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Numerical Validation of Emergent Behavior in Flocks

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1 Introduction

An emergent behaviour of a system appears when a number of simple system components interact with each other to create more complex behaviour as a collective. Examples of emergent behaviour are found in stock market, language evolution, flocking and swarming of animal species like birds and fish. Such behaviour of a system is hard to predict. This is mostly due to the fact that the number of interactions between components of a system increases exponentially with the number of components.

A simple model for a flock of birds was proposed by Cucker and Smale [1]. A population of birds moving in $\mathbb{R}^3$ and influencing each other depending on the distance between them was considered. The velocity of each bird was adjusted by adding to it a weighted average of the differences to the velocities of other birds. Moreover conditions on the initial state, namely positions and velocities of birds, were established under which the flock converges to one in which all birds have the same velocity. Additionaly, a similar model was provided which describes language evolution in primitive societies.

The goal of this paper is to validate these models by performing computer simulations. In Sections 2 and 3 we introduce mathematical models for the evolution of a flock of birds in continuous and discrete time. Section 4 introduces some preliminaries which are important for the analysis of convergence theorems in continuous and discrete time described in Sections 5 and 6. Finally the mathematical model and verification of the convergence theorem of the language evolution is discussed in Section 7.

2 Continuous Model

In the following we consider the population of birds in $\mathbb{R}^d$, where $d \in \mathbb{N}$ is the space dimension. Usually d is chosen to be 3. Each bird i has a position and a velocity at time t given by $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$. The entire flock of k birds is then described by the vector of positions $x \in (\mathbb{R}^d)^k$ and the vector of velocities $v \in (\mathbb{R}^d)^k$. Let a_{ij} be a weight that quantifies the influence of the bird i on the bird j which is given by

$$a_{ij} = \eta(\|x_i - x_j\|^2), \qquad (2.1)$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically decreasing function

$$\eta(y) = \frac{K}{(\sigma^2 + y)^\beta} \qquad (2.2)$$

with fixed values for $K \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\sigma > 0$ and $\beta \in \mathbb{R}$, $\beta \geq 0$. This function describes the influence of bird i on bird j depending on the distance between them, see Figure 1.

The change in velocity of bird i is modeled as a weighted sum of differences

to velocities of other birds, which can be formulated as

$$
\begin{aligned}
\dot{v}_i(t) &= -\sum_{j=1}^{k} a_{ij}\big(v_i(t) - v_j(t)\big) \\
&= -\Big(\sum_{j=1}^{k} a_{ij}\Big) v_i(t) + \sum_{j=1}^{k}\big(a_{ij} v_j(t)\big) \\
&= -[D_x v(t)]_i + [A_x v(t)]_i \\
&= -[L_x v(t)]_i.
\end{aligned} \qquad (2.3)
$$

Here $A_x \in \mathbb{R}^{k \times k}$ is a weighted adjacency matrix, $D_x \in \mathbb{R}^{k \times k}$ a diagonal matrix with the ith diagonal entry given by $\sum_{j \leqslant k} a_{ij}$ and L_x the Laplacian of the matrix A_x, given by

$$
L_x = D_x - A_x.
$$

The multiplication of $A_x \in \mathbb{R}^{k \times k}$ with $v \in (\mathbb{R}^d)^k$ is a usual matrix-vector multiplication whereas the elements of v are in $\mathbb{R}^d$.

The model for k birds is given by a system of differential equations, which describe the evolution of a swarm in continuous time,

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -L_x v.
\end{aligned} \qquad (2.4)
$$

It is known that L_x satisfies the following conditions, see the Section 2 in [1].

a) $\forall v_0 \in \mathbb{R}^d$ and $v = (v_0, v_0, \dots, v_0) \in (\mathbb{R}^d)^k$, $\quad L_x v = 0$.

b) If $\lambda_1, \dots, \lambda_k$ are eigenvalues of of L_x, then $0 = \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_k = \|L_x\|$.

c)

$$
\forall v, u \in (\mathbb{R}^d)^k, \quad \langle L_x v, u \rangle = \frac{1}{2} \sum_{i,j=1}^{k} a_{ij} \langle v_i - v_j, u_i - u_j \rangle \qquad (2.5)
$$

Since we want to analyse the interaction between birds depending only on their relative positions and not on the exact spatial positions, it is useful to transform our system into a center of momentum frame. This can be done by defining corresponding spaces as follows.

Let Δ be the diagonal in $(\mathbb{R}^d)^k$,

$$
\Delta := \{ v = (v_0, v_0, \dots, v_0) \in (\mathbb{R}^d)^k \mid v_0 \in \mathbb{R}^d \}
$$

and $\Delta^\perp$ the orthogonal complement of Δ,

$$
\Delta^\perp := \{ v \in (\mathbb{R}^d)^k \mid \langle v, u \rangle = 0 \ \forall u \in \Delta \} = \{ v \in (\mathbb{R}^d)^k \mid \sum_{i=1}^{k} v_i = 0 \in \mathbb{R}^d \}.
$$

Then every point $x \in (\mathbb{R}^d)^k$ can be uniquely decomposed as $x = x_\Delta + x_\perp$ where $x_\Delta \in \Delta$ and $x_\perp \in \Delta^\perp$.
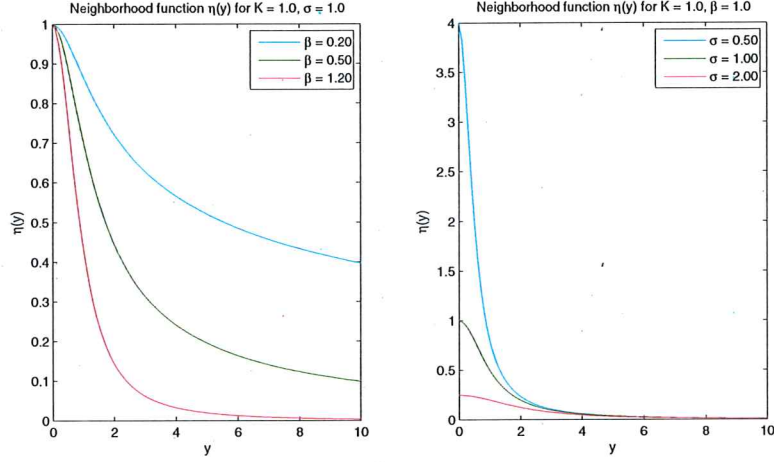
Figure 1: Neighborhood function $\eta(y)$ for different parameters σ and β. In the left figure β is varied. Large β-values decrease the distance of influence between neighbors. In the right figure σ is varied. σ represents the minimal typical distance between neighbors.

Proposition 2.1. *Let* $x(t), v(t)$ *be two functions* $\mathbb{R}^+ \to (\mathbb{R}^d)^k$, *which can be decomposed as* $x(t) = x_\Delta(t) + x_\perp(t)$ *where* $x_\Delta(t) \in \Delta$ *and* $x_\perp(t) \in \Delta^\perp$. *Similarly* $v(t)$ *is decomposed as* $v = v_\Delta(t) + v_\perp(t)$ *where* $v_\Delta(t) \in \Delta$ *and* $v_\perp(t) \in \Delta^\perp$.

(x, v) solves the system (2.4) is equivalent to $(x_\perp, v_\perp)$ and (x_Δ, v_Δ) solve the restricted systems

$$
\begin{aligned}
\dot{x}_\perp &= v_\perp \\
\dot{v}_\perp &= -L_{x_\perp} v_\perp
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
\dot{x}_\Delta &= v_\Delta \\
\dot{v}_\Delta &= 0.
\end{aligned}
\tag{2.7}
$$

Proof. Consider $v_\perp, x_\perp$ to be a solution of (2.6) with initial conditions $v_\perp(0) \in \Delta^\perp, x_\perp(0) \in \Delta^\perp$. First check that $x_\perp(t), v_\perp(t) \in \Delta^\perp$, as claimed.

· Using $x_{\Delta,i} = x_{\Delta,j}$, which follows from the definition of Δ, we see

$$
\|x_i - x_j\| = \|x_{\Delta,i} + x_{\perp,i} - x_{\Delta,j} - x_{\perp,j}\| = \|x_{\perp,i} - x_{\perp,j}\|.
\tag{2.8}
$$

Therefore $a_{ij} = \eta(\|x_i - x_j\|^2) = \eta(\|x_{\perp,i} - x_{\perp,j}\|^2) = a_{ij\perp}$
$\Rightarrow L_x = L_{x_\perp}$

· $\dot{v}_\perp = -L_{x_\perp} v_\perp \in \Delta^\perp$. From equation (2.5) it follows that $\langle L_x v, u \rangle = 0$ for $u \in \Delta$. Therefore, $L_x v \in \Delta^\perp$ for all $x, v \in (\mathbb{R}^d)^k$.

· As $\dot{v}_\perp \in \Delta^\perp$ and $v_\perp(0) \in \Delta^\perp$ for all t, $v_\perp$ remains in $\Delta^\perp$.

· With the same argument, $x_\perp$ remains in $\Delta^\perp$.

4

(a) domain for 1st point

(b) domain for 2nd point

(c) 1D domain of both points combined

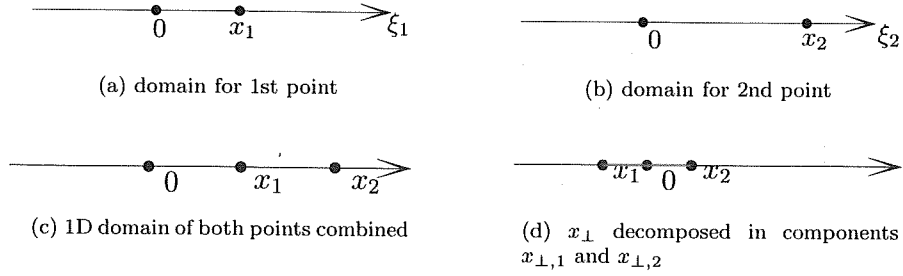(d) $x_\perp$ decomposed in components $x_{\perp,1}$ and $x_{\perp,2}$

Figure 2: domains for each point

For system (2.7), we observe that $\dot{v}_\Delta = 0 = -L_x v_\Delta$, which follows from (2.3) since $v_\Delta \in \Delta$. Therefore $v_\Delta(t) = v_\Delta(0) = const$ and $\dot{x}_\Delta = v_\Delta \in \Delta \Rightarrow x_\Delta(t) \in \Delta \; \forall t$.

We define $\hat{v} := v_\Delta + v_\perp$ and $\hat{x} := x_\Delta + x_\perp$, then $(\hat{v}, \hat{x})$ fulfills (2.4), because

$$\dot{\hat{x}} = \frac{d(x_\Delta + x_\perp)}{dt} = \dot{x}_\Delta + \dot{x}_\perp$$
$$= v_\Delta + v_\perp = \hat{v}$$
$$\dot{\hat{v}} = \dot{v}_\Delta + \dot{v}_\perp = 0 - L_{x_\perp} v_\perp$$
$$= \underbrace{-L_x v_\Delta}_{=0} - L_{x_\perp} v_\perp = -L_x(v_\Delta + v_\perp)$$
$$= -L_x \hat{v}.$$

The solution of (2.4) is unique, therefore $x = \hat{x}, v = \hat{v}$. $\qquad \square$

In order to obtain an intuition of the statement of Proposition 2.3 we present a simple example.

Example 2.2 (1D Example). Let $x = (x_1, x_2) \in (\mathbb{R}^1)^2$ be a vector of two one-dimensional birds represented by points. We can visualize them in a 2D plot, see Figure (3), by setting the ξ_1 axis to the domain where x_1 exists and the ξ_2 axis to the domain where x_2 exists, see Figures (2a),(2b) and (2c). The collection of these points (e.g the swarm of two birds) is identified by a two-dimensional vector $x \in \mathbb{R}^2$. Let $\Delta = \{(u,v) \in \mathbb{R}^2 | u = v\}$. By projecting x onto $\Delta^\perp$, see Figure (3), and decomposing $x_\perp$ in its components $x_{\perp,1}$ and $x_{\perp,2}$ you can see in (2d) that the distance between $x_{\perp,1}$ and $x_{\perp,2}$ remains the same as the distance between x_1 and x_2, see Figure (2c).

This example shows the distances between individuals to be conserved by projection onto orthogonal complement $\Delta^\perp$. This is also easy to see for general d and k since (2.8).

3 Discrete Model

The evolution of (2.4) can also be considered in discrete time by applying explicit Euler. We consider the approximate solutions $x^m \approx x(t_m)$ and $v^m \approx v(t_m)$ for
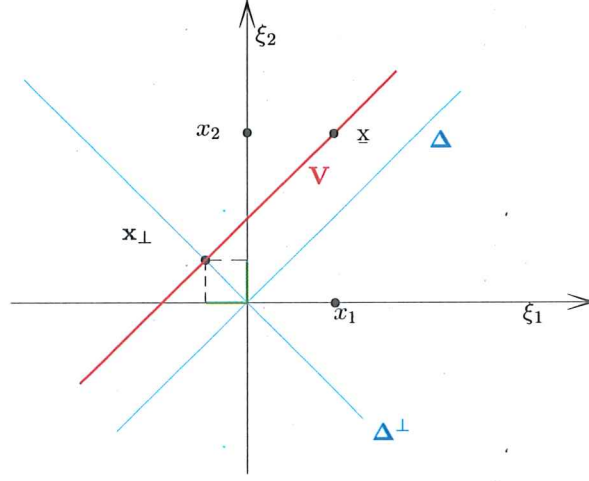
Figure 3: Visualisation of both points in 2D space

$t_m = t_0 + m\triangle t$, where $\triangle t$ is the step size and $0 \leq m \leq n$, where n is number of timesteps. The discrete model is obtained by discretizing the first derivatives and solving for $m + 1$,

$$x^{m+1} = x^m + \triangle t v^m$$
$$v^{m+1} = v^m - \triangle t L_{x^m} v^m. \tag{3.1}$$

The Laplacian matrix is the same as for the continuous model (2.4). Since we are interested in relative differences between individuals only, the discrete model as well as the continuous model can be transformed into the center of momentum frame, which corresponds to the $\triangle^\perp$ space.

Proposition 3.1. *Let (x, v) be a solution to system (3.1) and $(x_\perp, v_\perp)$ the projection of (x, v) onto $\triangle^\perp$. Then $(x_\perp, v_\perp)$ solves the system restricted to $\triangle^\perp$*

$$x_\perp^{m+1} = x_\perp^m + \triangle t v_\perp^m$$
$$v_\perp^{m+1} = v_\perp^m - \triangle t L_{x_\perp^m} v_\perp^m. \tag{3.2}$$

Proof. Each pair (x^m, v^m) can be uniquely decomposed into $x^m = x_\triangle^m + x_{\triangle_\perp}^m$ and $v^m = v_\triangle^m + v_{\triangle_\perp}^m$. Therefore

$$\underbrace{x_\triangle^{m+1}}_{\in \triangle} + \underbrace{x_\perp^{m+1}}_{\in \triangle^\perp} = x^{m+1} = x^m + \triangle t v^m$$

$$= \underbrace{x_\triangle^m + \triangle t v_\triangle^m}_{\in \triangle} + \underbrace{x_\perp^m + \triangle t v_\perp^m}_{\in \triangle^\perp}.$$

Due to the uniqueness of the decomposition, $x_\perp^{m+1} = x_\perp^m + \triangle t v_\perp^m$ (and $x_\triangle^{m+1} = x_\triangle^m + \triangle t v_\triangle^m$).
The same principle can be applied to the velocities:

$$\underbrace{v_\triangle^{m+1}}_{\in \triangle} + \underbrace{v_\perp^{m+1}}_{\in \triangle^\perp} = v^{m+1} = v^m - \triangle t L_{x^m} v^m$$

$$= \underbrace{v_\triangle^m}_{\in \triangle} - \underbrace{\triangle t L_{x^m} v_\triangle^m}_{=0} + \underbrace{v_\perp^m - \triangle t L_{x^m} v_\perp^m}_{\in \triangle^\perp}.$$

6

Therefore $v_{\perp}^{m+1} = v_{\perp}^m - \triangle t L_{x_{\perp}^m} v_{\perp}^m$ with $L_{x_{\perp}^m} = L_{x^m}$. $\qquad\qquad\square$

The discrete model (3.1) is the same model proposed by [1] for $\triangle t = 1$, given by

$$x_{\perp}^{m+1} = x_{\perp}^m + \triangle t v_{\perp}^m$$
$$v_{\perp}^{m+1} = v_{\perp}^m - \tilde{L}_{x_{\perp}^m} v_{\perp}^m. \qquad\qquad (3.3)$$

The model (3.1) can be transformed to (3.3) by scaling L_x^m in (3.1) such that $\tilde{K} = K\triangle t$.

4   Some Preliminaries

Since we are interested in differences $x_i - x_j, v_i - v_j$, it is useful to define terms for measuring these differences.

Let $Q : (\mathbb{R}^d)^k \times (\mathbb{R}^d)^k \to \mathbb{R}$ be defined as

$$Q(u,v) = \frac{1}{2} \sum_{i,j}^{k} \langle u_i - u_j, v_i - v_j \rangle.$$

$Q(u,v)$ is a symmetric bilinear form. If u,v are restricted to $\Delta^{\perp}$, then Q is additionally positive definite and therefore defines a scalar product $\langle \cdot, \cdot \rangle_Q$ on $(\mathbb{R}^d)^k$. We define $\Gamma := \frac{1}{2} \sum_{i,j}^{k} \|x_i - x_j\|_2^2 = \|x\|_Q^2$ and $\Lambda := \frac{1}{2} \sum_{i,j}^{k} \|v_i - v_j\|_Q^2 = \|v\|_Q^2$. $\Gamma(x) = \Gamma(x_{\perp})$ and $\Lambda(v) = \Lambda(v_{\perp})$, since (2.8).

On a finite-dimensional vector space all norms are equivalent, therefore constants ν and $\bar{\nu}$ can be found such that, restricted to $\Delta^{\perp}$, it holds

$$\nu \left\| \cdot \right\|_2^2 \le \left\| \cdot \right\|_Q^2 \le \bar{\nu} \left\| \cdot \right\|_2^2. \qquad\qquad (4.1)$$

It was shown by Cucker et al. [1] that bounds for ν and $\bar{\nu}$ are: $\nu \ge \frac{1}{3k}$ and $\bar{\nu} \le 2k(k-1)$ for d=3. We provide sharper bounds for $\nu, \bar{\nu}$ for arbitrary d.

Lemma 4.1. $\nu = \bar{\nu} = k$, i.e. $\left\| \cdot \right\|_Q^2 = k \left\| \cdot \right\|_2^2$ on $\Delta^{\perp}$.

Proof. We give the proof for $d = 1$ in order to simplify the notation. The proof for $d \in \mathbb{N}, d > 1$ is similar.

By rewriting (4.1) it follows $\nu \le \frac{\|v\|_Q^2}{\|v\|_2^2} = \frac{Q(v,v)}{\langle v, v \rangle}$. In order to achieve a sharp condition for ν, the right term of the inequlity should be minimized:

$$\nu = \min_{v \in \Delta^{\perp}} \frac{Q(v,v)}{\langle v, v \rangle}. \qquad\qquad (4.2)$$

For $a_{ij} = 1 \; \forall i,j, \quad Q(v,v) = \langle L_x v, v \rangle$ in (2.5).
Finding the minimal eigenvalue of L_x to the corresponding eigenvector $v \in \Delta^{\perp}$ is equivalent to minimizing (4.2).

Consider the matrix $L_x = D_x - A_x = \begin{pmatrix} k & & \\ & \ddots & \\ & & k \end{pmatrix} - \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$, where

$L_x, A_x, D_x \in \mathbb{R}^{k \times k}$.

The matrix A_x is symmetric and thus diagonalizable. k is the eigenvalue of A_x to

the eigenvector $v \in \Delta, v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, since $\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = k \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

All other eigenvectors $\hat{v}$ are orthogonal to v, so $\sum_{i=1}^{k} \hat{v}_i = 0$. The equation $A_x \hat{v} = \lambda \hat{v}$ is only fulfilled if $\lambda = 0 \; \forall \hat{v}$.

The eigenvalues of $L_x = k \cdot \mathrm{Id} - \hat{A}_x$ are $0 = k - k$ and $k = k - 0$. By restricting ourselves to $\Delta^\perp$, we obtain that k is the minimal and the maximal eigenvalue of L_x. Therefore $\nu = min(\lambda(L_x)) = k$ and $\bar{\nu} = max(\lambda(L_x)) = k$.

$\square$

5 Convergence in Continuous Time

In order to simplify notation, we denote the solution $(x_\perp, v_\perp)$ of (2.6) at time $t \in \mathbb{R}^+$ as $x(t), v(t) \in \Delta^\perp$. $x(t)$ determines the adjacecy matrix $A_{x(t)}$, the Laplacian $L_{x(t)}$ and $\Gamma(x(t))$ at timepoint t, which we denote as $A(t), L(t), \Gamma(t)$. Likewise $v(t)$ determines $\Lambda(v(t))$, which we will denote as $\Lambda(t)$.

It is useful to introduce the Fiedler number ϕ_t as the second smallest eigenvalue of $L_x(t)$.

Theorem 5.1. *Let $(x(t), v(t))$ be a solution of (2.4) for $d = 3$. Assume that*

$$a_{ij} = \eta(\|x_i - x_j\|^2) \tag{5.1}$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically decreasing function

$$\eta(y) = \frac{K}{(\sigma^2 + y)^\beta}$$

for some constants $K, \sigma > 0, \beta \geq 0$. Assume that one of the following initial conditions holds.

i) $\beta < \frac{1}{2}$

ii) $\beta = \frac{1}{2}$ and $\Lambda(0) < \frac{(kK)^2}{8}$

iii) $\beta > \frac{1}{2}$ and $[(\frac{1}{2\beta})^{\frac{1}{2\beta-1}} - (\frac{1}{2\beta})^{\frac{2\beta}{2\beta-1}}](\frac{(kK)^2}{8\Lambda(0)})^{\frac{1}{2\beta-1}} > 2\Gamma(0)$.

Then

· $\Lambda(t) \to 0$ when $t \to \infty$

· there exists $\hat{x} \in \Delta^\perp$ such that $x(t) \to \hat{x}$ when $t \to \infty$

Theorem 5.1 was introduced as Theorem 2 by [1] and proved for $d = 3$. Moreover, it was proven that

$$\Lambda(t) \leq \Lambda(0) \, e^{-2t\Phi_t} \tag{5.2}$$

for $\Phi_t = \min_{\tau \in [0,t]} \phi_\tau, \; \forall t \geq 0$, which shows exponential convergence of $\Lambda(t)$.

8

We carried out several simulations in order to evaluate the model proposed by F. Cucker et al. and to show the validity of Theorem 5.1.

The birds were uniformly distributed in a three dimensional space with a chosen maximal distance from each other, see Figure 4. The same was done for the velocity of birds. The different initial conditions were chosen by varying the initial velocities. For the remainder of this section we will present testcases, where the main modeling parameter β and the variance of the initial distribution of velocities and positions are varied. The parameters were selected to be $K = 0.0029, \sigma = 1, T_{end} = 1000$.

(a) Variance in velocity 0.05   (b) Variance in velocity 6
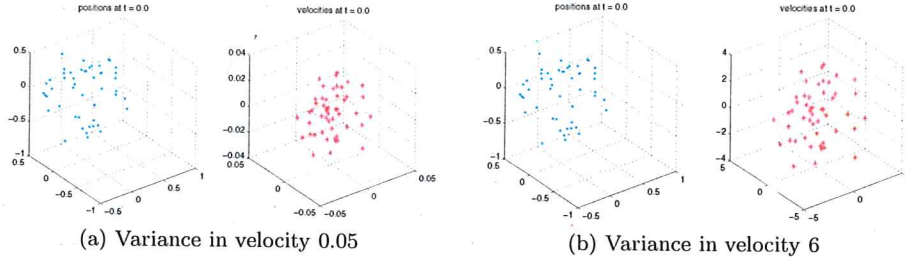
Figure 4: Initial positions and velocities of birds in $\Delta^\perp$. The positions of birds are centered around 0 with a variance of 1. The velocities of birds are chosen to be different in both figures. The velocities are again centered around 0 with the variance of 0.05 in the left figure and 6.0 in the right figure.

We first discuss the case $\beta < 1/2$, namely $\beta = 0.1$. After the simulation of (2.4) the velocities in $\Delta^\perp$ of all birds with the initial variance of 0.05 as well as 6.0 converged to 0, see Figures (8a),(8b). Figures (8c) and (8d) show the evolution of Λ in time. For both initial conditions we observed exponential convergence of $\Lambda(t)$. It is easy to derive the convergence of $\|v(t)\|_2$, $\|x(t+1) - x(t)\|_2$ and $\|x(t+1) - x(t)\|_Q$, namely

$\cdot\ \|v(t)\|_2 = \frac{1}{\sqrt{k}} \|v(t)\|_Q = \sqrt{\frac{\Lambda(t)}{k}}$,

$\cdot\ \|x(t+1) - x(t)\|_2 = \left\| \int_t^{t+1} v(s)ds \right\|_2 \leqslant \int_t^{t+1} \|v(s)ds\|_2 = \int_t^{t+1} \sqrt{\frac{\Lambda(s)}{k}}ds$, and

$\cdot\ \|x(t+1) - x(t)\|_Q = \left\| \int_t^{t+1} v(s)ds \right\|_Q \leqslant \int_t^{t+1} \sqrt{\Lambda(s)}ds$.

Figures (5a) and (5b) show exponential convergence of velocity and change of positions in Q-norm and in the Euclidean norm. In Figures (8e) and (8f) the evolution of the Fiedler number is presented. If the Fiedler number $\phi_t > 0$ is bounded away from zero, then the convergence of $\Lambda(t)$ is certain because (5.2) holds.

The summary for the cases $\beta = \frac{1}{2}$ and $\beta > \frac{1}{2}$, namely $\beta = 1.2$, is given in Figures (9) and (10). For the variance in initial velocities of 0.05 similar results as for the case $\beta < \frac{1}{2}$ are obtained, in fact $\Lambda(t)$ converges to 0 and the positions of birds converge to a fixed value. However, in the case where the variance of intial velocities is 6.0, convergence of $\Lambda(t)$ is not observed. The birds flew apart as can be seen in Figures (9b) and (10b). The lower bound for ϕ_t is not

(a) Evolution of $\|v(t)\|_2$ and $\|x(t+1) - x(t)\|_2$, $\|x(t+1) - x(t)\|_Q$ for variance = 0.05 in intitial velocity

(b) Evolution of $\|v(t)\|_2$ and $\|x(t+1) - x(t)\|_2$, $\|x(t+1) - x(t)\|_Q$ for variance = 6.0 in intitial velocity
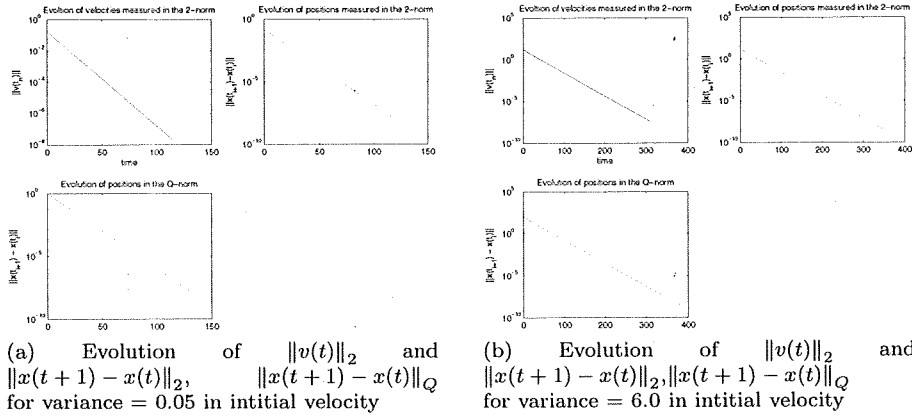
Figure 5

identifiable in Figures (9e) and (10e). In all discussed cases for $\beta = \frac{1}{2}$ and $\beta > \frac{1}{2}$ the conditions of Theorem 5.1 are not fulfilled.

Furthermore, the model (2.4) with $\beta = 0.7$ was tested for convergence, even though the conditions in Theorem 5.1 are not fulfilled. In Figure (6) we varied the variance in initial conditions for velocities. In the upper figure the last Fiedler number $\phi_{T_{end}}$ was plotted. Moreover, the Fiedler numbers $\phi_{T_{end}}$ for which the constrains of Theorem 5.1 are violated are highlighted by pink circles. The center figure depicts the slope of the last Fiedler numbers $\frac{d \log \phi}{dt}$. If $\frac{d \log \phi}{dt} \geqslant 0$ the convergence of $\Lambda(t)$ is assured, otherwise it is not clear whether the Fiedler number is bounded or not. In the bottom figure, $\Lambda(T_{end})$ is presented and the $\Lambda(T_{end})$ for which convegence is observed are highlighted by blue circles. An identical procedure was carried out for the variation of initial distances between birds whereas the initial variation of velocity was set to a fixed value 0.006, see Figure (7). We could observe the conditions in Theorem 5.1 to be too pessimistic and model (2.4) to converge often even if the conditions of the theorem were not fulfilled. Another observation is that the initial velocity determines the behaviour of a swarm more critically than the initial distances between birds. The model converges mostly for small variations in velocity and the variations in distance do not have a pronounced effect on convergence. Even for large differences in the initial positions, as in Figure (7), convergence can be observed.
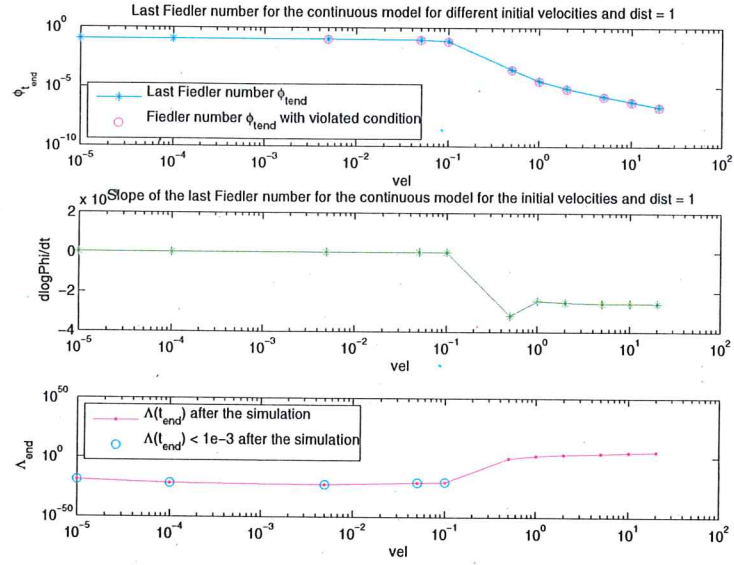
Figure 6: Summary of results for the continuous model for the variation initial variance in velocities denoted by vel.
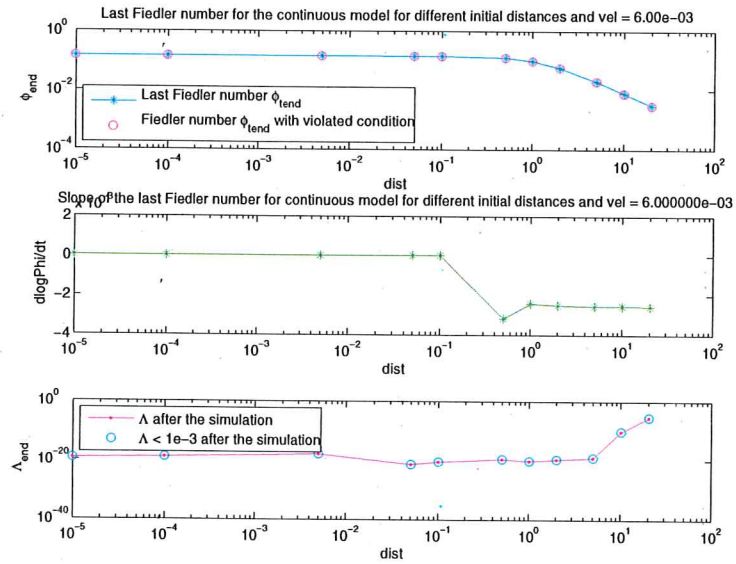
Figure 7: Summary of results for the continuous model for the variation initial variance in positions denoted by dist.
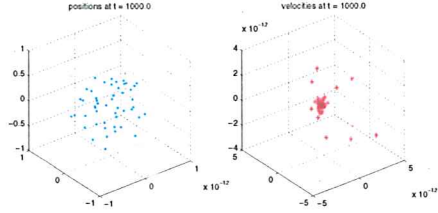
(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-12}.
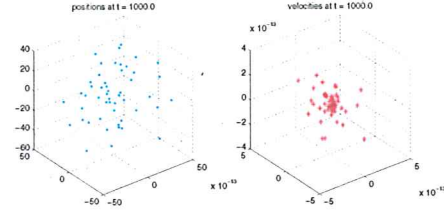
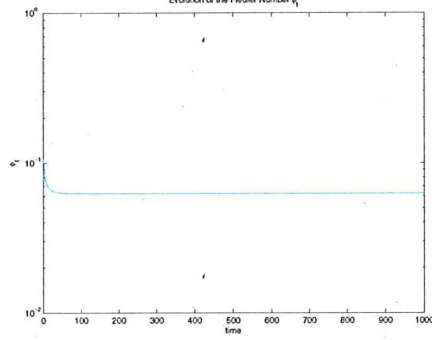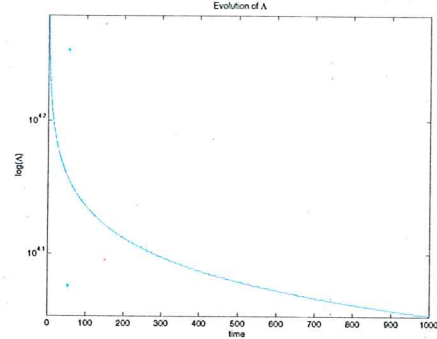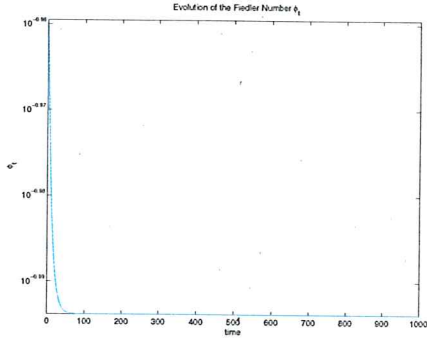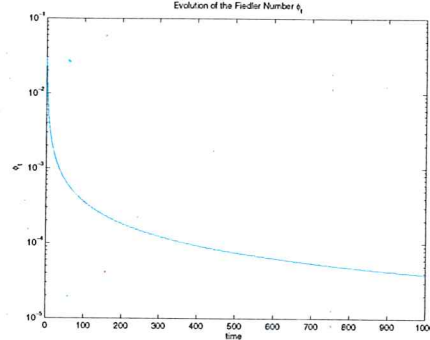(b) End positions and velocities for variance = 6.0 in initial velocities. Velocities converged to 10^{-13}.
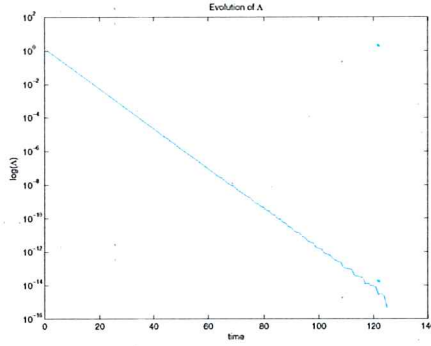
(c) Evolution of $\Lambda(t)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t)$ is observed.
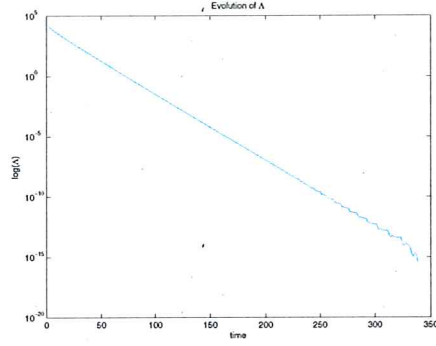
(d) Evolution of $\Lambda(t)$ for variance = 6.0 in initial velocities. Exponential convergence of $\Lambda(t)$ is observed.
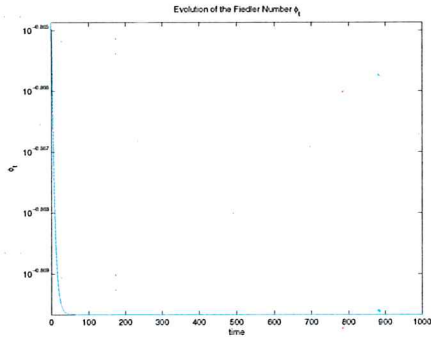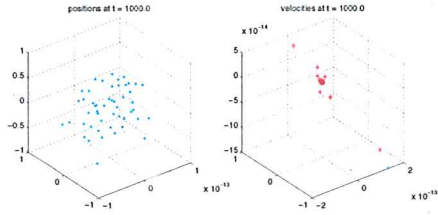
(e) Evolution of Fiedler number ϕ_t for variance = 0.05 in initial velocities. ϕ_t is bounded.

(f) Evolution of Fiedler number ϕ_t for variance = 6.0 in initial velocities. ϕ_t is bounded.

Figure 8: Summary of results for $\beta = 0.1$ for the continuous model. The left figures present the results for the initial variance in velocity of 0.05. These simulation results are compared to results when the initial variance in velocity is 6.0. These are presented on the right. Both results are similar, convergence is observed for both initial conditions.
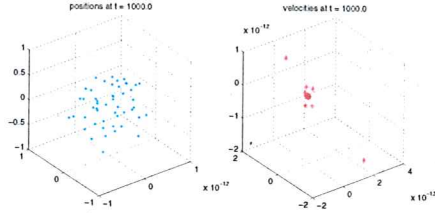
12

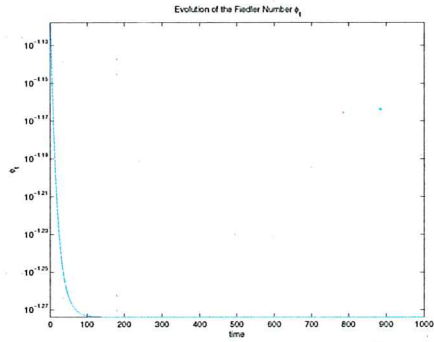(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-12}.
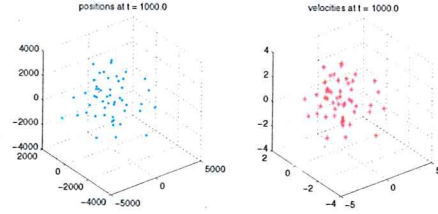
(b) End positions and velocities for variance = 6.0 in initial velocities. Velocities did not converge and the birds flew apart.
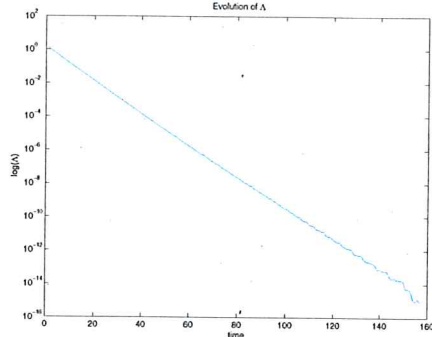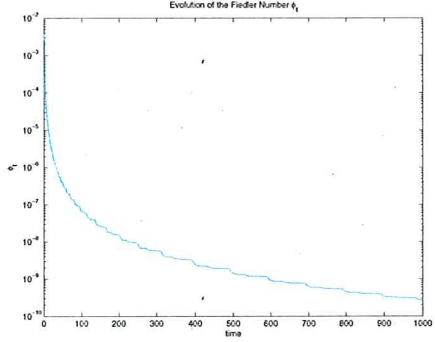
(c) Evolution of $\Lambda(t)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t)$ is observed.

(d) Evolution of $\Lambda(t)$ for variance = 6.0 in initial velocities. Exponential convergence of $\Lambda(t)$ is not observed. Moreover, $\Lambda(t)$ did not turn to 0.

(e) Evolution of Fiedler number ϕ_t for variance = 0.05 in initial velocities. ϕ_t is bounded.

(f) Evolution of Fiedler number ϕ_t for variance = 6.0 in initial velocities. The positive lower bound for ϕ_t is not evident.

Figure 9: Summary of results for $\beta = 0.5$ for the continuous model. Like in Figure (8) the simulations results for initial velocity variance of 0.05 and 6.0 are compared. Convergence for the latter is not observed.

13

(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-14}.
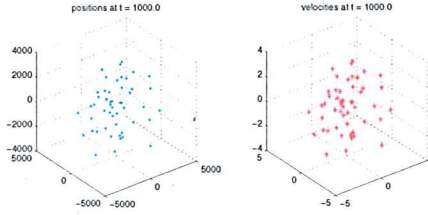
(b) End positions and velocities for variance = 6.0 in initial velocities. The positions of birds diverged and the velocities did not turn to 0.
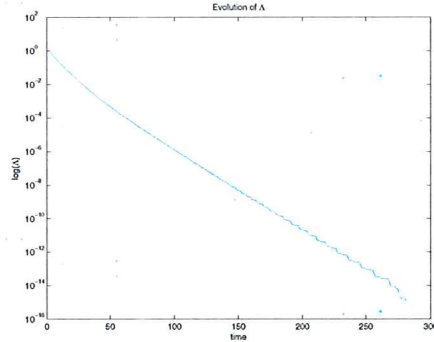
(c) Evolution of $\Lambda(t)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t)$ is observed.
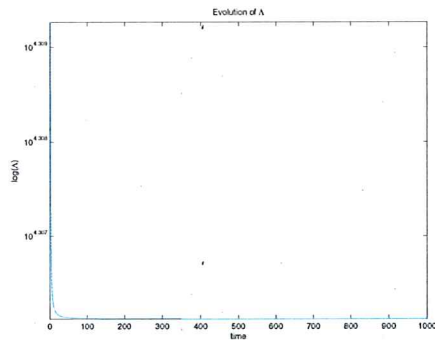
(d) Evolution of $\Lambda(t)$ for variance = 6.0 in initial velocities. $\Lambda(t)$ did not turn to 0.

(e) Evolution of Fiedler number ϕ_t for variance = 0.05 in initial velocities. ϕ_t is bounded.

(f) Evolution of Fiedler number ϕ_t for variance = 6.0 in initial velocities. The positive lower bound for ϕ_t is not evident.

Figure 10: Summary of results for $\beta = 1.2$ for the continuous model. The simulation results are similar to these presented in Figure (9).

14

6 Convergence in Discrete Time

The theorem and the proof for evolution in discrete time was provided by [1] as Theorem 3.

Theorem 6.1. *Let (x_n, v_n) be a solution of (2.4) for $\triangle t = 1$. Assume that*

$$a_{ij} = \eta(\|x_i - x_j\|^2)$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotonically decreasing function

$$\eta(y) = \frac{K}{(\sigma^2 + y)^\beta}$$

for some constants $K < \frac{\sigma^{2\beta}}{(k-1)\sqrt{k}}, \sigma > 0, \beta \geq 0$. Assume that one of the following initial conditions holds.

i) $\beta < \frac{1}{2}$

ii) $\beta = \frac{1}{2}$ and $\|v_0\|_2 \leqslant \frac{\sqrt{k}K}{2}$

iii) $\beta > \frac{1}{2}$ and

$$\left(\frac{1}{a}\right)^{\frac{-2}{\alpha-1}} \left[\left(\frac{1}{\alpha}\right)^{\frac{-2}{\alpha-1}} - \left(\frac{1}{\alpha}\right)^{\frac{1+\alpha}{\alpha-1}}\right] > k\left(V_0^2 + 2V_0((\alpha a)^{\frac{-2}{2}\alpha-1} - \sigma^2)\frac{1}{\sqrt{k}}\right) + b.$$

Here, $\alpha = 2\beta, V_0 = \|v(0)\|_2, a = \frac{2}{\sqrt{k}K}V_0$ and $b = \sqrt{k}\|x(0)\|_2 + \sigma$.

Then

- *$\|v(t_m)\| \to 0$ when $m \to \infty$*

- *there exists $\hat{x} \in \Delta^\perp$ such that $x(t_m) \to \hat{x}$ when $m \to \infty$*

The claim of this theorem is the same as in Theorem 5.1, but the conditions for convergence are different. This theorem is true for $\triangle t = 1$, i.e. for the system

$$x^{m+1} = x^m + \tilde{v}^m$$
$$\tilde{v}^{m+1} = \tilde{v}^m - \tilde{L}_{x^m}\tilde{v}^m,$$

which corresponds to the discrete model in [1] with $\triangle t = 1$.

In order to apply this theorem to the model in [1] for arbitrary $\triangle t$, this system should be transformed in the following way: $\tilde{v}^m = v^m \triangle t$. This results in the following system

$$x^{m+1} = x^m + \triangle t v^m$$
$$v^{m+1} = v^m - \hat{L}_{x^m}v^m.$$

In order to apply this theorem to system (3.1) the transformation $\hat{K} = \triangle tK$, in addition to the previous transformation, should be done.

The simulation results for discrete model (3.1) for $\triangle t = 1$ are presented in Figures (13), (14) and (15). The results are very similar to the ones for the

15

continuous model. A very small difference is visible for the case $\beta = \frac{1}{2}$, see Figure (14). Still no convegence is observed, but the divergence of the birds' positions is twice as slow.

We also tested when the conditions in Theorem 6.1 are fulfilled for different initial conditions on the maximal distance and maximal velocities difference between birds. The results are presented in Figures 11 and 12.
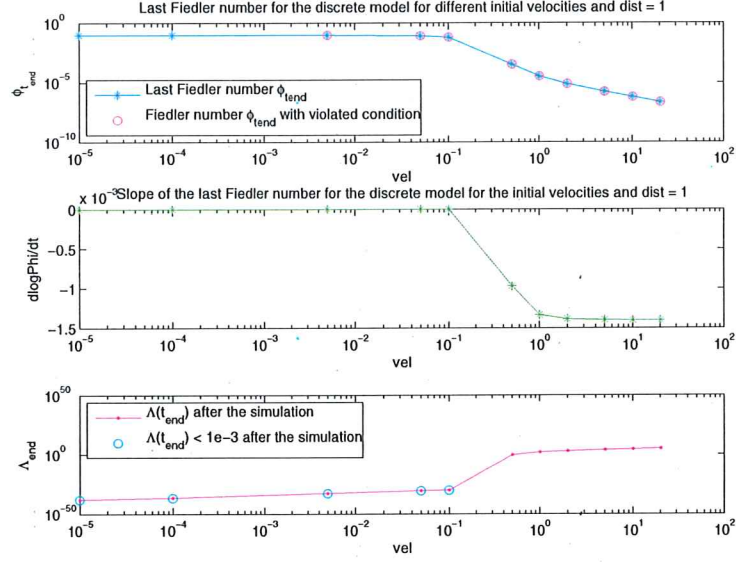

Figure 11: Summary of results for the discrete model for the variation initial variance in velocities denoted by vel.
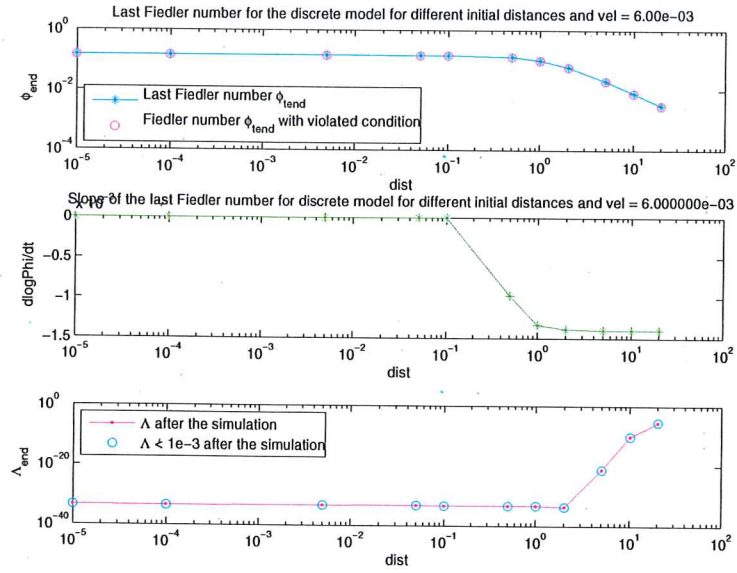
Figure 12: Summary of results for the discrete model for the variation initial variance in positions denoted by dist.
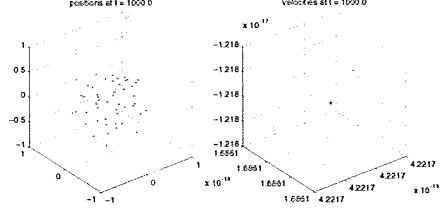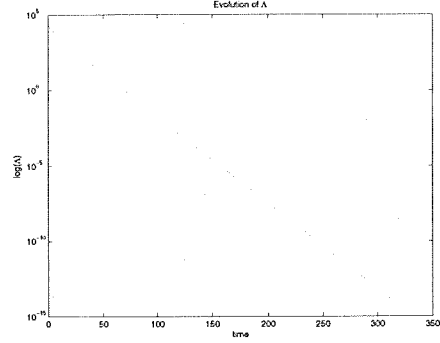
(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-17}.
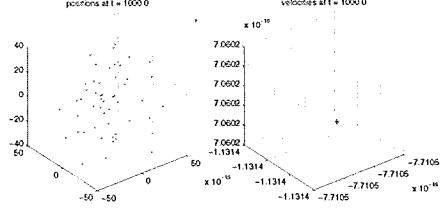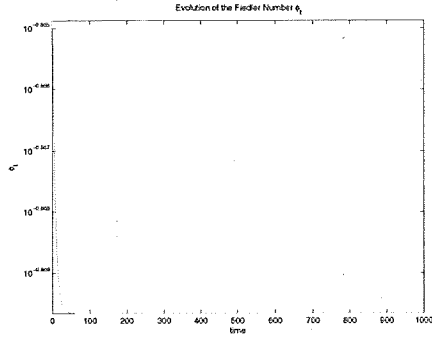
(b) End positions and velocities for variance = 6.0 in initial velocities. Velocities converged to 10^{-16}.
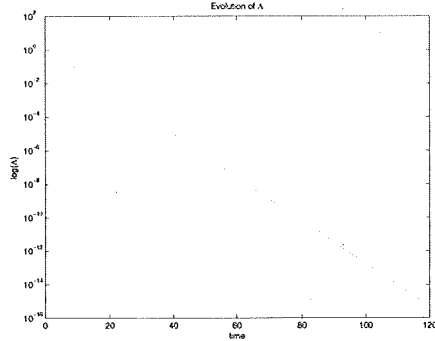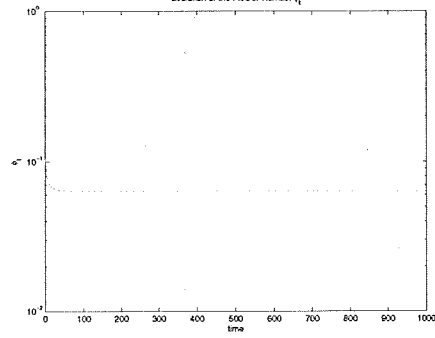
(c) Evolution of $\Lambda(t_m)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is observed.

(d) Evolution of $\Lambda(t_m)$ for variance = 6.0 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is observed.
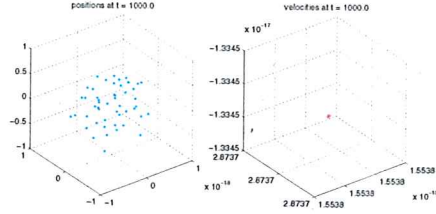
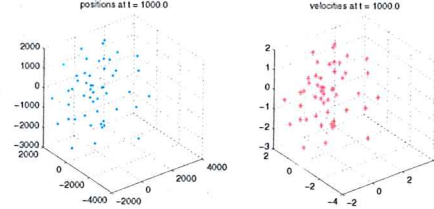(e) Evolution of Fiedler Number ϕ_{t_m} for variance = 0.05 in initial velocities. ϕ_{t_m} is bounded.

(f) Evolution of Fiedler Number ϕ_{t_m} for variance = 6.0 in initial velocities. ϕ_{t_m} is bounded.
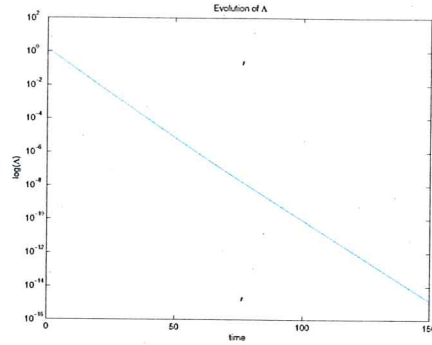
Figure 13: Summary of results for $\beta = 0.1$ for the discrete model. The results are similar to the ones in Figure (8).
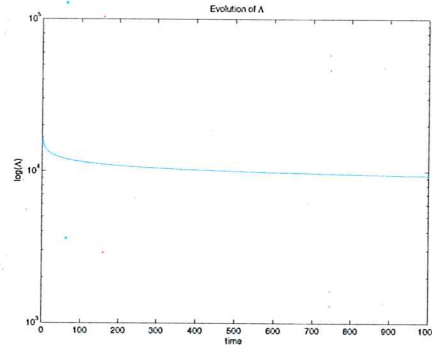
18

(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-17}.
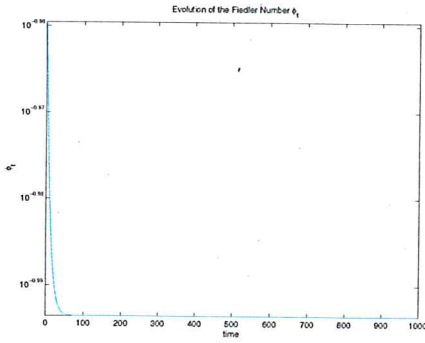
(b) End positions and velocities for variance = 6.0 in initial velocities. Velocities did not converge and the birds flew apart.
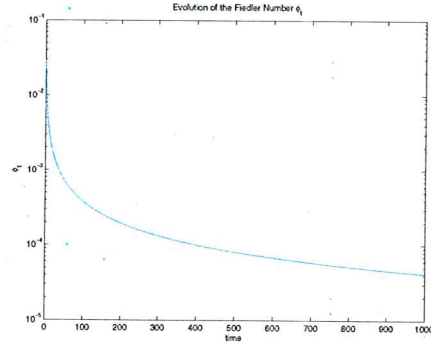
(c) Evolution of $\Lambda(t_m)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is observed.

(d) Evolution of $\Lambda(t_m)$ for variance = 6.0 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is not observed. Moreover, $\Lambda(t_m)$ did not turn to 0.

(e) Evolution of Fiedler Number ϕ_{t_m} for variance = 0.05 in initial velocities. ϕ_{t_m} is bounded.

(f) Evolution of Fiedler Number ϕ_{t_m} for variance = 6.0 in initial velocities. The positive lower bound for ϕ_{t_m} is not evident.

Figure 14: Summary of results for $\beta = 0.5$ for the discrete model. The results are similar to Figure (9).

(a) End positions and velocities for variance = 0.05 in initial velocities. Velocities converged to 10^{-17}.

(b) End positions and velocities for variance = 6.0 in initial velocities. Velocities did not converge and the birds flew apart.

(c) Evolution of $\Lambda(t_m)$ for variance = 0.05 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is observed.

(d) Evolution of $\Lambda(t_m)$ for variance = 6.0 in initial velocities. Exponential convergence of $\Lambda(t_m)$ is not observed. Moreover, $\Lambda(t_m)$ did not turn to 0.
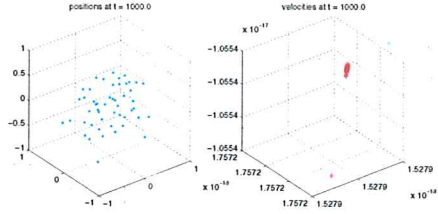

(e) Evolution of Fiedler Number ϕ_{t_m} for variance = 0.05 in initial velocities. ϕ_{t_m} is bounded.
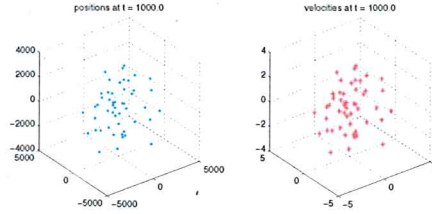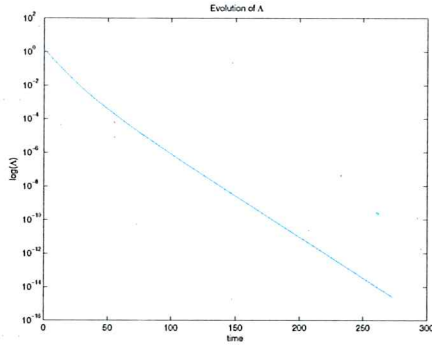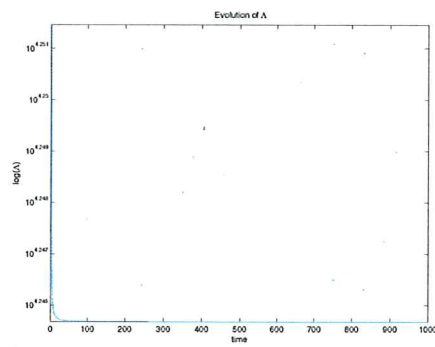

(f) Evolution of Fiedler Number ϕ_{t_m} for variance = 6.0 in initial velocities. The positive lower bound for ϕ_{t_m} is not evident.

Figure 15: Summary of results for $\beta = 1.2$ for the discrete model. The results are similar to Figure (10).

7 Language Evolution

Another intriguing topic which can be simulated with a similar model is the evolution of languages. We again consider the model proposed by F. Cucker et al. [1]. The linguistic population consists of k agents in $\mathbb{R}^d$.

Let $Z = \{z_1, z_2, \ldots, z_r\} \subset \mathbb{R}^r$ be a set of objects, $\rho_z : Z \to [0, \infty]$ an importance function and $Y = \mathbb{R}^n$ the space of sounds, where $y = (y_1, y_2, \ldots, y_n) \in Y$ are the frequences of a sound. A composition of sounds y is the linguistic representation of an object. The importance function ρ_z assigns to each object the corresponding weight. Then $f : Z \to Y$ is the language function, which assigns to each object its linguistic representation. So we consider the language as an element of the space $H = \{\overrightarrow{f} \in (\mathbb{R}^n)^r\}$.

For the evolution of languages we consider k agents speaking k different languages evolving in time. At time t the state of a population is given by $(x(t), F(t)) \in (\mathbb{R}^d)^k \times (H)^k$. In contrast to the previos models described in Sections 2 and 3, the functions $x(t), F(t)$ do not belong to the same space.

The language model is given by

$$
\begin{aligned}
\dot{x} &= -L_F x \\
\dot{F} &= -L_x F.
\end{aligned}
\tag{7.1}
$$

$L_x \in \mathbb{R}^{nrk \times nrk}$ is again a Laplacian of a matrix A_x given by $a_{ij} = \eta_x(\|x_i - x_j\|^2)$ for a distance function $\eta_x : \mathbb{R}^+ \to \mathbb{R}^+$ defined as in (2.2). Similarly $L_F \in \mathbb{R}^{dk \times dk}$ is a Laplacian of the language adjacency matrix A_F given by $a_{ij} = \eta_H(\|F_i - F_j\|_H^2)$ where $\eta_H : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonically decreasing function, again defined as in (2.2). The distance between two language functions $f, g \in H$ is defined by [2] as

$$
\|f - g\|_H := \sqrt{\sum_{i=1}^r \|f(z_i) - g(z_i)\|_2^2 \, \rho_z(z_i)}
$$

which corresponds to a weighted Eucledian norm.

The interpretation of the model is the following. Agents tend to move towards other agents with similar languages in order to be able to communicate. This is described by the first equation of 7.1. The second equation describes language evolution itself because of the influence from other agents' languages. Therefore, the linguistic distance diminishes.

Like in the simulation of flock formation of birds, we are only interested in relative differences in agents' positions and languages, therefore we transform the model into a center of momentum frame. $\Lambda(x), \Lambda(F)$ can be similarly defined as

$$
\Lambda(x) = Q(x, x) := \frac{1}{2} \sum_{i,j}^k \|x_i - x_j\|_2^2,
$$

where $x_i \in \mathbb{R}^d$ and

$$
\Lambda(F) = Q(F, F) := \frac{1}{2} \sum_{i,j}^k \|F_i - F_j\|_H^2,
$$

21

with $F_i \in \mathbb{R}^{nr}$. The Fiedler numbers ϕ_x, ϕ_F are the second smallest eigenvalues of L_x and L_F.

Theorem 7.1. *Let $\eta_x : \mathbb{R}^+ \to \mathbb{R}^+$ and $\eta_H : \mathbb{R}^+ \to \mathbb{R}^+$ be monotonically decreasing functions, then the state (x, F) tends to a fixed point in $(\mathbb{R}^d \times H)^k$ for $t \to \infty$, such that $\Lambda(x(t)) \to 0$ and $\Lambda(F(t)) \to 0$.*

Theorem 7.1 was inroduced and proved as Theorem 4 by [1]. As was shown in Section 5, the convergence of $\Lambda(x(t))$ and $\Lambda(F(t))$ turns out to be exponential if the Fiedler numbers ϕ_x, ϕ_F are bounded away from zero.

We simulated the system (7.1) for different parameters. We chose $r = 10, \rho_{z1} = \rho_{z2} = \ldots = \rho_{z10} = 1$ and $d = 2$ since we consider agents living in a two dimensional space. For simplicity we chose the number of possible sounds n to be 1, 2 or 4 and varying number of agents k, namely $k = 5, 10, 20$. The agents were placed randomly with a maximal distance of 50 from each other. The language of each agent were also chosen randomly with a maximal differ-ence between them set to 30, the simulation time was set to 75 and the model parameters to $K = 1, \beta = 1, \sigma = 1$. Therefore, both neighborhood functions are identical. The simulation results for $n = 1, k = 10$ are presented in Figure (16). In the upper left figure the convergence of 10 agents to a fixed point is observed. The upper right figure demonstrates the convergence of the word which represents one of the objects. The evolution of the two Fiedler numbers ϕ_x and ϕ_F is presented. The bottom figures show the evolution of $\Lambda(x(t))$ and $\Lambda(F(t))$. After a certain time values of $\Lambda(x(t))$ and $\Lambda(x(t))$ approach zero. The time for $\Lambda(x(t))$ and $\Lambda(F(t)$ to reach 10^{-16} is the same as the time for both Fiedler numbers to converge to a fixed value.

The figures for other parameter combinations of n and k look very similar to Figure (16). The convergence of agents and their languages could always be observed. The only difference was the time needed to converge to a fixed point, see Figure 17. An increasing number of agents leads to a faster convergence. There is no clear conclusion in the convergence for increasing values for n.

8 Conclusion

In the present work we studied the models for the flock formation and language evolution proposed by [1] and [2]. We simulated continuous and discrete models for flock formation and tested the convergence propositions of Theorems 5.1 and 6.1 for different values of β. Simulations verified the theorems in case of the fulfillment of these conditions. We discovered that the conditions for the convergence of a flock are too pessimistic. Different cases were presented for the convergence of the flock in case where conditions were not fulfilled. Additionaly we could observe the convergence of a flock to depend on the differences in initial conditions more critically than it does on the initial distances between the birds.

We performed simulations for language evolution in primitive societies for different number of agents, objects and sounds used to represent these. The convergence of different agents and languages was observed throughout.

The theorems proposed by [1] were successfully validated with computer sim-ulations. As flock formation is an interesting research area which can be applied
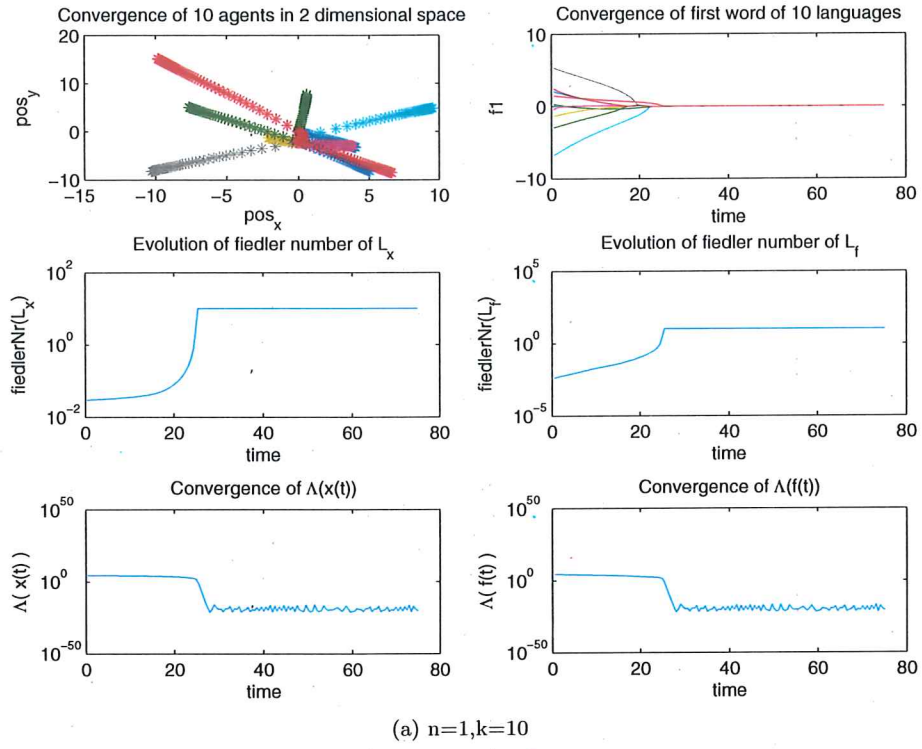
Convergence of 10 agents in 2 dimensional space

Convergence of first word of 10 languages

Evolution of fiedler number of L_x

Evolution of fiedler number of L_f

Convergence of $\Lambda(x(t))$

Convergence of $\Lambda(f(t))$

(a) n=1,k=10

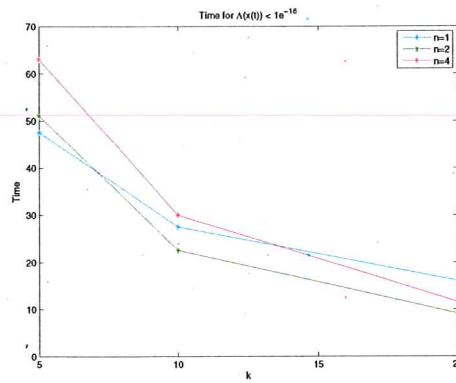Figure 16: Summary of language evolution for different for n=1,k=10

Figure 17: Convergence time for $\Lambda(x(t))$. For different parameters k and n the time until $\Lambda(x(t)) < 10^{-16}$ is plotted.

23

to different dynamic systems with emergent behaviour, further improvements of the sharpness of the conditions are essential.

References

[1] F.Cucker and S. Smale. Emergent behavior in flocks. *IEEE Transactions on automatic control*, 52, No 5, 2007.

[2] F.Cucker, S. Smale, and D. Zhou. Modeling language evolution. *Foundations of Computational Mathematics*, 4, 2004.