Modeling Credit Risk of Loan Portfolios in the Presence of Autocorrelation (Part 2)

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Section 1

Why is autocorrelation present in default rates?

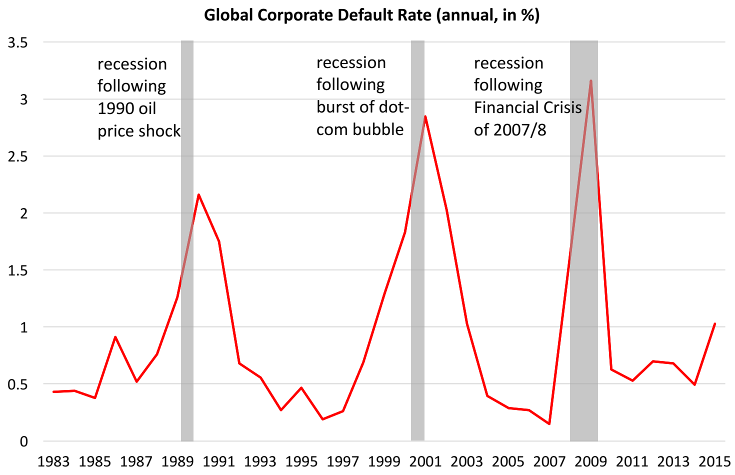
Rating and sectoral buckets

- Typically, obligors are grouped into rating and sectoral buckets.

		$\leftarrow$ sectors/business lines/regions $\rightarrow$			
		Swiss mortgages	Swiss small businesses	US credit cards	$\ldots$
$\uparrow$ ratings $\downarrow$	AAA	bucket (1,1)	bucket (1,2)	bucket (1,3)	$\ldots$
	AA+	bucket (2,1)	$\ddots$		
	AA	bucket (3,1)			
	AA−	bucket (4,1)			
	$\vdots$	$\vdots$			

- It is assumed that the obligors in a bucket are homogeneous: same default probabilities and default correlation within a bucket and to other buckets.

Default rates are cyclical



Gray areas: recessions as defined by US National Bureau of Economic Research

Source: Moody's

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Point-in-time versus through-the-cycle ratings

Based on cyclicality of default rates, there are two types of ratings:

- Point-in-time (PiT) ratings: evaluate the credit quality by taking into account all currently available information

  $\Rightarrow$ the credit cycle affects the rating of an obligor and thus obligors move to different rating buckets in the credit cycle

  $\Rightarrow$ default prob. in rating bucket not dependent on credit cycle

- Through-the-cylce (TtC) ratings: focus on the permanent component of creditworthiness

  $\Rightarrow$ obligor's rating does not depend on credit cycle

  $\Rightarrow$ default prob. in rating bucket changes through credit cycle

Meaning and (dis)advantages of ratings as PiT versus TtC credit indicators are widely discussed in industry and academia.

Main disadvantage of PiT ratings: loss estimates (and thus the capital buffer) reduce in good times and expand in recessions.

Through-the-cycle ratings in the Basel Accords

TtC ratings are often used in practice, for example, in the Basel's advanced internal rating-based approach:

- Current: "Although the time horizon used in PD [probability of default] estimation is one year... banks must use a longer time horizon in assigning ratings."[1]
- Proposed addition: "Rating systems should be designed in such a way that assignments to rating categories generally remain stable over time and throughout business cycles. Migration from one category to another should generally be due to idiosyncratic or industry-specific changes rather than due to business cycles."[2]

[1] Paragraph 414 in Basel Committee on Banking Supervision. International convergence of capital measures and capital standards, 2006.

[2] Section 4.1 in Basel Committee on Banking Supervision. Consultative document: Reducing variation in credit risk-weighted assets — constraints on the use of internal model approaches, 2016.

Presence of autocorrelation in default rates

In practice, when analyzing time series of default rates, we often observe autocorrelation. Underlying reasons:

- Credit cycles are driven by economic factors, which typically exhibit autocorrelation.
- When using TtC ratings, credit cycle and thus the autocorrelation directly affect default rates.
- When using PiT ratings, default rates for a given rating bucket should (in theory) not exhibit autocorrelation because they are not dependent on the credit cycle since all available information on the state of the economy is reflected in the current ratings.
- Attempting to use PiT ratings in practice, we still see autocorrelation due to TtC "dampening" of rating transitions: a change in credit quality may not lead immediately to a change in rating.

$\Rightarrow$ Autocorrelation should be taken into account when using default rates as input of estimators.

Section 2

Convergence results for autocorrelated time series

Questions on asymptotic properties

Consider the sample mean $M_T = \frac{1}{T}\sum_{t=1}^{T} Z_t$ of an autocorrelated time series $(Z_t)_{t=-\infty}^{t=\infty}$.

- Does M_T converge almost surely as $T \to \infty$?
  (Law of large numbers)

  Under mild assumptions: yes, but more slowly than i.i.d. sequence.

- Is $\sqrt{T} M_T$ asymptotically normally distributed?
  (Central limit theorem)

  Under mild assumptions, yes, but with greater variance than i.i.d. sequence.

Illustration: AR(1) versus i.i.d.

Consider an AR(1) process:

$$Z_t = cZ_{t-1} + \epsilon_t$$

for $\epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $c = 0.7$.

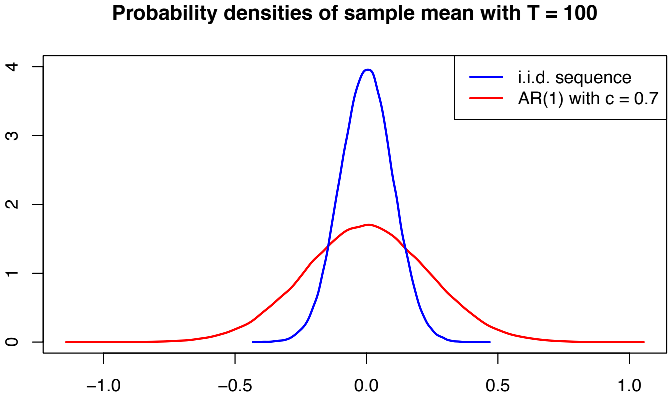
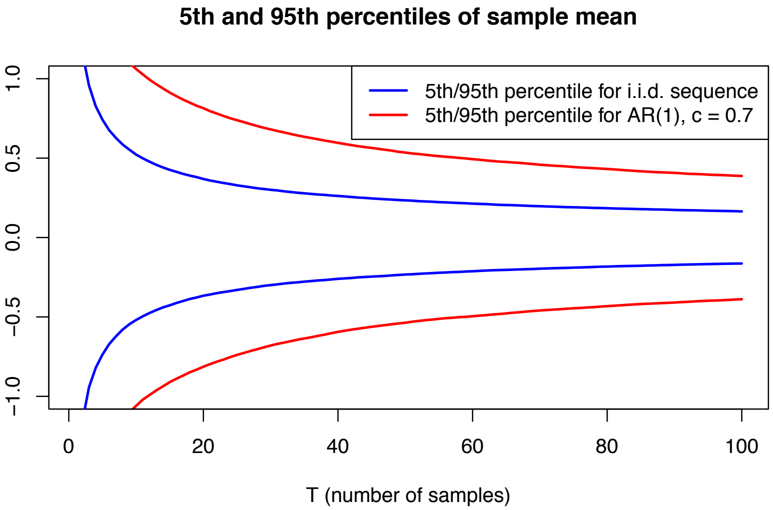


Illustration: AR(1) versus i.i.d. (cont'd)



Assumptions on time series

We consider the following assumptions:

A1 Stationarity: any k subsequent random variables $Z_{t+1}, \dots, Z_{t+k}$ have the same distribution regardless of the starting point t.

A2 Absolute summability of autocovariances: there exists a constant $C < \infty$ such that $\sum_{t=-\infty}^{\infty} |\gamma_{s,t}| \leq C$ for all s, where the autocovariances $\gamma_{s,t}$ are defined by

$$\gamma_{s,t} = E\Big[\big(Z_s - E[Z_s]\big)\big(Z_t - E[Z_t]\big)\Big].$$

A3 Asymptotic uncorrelatedness: $E[Z_t | Z_{t-k}, Z_{t-k-1}, \dots]$ converges in mean square to zero as $k \to \infty$.

A4 Asymptotic negligibility of innovations: $\sum_{k=0}^{\infty} E[r_{t,k}^2]$ is finite for fixed t, where

$$r_{t,k} = E[Z_t | Z_{t-k}, Z_{t-k-1}, \dots] - E[Z_t | Z_{t-k-1}, Z_{t-k-2}, \dots].$$

For example, all assumptions are satisfied for an AR(1) process.

Convergence results

Law of large numbers for autocorrelated time series: under assumptions A1 and A2,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} Z_t = E[Z_1] \text{ almost surely.}$$

Gordin's central limit theorem: under assumptions A1–A4,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} Z_t \xrightarrow{T\to\infty} \mathcal{N}\left(0, \sum_{t=-\infty}^{\infty} \gamma_{0,t}\right) \text{ in distribution.}$$

Biased estimator for finite time series

- We consider a moment estimator of the form $g(\theta) = \mu$, where θ is a parameter to be estimated, μ is the unknown mean of the stationary time series $(Z_t)_{t=1,\ldots,T}$.
- We assume that g is three times continuously differentiable and invertible with inverse $\tilde{g} = g^{-1}$.
- A natural estimation for θ is the moment estimator

$$\hat{\theta}_1 = \tilde{g}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right).$$

- For finite T, there is an estimation bias because

$$E[\hat{\theta}_1] = E\left[\tilde{g}\left(\frac{1}{T}\sum_{t=1}^{T} Z_t\right)\right] \neq \tilde{g}\left(E\left[\frac{1}{T}\sum_{t=1}^{T} Z_t\right]\right) = \tilde{g}(\mu) = \theta.$$

Adjusting for shortness and autocorrelation

- We introduce a new estimator

$$\hat{\theta}_2 = \underbrace{\tilde{g}(\bar{\mu})}_{\text{original estimator}} + \underbrace{\frac{g''(\tilde{g}(\bar{\mu}))}{2T(g'(\tilde{g}(\bar{\mu})))^3}\alpha_0}_{\text{adjustment for pure shortness}} + \underbrace{\frac{g''(\tilde{g}(\bar{\mu}))}{T(g'(\tilde{g}(\bar{\mu})))^3}\sum_{\ell=1}^{k}(1-\ell/T)\alpha_\ell}_{\text{adjustment for shortness and autocorrelation}},$$

where $\bar{\mu} = \frac{1}{T}\sum_{t=1}^{T} Z_t$ is the sample mean and

$$\alpha_\ell = \frac{1}{T}\sum_{t=1+\ell}^{T}(Z_t - \bar{\mu})\big(Z_{t-\ell} - \bar{\mu}\big), \quad \ell = 0, 1, \ldots, k$$

is the lag-ℓ sample autocovariance.

- We can show an explicit bound for the error of $\hat{\theta}_2$ in estimating θ, and find approximate confidence intervals for $\hat{\theta}_2$.

Section 3

Adjusted estimators in credit risk

Revisiting the Merton framework

- We consider a fixed rating and sectoral bucket.
- The normalized asset return of obligor i is given by

$$R_i = \sqrt{\varrho}\, Y + \sqrt{1-\varrho}\, \epsilon_i$$

  where
  - $\varrho \in [0,1]$ is the latent return correlation
  - $Y \sim \mathcal{N}(0,1)$ is the systematic factor common to all obligors
  - $\epsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ is the idiosyncratic component, indep. of Y.
- Obligor i defaults if his/her return is below a threshold s.
- If the unconditional default probability is p,

$$p = P[R_i \leq s] = \Phi(s) \implies s = \Phi^{-1}(p)$$

- The loss rate conditional on the systematic factor Y is given by

$$p(Y) = \Phi\left(\frac{\Phi^{(-1)}(p) - \sqrt{\varrho}\, Y}{\sqrt{1-\varrho}}\right).$$

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Classical estimator for latent return correlation

- In practice, we observe a time series $Z_t = p(Y_t)$ for $t = 1, \ldots, T$ and want to estimate latent correlation ϱ.
- Classical result:

$$E\Big[(p(Y_t))^2\Big] = \Phi_2\big(\Phi^{(-1)}(p), \Phi^{(-1)}(p); \varrho\big),$$

  where $\Phi_2(.,.;\varrho)$ denotes the bivariate normal cumulative distribution function with correlation ϱ.
- Thus, we obtain an estimator $\hat{\varrho}_1$ from

$$\frac{1}{T}\sum_{t=1}^{T}(p(Y_t))^2 = \Phi_2\big(\Phi^{(-1)}(\hat{p}), \Phi^{(-1)}(\hat{p}); \hat{\varrho}_1\big).$$

  where $\hat{p} = \frac{1}{T}\sum_{t=1}^{T} p(Y_t)$.

Classical estimator is biased

- We can show $g(.) = \Phi_2\big(\Phi^{(-1)}(\hat{p}), \Phi^{(-1)}(\hat{p})); .\big)$ is invertible.
- Estimator given by

$$\hat{\varrho}_1 = g^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}(p(Y_t))^2\right)$$

  - has bias because of shortness and autocorrelation of time series $(p(Y_t))^2$, $t = 1, \ldots, T$
  - is of the same form as in the previous section
  - can be adjusted for shortness and autocorrelation

New estimator for latent return correlation

- Applying results from previous section, we obtain a new estimator

$$\hat{\varrho}_2 = \hat{\varrho}_1 + \frac{g''(\hat{\varrho}_1)}{T(g'(\hat{\varrho}_1))^3}\left(\alpha_0/2 + \sum_{\ell=1}^{k}(1-\ell/T)\alpha_\ell\right),$$

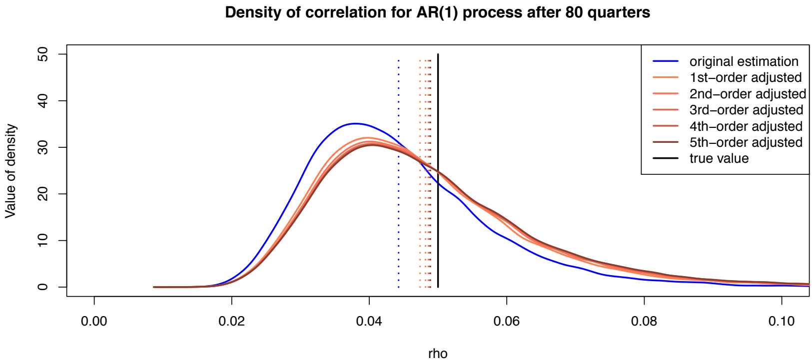
  where derivatives in correction term are explicitly determined, and α_0 and α_ℓ are the sample variance and covariances.

- Adjusting for autocorrelation crucially depends on the length T of the time series.

- Based on the results of the previous section, we can also find confidence intervals for the estimators.

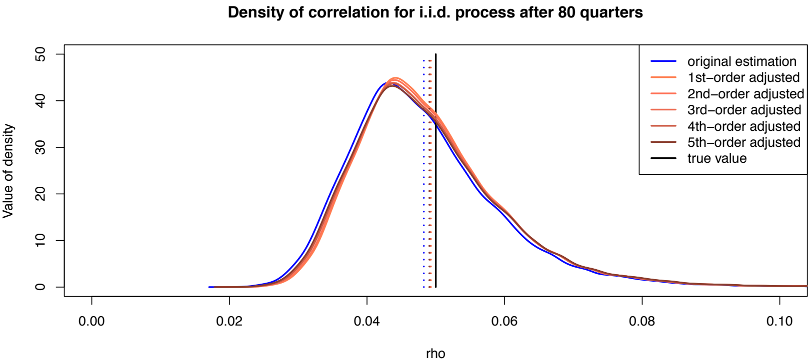
- We have similarly adjusted estimators for the correlation estimator between two different buckets.

Illustration of adjustment for AR(1)



- The underlying factors are simulated based on an AR(1) process with coefficient 0.7 and 50,000 simulations.
- The adjustments remove a big part of the bias so that the adjusted means are much closer to the true value of $\varrho = 0.05$

Illustration of adjustment for i.i.d.



- The underlying factors are simulated based on i.i.d. observations and 50,000 simulations.
- Also for independent observations, the adjusted means are much closer to the true value of $\varrho = 0.05$

Conclusion

- We explained why and how autocorrelation is present in time series of default rates.
- We showed that classical estimators used in credit risk modelling suffer from bias due to shortness and autocorrelation of default series.
- We suggested new estimators based on adjustments for general autocorrelated time series, removing a big portion of the bias.
- Alternatives are maximum likelihood estimators, which, however, are much harder to adjust for autocorrelation than method of moment estimators.
- Thank you very much for your attention!

Disclaimer

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