 + \alpha T) - 1 = 0 \), i.e. \( z_p = \frac{1}{1 + \alpha T} \). From the condition above, \( \alpha \) must satisfy

\[-1 < \frac{1}{1 + \alpha T} < 1.\]

This inequality constraint leads to the solutions \( \alpha > 0 \) and \( \alpha < -\frac{2}{T} \).

c) For \( C(s) \) being asymptotically stable, its pole \( s_p = -\alpha \) must lie in the left half of the complex plane, i.e. \( \text{Re}\{s_p\} < 0 \Rightarrow \alpha > 0 \).

Together with the results from b), the condition on \( \alpha \) is that \( \alpha > 0 \).
Question 3  (Plant Discretization and Analysis)  8 Points

a)  (3 points) Derive the discrete-time transfer function of the following continuous-time plant:

\[ P(s) = \frac{1}{s^2 + 3s + 2} \]  \hspace{1cm} (1)

Use a zero-order hold element and assume the sampling time \( T \) to be known.

b)  (3 points) Determine the state-space representation of the above system.

Introduce a state vector \( x \) and find the matrices \( F, G, C, D \) describing the state-space representation of the discretized system:

\[ x_{k+1} = Fx_k + Gu_k \]  \hspace{1cm} (2)

\[ y_k = Cx_k + Du_k \]  \hspace{1cm} (3)

c)  (2 points) Assess the stability of the discrete-time system.

Solution 3

a)  Discretizing a continuous-time transfer function using zero-order hold as sampling method can be done by evaluating the following formula:

\[ P(z) = \frac{z - 1}{z} Z \left\{ \frac{1}{s} \cdot P(s) \right\} \bigg|_{t=kT} \]  \hspace{1cm} (4)

Partial fraction decomposition is required in order to apply the transformation table:

\[ \frac{1}{s} \cdot P(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \]  \hspace{1cm} (5)

\[ = \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)} \]  \hspace{1cm} (6)

\[ = \frac{s^2(A + B + C) + s(3A + 2B + C) + 2A}{s(s+1)(s+2)} \]  \hspace{1cm} (7)

Comparing coefficients of the numerator obviously yields \( A = 0.5 \) and solving the remaining two equations for \( B \) and \( C \) results in \( B = -1 \), \( C = 0.5 \). By inserting these values we obtain a transfer function in a form suitable for direct transformation:

\[ \frac{1}{s} \cdot P(s) = \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2} \]  \hspace{1cm} (8)

Using the transformation table results in:

\[ P(z) = \frac{z - 1}{z} \left( 0.5 \cdot \frac{z}{z - 1} - \frac{z}{z - e^{-T}} + 0.5 \cdot \frac{z}{z - e^{-2T}} \right) \]  \hspace{1cm} (9)

In order to simplify the notation for further calculations, the abbreviation \( a = e^{-T} \) is introduced:

\[ P(z) = \frac{z - 1}{z} \left( 0.5 \cdot \frac{z}{z - 1} - \frac{z}{z - a} + 0.5 \cdot \frac{z}{z - a^2} \right) \]  \hspace{1cm} (10)
b) Based on the discrete-time transfer function derived in the foregoing question, the input/output system description is readily obtained by applying the shift property of the Z transformation:

\[ \begin{align*}
Y(z) \cdot (z^2 - z(a^2 + a) + a^3) &= U(z) \cdot (z(0.5a^2 - a + 0.5) + (0.5a^3 - a^2 + 0.5a)) \\
y_{k+2} - (a^2 + a)y_{k+1} + a^3y_k &= (0.5a^2 - a + 0.5)u_{k+1} + (0.5a^3 - a^2 + 0.5a)u_k.
\end{align*} \]

Solving for \( y_{k+2} \) and shifting one timestep backwards yields:

\[ y_{k+1} = (a^2 + a)y_{k} - a^3y_{k-1} + (0.5a^2 - a + 0.5)u_k + (0.5a^3 - a^2 + 0.5a)u_{k-1}. \]  

Rearranging the elements in equation (15) in a chronological order, one gets

\[ y_{k+1} = (a^2 + a)y_{k} + (0.5a^2 - a + 0.5)u_k - a^3y_{k-1} + (0.5a^3 - a^2 + 0.5a)u_{k-1}. \]

By looking at equation (16), the system can be described using two states \( \nu_k = y_k \) and \( \xi_k \) as designated (which contains all information from the previous step), yielding:

\[ \begin{align*}
\nu_{k+1} &= (a^2 + a)\nu_k + (0.5a^2 - a + 0.5)u_k + \xi_k \\
\xi_{k+1} &= -a^3\nu_k + (0.5a^3 - a^2 + 0.5a)u_k.
\end{align*} \]

This is equivalent to the following state space representation:

\[ x_{k+1} = \begin{bmatrix} (a^2 + a) & 1 \\ -a^3 & 0 \end{bmatrix} \cdot x_k + \begin{bmatrix} 0.5a^2 - a + 0.5 \\ 0.5a^3 - a^2 + 0.5a \end{bmatrix} \cdot u_k 
\]

\[ y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x_k + \begin{bmatrix} 0 \end{bmatrix} \cdot u_k. \]

\[ \begin{align*}
(i) \quad & \text{Eigenvalues of matrix } F, \text{ i.e. solve } \det(\pi \cdot I - F) = 0; \\
(ii) \quad & \text{Zeros of the denominator of } P(z) \text{ calculated above (most direct way);} \\
(iii) \quad & \text{Calculate the poles of the continuous-time plant (e.g. from its transfer function } P(s) \text{) and use } \pi_1^* = e^{-\pi_1 T} \text{ to obtain the poles of the discrete-time system.}
\end{align*} \]

All three ways lead to \( \pi_1 = a = e^{-T} \) and \( \pi_2 = a^2 = e^{-2T} \). Since \( T > 0 \), it can be concluded that \( |\pi_{1,2}| < 1 \) and hence the system is asymptotically stable.
Question 4 (Anti–Aliasing Filter and Signal Sampling) 6 Points

a) Signal Sampling without Anti-Aliasing Filter

Given a signal \( x(t) \) which is corrupted by noise \( n(t) \):
\[
y(t) = x(t) + n(t)
\]
\[
x(t) = 2 \cdot \sin(\omega_0 t) + 1 \cdot \sin(\omega_1 t)
\]
\[
n(t) = 0.5 \cdot \sin(\omega_n t)
\]

with: \( \omega_0 = \frac{20 \text{ rad}}{\text{s}}, \omega_1 = \frac{30 \text{ rad}}{\text{s}}, \omega_n = \frac{70 \text{ rad}}{\text{s}} \)

i) (1 point) Draw the amplitude spectrum \( |Y(\omega)| \) of \( y(t) \).
   (Use the first prepared frame on the next page)

ii) (2 points) Now assume that the signal is sampled with the sampling time \( T = \frac{\pi}{40} \text{s} \).
    Calculate the sampling frequency as well as the Nyquist frequency. Draw vertical lines to indicate both frequencies. Sketch the amplitude spectrum of the sampled signal \( |\bar{Y}(\omega)| \). Indicate where the amplitude spectrum of the original signal \( |Y(\omega)| \) is distorted by the aliasing of the noise.
   (Use the second prepared frame on the next page)

b) Signal Sampling with Anti-Aliasing Filter

To avoid aliasing, the following idealized filter \( F \) is placed before the sampling

![Filter Diagram](image)

The filter parameters are \( \alpha > 0 \in \mathbb{R} \) and \( \omega_F > 0 \in \mathbb{R} \).

i) (1 point) Now apply the filter \( F \) to the signal \( y(t) \). Use the filter parameters \( \alpha = 0.05 \frac{\text{rad}}{\text{s}} \) and \( \omega_F = 40 \frac{\text{rad}}{\text{s}} \). Draw the magnitude of the filter \( |F(\omega)| \) and the amplitude spectrum \( |Y_F(\omega)| \) of the filtered signal \( y_F(t) \).
   (Use the third prepared frame on the next page)

ii) (1 point) Now assume that the filtered signal \( y_F(t) \) from question c) i) is also sampled with sampling time \( T = \frac{\pi}{40} \text{s} \). Sketch the amplitude spectrum of the filtered and sampled signal \( |\bar{Y}_F(\omega)| \).
   (Use the fourth prepared frame on the next page)

iii) (1 Point) You want to use the filter described in b) i) for another unknown signal. What is the maximum sampling time you have to use in order to suppress aliasing completely?
Solution 4

a) i) (1 point) The frequencies $\omega_i$ and amplitudes $A_i$ can be read directly from the signal definition:

$\omega_0 = 20 \frac{\text{rad}}{s}$, $\omega_1 = 30 \frac{\text{rad}}{s}$ and $\omega_n = 70 \frac{\text{rad}}{s}$

$A_0 = 2$, $A_1 = 1$ and $A_n = 0.5$

ii) (2 points) The sampling frequency $\omega_S$ can be calculated by

$$\omega_S = \frac{2\pi}{T} = 80 \frac{\text{rad}}{s}$$

and the Nyquist frequency $\omega_N$ is

$$\omega_N = \frac{1}{2} \omega_S = 40 \frac{\text{rad}}{s}$$

The sampling has the following effect on the sampled signal: The signal $y(t)$ gets superimposed with infinitely many copies of the signal $y_n(t)$, with each of their amplitude spectra shifted by $\frac{2\pi n}{T}$ respectively. This creates the signal components at the frequencies $\omega = \{10, 50, 60\} \frac{\text{rad}}{s}$. The alias of the noise at $\omega = 10 \frac{\text{rad}}{s} < \omega_N$ lies in the frequency range of interest and thereby distorts the signal.

b) i) (1 point) The filter suppresses all signal components above $60 \frac{\text{rad}}{s}$. The noise is suppressed, whereas the two signal components at frequencies lower than the Nyquist frequency are left untouched.

ii) (1 point) Now, that there are no more components above the Nyquist frequency (noise) in the filtered signal, no aliases appear for frequencies lower than the Nyquist frequency. In the frequency range of interest the signal is not distorted anymore.

iii) (1 Point) The filter cuts off all signal content above a frequency of $60 \frac{\text{rad}}{s}$. This is the lower limit for the Nyquist frequency $\omega_N$:

$$\omega_{N, \text{min}} = \frac{1}{2} \omega_{S, \text{min}} = 60 \frac{\text{rad}}{s}$$

$$T_{\text{max}} = \frac{2\pi}{\omega_{S, \text{min}}} = \frac{\pi}{\omega_{N, \text{min}}} = \frac{\pi}{60} \text{s}$$
Solution for question b) i)

Solution for question b) ii)

Solution for question c) i)

Solution for question c) ii)
**Question 5 (Controller Synthesis) 10 Points**

You have to design a discrete controller \( C(z) \) for the plant \( P(z) \) and the given feedback structure in figure 3.

![Figure 3: Feedback Control](image)

\[ P(z) = \frac{\alpha z + \beta}{z^2 + \gamma z + \delta}, \quad \text{with } \alpha \neq 0, \beta \neq 0, \gamma, \delta \in \mathbb{R}, \text{ and } T = 1s \]

**a) Root-Locus Design:**

Assume the controller is \( C(z) = k_p \), with \( k_p \in \mathbb{R}^+ \)

i) (2 Points) Where do the poles of the transfer function of the closed-loop control system \( T(z) = \frac{L(z)}{1+L(z)} \) converge to for \( k_p = 0 \) and for \( k_p \to \infty \)?

ii) (2 Points) Assume \( \alpha = 1.05, \beta = -0.45, \gamma = -1.2, \delta = 0 \). Figure 4 (next page) shows the Nyquist diagram of the plant \( P(z) \). Does the Nyquist diagram ensure stability of \( T(z) \) for \( k_p = 1 \)? Give reasons for your answer.

**b) Plant Inversion:**

i) (2 Points) Assume you want to control \( P(z) \) such that the transfer function \( T(z) \) becomes:

\[ T(z) = \frac{A}{z^2 + Bz + 0.1}, \quad \text{with } A, B \in \mathbb{R} \]

Find a pair of numerical values for \( A \) and \( B \) such that all poles of \( T(z) \) are inside the "nice pole region" and the steady state gain of \( T(z) \) is one.

ii) (2 Points) Determine the controller \( C(z) \) which realizes \( T(z) \) for the plant \( P(z) \) with the poles in the nice pole region and the steady state gain of \( T(z) \) equal to one. Does \( C(z) \) contain an open integrator?

**c) MIMO-tuning:**

Assume now, that a discrete-time plant of the following form is given:

\[ x_{k+1} = Fx_k + Gu_k, \quad y_k = Cx_k \]
An engineer designed a discrete-time state-feedback-controller (controller gain $K$) with an observer (observer gain $H$) to control this system. To find the gains $K$ and $H$, he solved the LQR problem for $\{F,G,Q,\rho \cdot I\}$ and $\{F^T,C^T,GG^T,\mu \cdot I\}$, respectively. He proposed three different designs which only differ in $\rho$ and $\mu$. The poles of these designs are given in the plots below, with: ("*" = $F$, "o" = $F - GK$, "+" = $F - HC$).

**Note:** Please answer the following questions with a,b,c or none and explain why.

i) (1 point) Which of the designs has the slowest state-feedback-controller?

ii) (1 point) Which design has the slowest state-observer?

**Solution 5**

a) i) The poles of the closed-loop system $T(z)$ are

$$z_{p,1,2} = -\left(\gamma + k_p\alpha\right) \pm \sqrt{(\gamma + k_p\alpha)^2 - 4(\delta + k_p\beta)}$$
$k_p = 0$: The poles of the closed-loop system $T(z)$ are equal to the poles of the plant $P(z)$:

$$k_p = 0 \quad z_{p,1,2} = -\gamma \pm \sqrt{\frac{\gamma^2 - 4\delta}{2}}$$

$k_p \to \infty$: The poles of the closed-loop system $T(z)$ have to fulfill the following equation:

$$z_{p,1,2}^2 + \gamma z_{p,1,2} + \delta + k_p (\alpha z_{p,1,2} + \beta) = 0$$

For $k_p \to \infty$ the poles of $T(z)$ move to the zeros of the plant $P(z)$:

$$k_p \to \infty \quad z_{p,1} \to -\text{sign}(\alpha) \cdot \infty, \quad z_{p,2} \to -\frac{\beta}{\alpha}$$

ii)

$$P(z) = \frac{\ldots}{z^2 + \gamma z + \delta}$$

$$\Rightarrow \text{poles:} \quad z_{p,1,2} = -\gamma \pm \frac{\gamma^2}{2} = \{0, -\gamma\} = \{0, 1.2\}$$

$P(z)$ has one unstable pole and the gain $P(e^{j\varphi})$ for $\varphi \in [\epsilon, 2\pi - \epsilon]$ encloses the critical point -1 one time ccw. According to the Nyquist criterion the closed-loop system with $k_p = 1$ is stable.

b) i) Steady state unity gain condition:

$$A = 1 + B + 0.1 \quad (21)$$

Poles: $p_{1,2} = -\frac{B \pm \sqrt{B^2 - 0.4}}{2}$ make a simple choice and choose $B = -\sqrt{0.4}$ to obtain a double real pole at $\frac{\sqrt{0.4}}{2} = 0.3162$ which is clearly inside the "nice pole region" and calculate $A = 1 - \sqrt{0.4} + 0.1 = 0.4657$.

ii)

$$C(z) = \frac{1}{P(z)} \frac{T(z)}{1 - T(z)} = \frac{z^2 + \gamma z + \delta}{\alpha z + \beta} \cdot \frac{\alpha z + \beta}{z^2 + Bz + 0.1 - A}$$

an open integrator means a pole at $z = 1$. Looking at the denominator we obtain the condition:

$$(\alpha z + \beta) \cdot (z^2 + Bz + 0.1 - A)|_{z=1} = 0$$

$\alpha + \alpha B + 0.1\alpha - \alpha A + \beta + \beta B + 0.1\beta - \beta A = 0$

$\alpha (1 + B + 0.1 - A) + \beta (1 + B + 0.1 - A) = 0$

And from equation 21 follows that $1 + B + 0.1 - A = 0$. The controller $C(z)$ therefore contains an open integrator.

c) i) $b)$ is the slowest state-feedback-controller because this design has its eigenvalues of $F - GK$ in the "slowest location" (compared to the other designs) in the complex plane.

ii) $c)$ is the slowest state-observer because this design has its eigenvalues of $F - HC$ in the "slowest location" (compared to the other designs) in the complex plane.