Power System Reduction Techniques for Dynamic Analysis

Master Thesis

Peiyan Li

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Advisors: Dr. Petros Aristidou, Dr. Luigi Busarello, Stavros Karagiannopoulos

Dept. of Information Technology and Electrical Engineering, ETH Zürich
Abstract

The thesis presents some methods for solving the problem of power system reduction for proposals of stability analyses, especially the stability analyses involving interarea mode oscillation. To this end, three aspects are considered, the coherent area identification, network aggregation and synchronous generator dynamic aggregations.

For the first aspect, it involves two tasks—grouping synchronous generators and dividing power system networks. The power system can be seen as a network of synchronous oscillators. The stability, including transient stability and small signal stability, is analog to the synchrony in coupled oscillators. The eigenanalysis on its linearization reveals the coherence structure, which is mainly decided by the network graph and is complicated by power system controllers and various properties of generators, like the damping, working conditions, etc. By making use of linearized model, the well developed loose/tight coherent area identification method is introduced to group coherent generators. The second task is to extend the grouping of the synchronous generators, their buses, to load buses. For this purpose, some new methods are developed, based on 1) how much a load bus is influenced by synchronous generators or 2) the concept of edge betweenness borrowed from the study of complex network and graph theory.

For the second, the thesis has some discussions between the Kron reduction (Ward reduction) widely applied in circuit analysis and Schur complement in algebra. Finally a method of the ZIP load aggregation is suggested, which may provide ways to aggregate power electronics interfaced power source.

For the last, three methods are introduced to aggregate synchronous generators based on three different backgrounds. The first is derived with perturbation theory and center manifold theorem, which is closed related with the two time-scale properties of the dynamic structure. A relatively more accurate equation is concluded. The second is the widely applied Podmore’s method. The third one is based on nonlinear system identification solved by global optimization techniques.
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Chapter 1

Introduction

1.1 Backgrounds

With the increasing consumption of electric power and demand on clean energy in the modern society, the power transmission network is revamped with more and more application of cutting-edge technologies, which bring with a great challenge to the power networks. The tendency to integrate large scale of power networks, for commercial and stability reasons, also post some difficulties. One of these challenges is on the power system modeling, which the power system analysis program relies on for dynamic simulation and stability analysis. The [35] gives some details about these challenges, they are summarized as following,

1) the memory limitation for storing and maintaining big network data.

2) prolonged simulation time, particularly for dynamic analysis and stability analysis.

3) the reluctance to exchange modeling details between different Transmission System Operators (STOs) due to commercial and security reasons.

4) only parts of network system with certain distance away from the disturbance can have noticeable effects on some occasions, like the local oscillation modes,

5) not all system studies need modeling details. Depending on the purposes, a “rough” system model are enough in many cases.

Based on the 4) aforementioned, the power system networks can be divided into two parts, internal part and external part. The internal part is much more close to the disturbance physically and electrically, thus needs more details on modeling, whereas the external part can be represented by simple model, often called as equivalent system.
1.2 Methods categories

To engage with this new challenges, there are basically two general principles. The first one is based on the measurements of electrical quantities inside the internal subsystem and across borders, which doesn’t require any knowledge of the external subsystem. By contrast, the second method requires model information of external subsystem. The later method, usually referred to as model reduction method, is the focus of the remaining discussion.

The main work of the model reduction is to simplify the external networks either by eliminating unnecessary nodes, which are usually far away from the disturbance “spots” electrically or are much less important in the network behavior, or by aggregating several nodes to one equivalent. The method of model reduction can be again subdivided into three categories [35],

- Physical reduction.
- Topology reduction.
- Modal reduction.

For the last method modal reduction, it typically involves eliminating specific modals on the linearized models, having little to do with the disturbance. However, such methods usually result in simplification in “mathematical sense”, i.e. it does not retain the physical structures like generating units, lines, loads etc. Thus, its simplified models are not allowed by typical power system simulation or analysis software.

The topology methods combined with the physical simplification methods are however able to retain the standard system element structure suitable for simulation.

1.2.1 Traditional Topology Reduction Method

The main purpose of topology reduction is to reduce the order of the network model. One way to achieve this is by eliminating unnecessary nodes and by aggregating several nodes into an equivalent one. The elimination technique usually reduces the sparsity of the network model matrices, so special attention should be drawn when choosing which nodes shall be eliminated to preserve the sparsity as much as possible. Aggregation method helps to group some electrically much relate units together to compose an equivalent having similar significance in power system study. However, it may sacrifice some degree of accuracy.

Eliminating nodes

Figure 1.1 depicts the ideal of elimination method. Two sets of nodes \( \{R\} \) and \( \{E\} \) are labeled in the “original” network. Here the sets of unwanted nodes,
1.2. Methods categories

\{R\}
\\vdots
\{E\}

\[ \Rightarrow \]
\{R\}
\\vdots
\{E\}

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (original) at (0,0) {
\begin{tabular}{c|c}
\hline
\textbf{original network} & \textbf{reduced network} \\
\hline
\end{tabular}
};
\node (original1) at (0,1) {\ldots};
\node (original2) at (0,2) {\ldots};
\node (reduced) at (2,0) {
\begin{tabular}{c|c}
\hline
\textbf{original network} & \textbf{reduced network} \\
\hline
\end{tabular}
};
\node (reduced1) at (2,1) {\ldots};
\node (reduced2) at (2,2) {\ldots};
\end{tikzpicture}
\caption{Elimination of nodes}
\end{figure}

\{E\}, are removed while keeping the retained notes, \{R\}. For this purpose, the nodal equations before (1.1) and after (1.2) the elimination result need to be verified,

\[ \begin{bmatrix}
i_R \\
i_E \end{bmatrix} = \begin{bmatrix} Y_{RR} & Y_{RE} \\ Y_{ER} & Y_{EE} \end{bmatrix} \begin{bmatrix} v_R \\
v_E \end{bmatrix} \]

(1.1)

where the subscripts \(E\) and \(R\) refer to the parts to be eliminated and kept. With some linear algebra manipulations, it can be represented as,

\[ \begin{bmatrix}
i_R \\
v_E \end{bmatrix} = \begin{bmatrix} Y_{R} & K_i \\ K_V & Y_{EE}^{-1} \end{bmatrix} \begin{bmatrix} v_R \\
i_E \end{bmatrix} \]

(1.2)

where the sub-matrix, \(Y_{R} = Y_{RR} - Y_{RE} Y_{EE}^{-1} Y_{ER}\) and \(K = Y Y_{EE}^{-1}\) is of particular interest. If neglecting the second equation in (1.2),

\[ i_R = Y_{R}v_R + \Delta i_R \]

(1.3)

where,

\[ \Delta i_R = K_i i_E \]

(1.4)

(1.3) and (1.4) are the description of the eliminated networks. In other words the eliminated parts of the is summarized in the current injection vector \(\Delta i_R\).

Traditionally, the elimination is processed by eliminating one node at one time. By adjusting the dimension of poetical sub-matrix,

\[ Y_R = Y_{RR} - y_{Re} y_{ee}^{-1} y_{eR}^T \]

(1.5)

\[ = \frac{1}{y_{ee}} \begin{bmatrix} y_{1e} \\
y_{2e} \\
\vdots \\
y_{ne} \end{bmatrix} \begin{bmatrix} y_{e1} & y_{e2} & \cdots & y_{en} \end{bmatrix} \]

(1.6)

Thus, each entity \(y_{ij}^{\text{new}}\) in the new matrix \(Y_R\) can be simply written as,

\[ y_{ij}^{\text{new}} = y_{ij}^{\text{old}} - \frac{y_{ie} y_{ej}}{y_{ee}} \]
1. Introduction

**Dimo’s method**

The aggregation method can replace the set of nodes, \( \{A\} \), by a signal equivalent node \( a \), as illustrated in Figure 1.2. Usually, the power injection from the equivalent node \( a \) is the same as in aggregated nodes.

In the first step of this method, a fictitious node is inserted between the aggregated node set, \( \{A\} \) and the equivalent node. If total power injection from the aggregated nodes is assumed equal to the injection from the equivalent node, \( a \), the admittance linking aggregated nodes and the fictitious node can be calculated as following,

\[
y_{if} = \frac{S_i}{v_i^2} \tag{1.7}
\]

Certainly the voltage at the fictitious node \( f \) is zero. A zero voltage node might undermine the performance of power system analysis software. To avoid this, another admittance is added between node \( f \) and \( a \). Typically the admittance \( y_{fa} \) is,

\[
y_{fa} = -\frac{S_a}{v_a^2} \tag{1.8}
\]

From these set-ups of the method, the voltage at node \( a \),

\[
v_a = \frac{S_a}{i} = \frac{\sum_{k \in \{A\}} S_k}{\sum_{k \in \{A\}} \frac{S_k}{v_k}} \tag{1.9}
\]

Then, the admittance between the aggregated node and the fictitious node can be eliminated together with the \( y_{fa} \). The resulting network is referred to as radial equivalent independent (REI) circuit [35]. Due to the modifications to the original network, the eliminating procedure adds new mutual connection to the retained nodes, i.e. the sub-matrix, \( Y_{RR} \), in (1.1). On the other hand,
1.2. Methods categories

the admittance \( y_{if} \) and \( y_{af} \) in (1.7) and (1.8) respectively are derived from the power injection, thus having large part of real value, i.e. resistive value. Plus, the negative admittance that may exist in the resulting furthermore undermines the performance of the analysis software.

**Zhukov’s method**

To remedy several drawbacks from the Dimo’s method, the Zhukov’s method is prevalently used, especially for aggregating coherent generating units. This method is illustrated in Figure 1.3, where the node set \( \{A\} \) is represented by a signal equivalent node \( a \) while 1) keeping the node in \( \{R\} \) retains the current and voltage values, and 2) the power injection from node \( a \) remains the same as from the aggregated nodes.

Firstly, inspect the nodal equations before and after the aggregation,

\[
\begin{bmatrix}
i_R \\ i_A
\end{bmatrix} =
\begin{bmatrix}
Y_{RR} & Y_{RA} \\ Y_{AR} & Y_{AA}
\end{bmatrix}
\begin{bmatrix}
v_R \\ v_A
\end{bmatrix}
\]

\( (1.10a) \)

\[
\begin{bmatrix}
i_R \\ i_a
\end{bmatrix} =
\begin{bmatrix}
Y_{RR} & Y_{RA} \\ y_{aR}^T & y_{aa}
\end{bmatrix}
\begin{bmatrix}
v_R \\ v_a
\end{bmatrix}
\]

\( (1.10b) \)

To satisfy the two criteria mentioned above, there must have two equations as necessary conditions,

\[
Y_{RR}v_R + Y_{RA}v_a = Y_{RR}v_R + Y_{Ra}v_a
\]

\( (1.11a) \)

\[
v_A^T i_A = v_a^T i_a
\]

\( (1.11b) \)

From (1.11a),

\[
Y_{RA}v_a = Y_{RA}v_a
\]

where the vector \( \theta \) is of a set of ratios between \( v_k \) (\( k \in \{A\} \)) and \( v_a \). The second condition, (1.11b) implies,

\[
v_A^T Y_{AR} v_R + v_A^T Y_{AA} v_A = v_a^T Y_{aR} v_R + v_a Y_{aa} v_a
\]
1. **Introduction**

In summary, the resulting nodal equations are,

\[
\begin{bmatrix}
    i_R \\
    i_a
\end{bmatrix} = \begin{bmatrix}
    Y_{RR} & y_{RA} \theta \\
    \theta^T Y_{AR} & \theta^T Y_{AA} \theta
\end{bmatrix} \begin{bmatrix}
    v_R \\
    v_a
\end{bmatrix}
\]

(1.12)

Some attention should be drawn into the voltage ratio vector \( \theta \). Firstly, this is a complex vector with its angle depending on equivalent node voltage, \( v_a \). Its voltage angle \( \delta_a = \arg(v_a) \) is usually decided via weighing voltage angles of aggregated nodes by power injection (1.13) or inertia coefficients (1.14),

\[
\delta_a = \frac{\sum_{k\in\{A\}} S_k \delta_k}{\sum_{k\in\{A\}} S_k}
\]

(1.13)

\[
\delta_a = \frac{\sum_{k\in\{A\}} M_k \delta_k}{\sum_{k\in\{A\}} M_k}
\]

(1.14)

(weight) averaging angle by power injections (1.13) is primarily used in steady stability analysis, whereas averaging by inertia coefficients is mainly used for analyzing transient stability.

One advantage of Zhukov’s method over Dimo’s method is that the admittance topology in the original network has yet been modified, except the blocks \( Y_{RA}, y_{aR}^T \) and \( y_{aa} \) in (1.10a) are modified via \( \theta \). However, it does introduces self admittance to the retained nodes [35].

Since the \( \theta \) is complex in general, the resulting matrix (1.12) generally lacks of the symmetric property. One way to remedy it is to modify the (1.10a), which is equal to

\[
\begin{bmatrix}
    i_R \\
    i_a
\end{bmatrix} = \begin{bmatrix}
    Y_{RR} & y_{RA} \theta \\
    \theta^T Y_{AR} & \theta^T Y_{AA} \theta + (\theta^T Y_{AR} - \theta^T Y_{AR} v_a^T) v_a
\end{bmatrix} \begin{bmatrix}
    v_R \\
    v_a
\end{bmatrix}
\]

(1.15)

\[
= \begin{bmatrix}
    Y_{RR} & y_{RA} \\
    y_{Ra}^T & y_{aa} + (\theta^T Y_{AR} - \theta^T Y_{AR} v_a^T) v_a
\end{bmatrix} \begin{bmatrix}
    v_R \\
    v_a
\end{bmatrix}
\]

(1.16)

The (1.16) is a linear (it will be seen that even (1.12) is not linear) because the first matrix on the right side of equality also contains \( v_a \). However, the correction term, \( (\theta^T Y_{AR} - \theta^T Y_{AR} v_a^T) v_a \), is generally small compared with \( i_a \). Thus, its variation is usually neglected and this term is replace by a constant determined at initial condition (steady state condition).

1.3 **Coherency Recognition**

1.3.1 **Methodology**

In the process of generating unit aggregation for the external subsystem, one important part is to find out the most related units based on their coherency.
1.3. Coherency Recognition

Coherency recognition is such process to group the coherent generating unit without the need of dedicated dynamic simulation work. The following method is based on [35, p. 569] for the case when the voltage angle at boundary nodes changes.

Providing the external subsystems are simplified by equivalent network, the mutual influence between the generating units in the external network and the delicately represented internal subsystem can be studied through the transfer admittance, the power productions from the generators in external subsystem are (neglecting the mutual conductance $G_{kl}$),

$$P_k = E_k' G_{ii} + \sum_{j \in \{B\}} E_k' V_j \sin(\delta'_k - \delta_j) + \sum_{i \in \{G\}} E_k' E_i' \sin(\delta'_k - \delta_i').$$  \hspace{1cm} (1.17)

By linearizing the system at initial steady state point, the variance of power output of a generator due to a small change in the voltage angle in boundary nodes, $\Delta \delta_j$, is

$$\Delta P_k(\Delta \delta_j) \approx E_k' V_j \cos([\delta'_k] - [\delta_j]) \Delta \delta_j.$$  \hspace{1cm} (1.18)

where $E_k' V_j \cos([\delta'_k] - [\delta_j])$ is called synchronizing power. The accelerations of the generator’s rotor caused by such change in voltage angle is,

$$\varepsilon = \frac{\Delta P_k(\Delta \delta_j)}{M_k} = \frac{E_k' V_j \cos([\delta'_k] - [\delta_j])}{M_k} \Delta \delta_k.$$  \hspace{1cm} (1.19)

In ideal case, if two generators in the external subsystem having the same acceleration in response to any single change in the voltage angle of boundary nodes, they are exact coherent. Hence, these two generators can be seen sitting in a single shaft. In practice, the exact coherent is not practical, except in the trivial case where two generators are connected in parallel on the same busbar. Thus, based on experience of simulation studies, an admissible error, $\rho_h$, is allowed in grouping approximately coherent group,

$$\max_{k \in \{G\}} \frac{H_{kl}^l - \max_{j \in \{G\}} H_{kj}^j}{d_{\{G\}}} \leq \rho_h \quad \forall l, j \in \{G\}, k \in \{B\},$$  \hspace{1cm} (1.20)

where the $d_{\{G\}}$ is the density measurement of the group of generators to be aggregated into one equivalent generating unit. To define the density measurement, the case containing only two generators, number $k$ and $l \in \{G\}$, is considered in the first place, define $d_{kl}$,

$$d_{kl} = d_{lk} = \min \left\{ \frac{E_k' E_l' \cos([\delta'_k] - [\delta'_l])}{M_k}, \frac{E_l' E_k' \cos([\delta'_l] - [\delta'_k])}{M_l} \right\} \quad \forall k, l \in \{G\}.$$  \hspace{1cm} (1.21)
1. **Introduction**

If one generator group \(\{G\}\) has only two generator, the \(d_{kl}\) is the density measure of the group. Generally for a group having any possible number of generators, the density measure is defined as,

\[
    d_{\{G\}} = \min_{i,j \in \{T\}} d_{ij},
\]

where the \(\{T\}\) is the tree of the group, which comprises of branches having highest value of \(d_{ij}\). The physical meaning of the measure can be understood in such a way that every nodes (generators) in the group are coupled in the dynamic response by the various links connecting among them. One motion in generator will be transferred into motions of other generators in the same connected group through the power transfer among the branches. Among these branches, the ones with higher \(d_{ij}\) play the dominant roles, and they are referred to as the tree. Another important observation from the experience is that the generators located far away from the disturbance have reduced effect. Based on this important observation, larger admissible error should be allowed for generators at greater distance. Thus, the \(\rho_h\) can be distance dependent, one among the simplest definition is in linear function,

\[
    \rho_h = \rho_{h0} + k_{\rho_h} \frac{d_{\{G\}}}{\max_{i \in \{G\}, k \in \{B\}} d_{ik}},
\]

where the \(k_{\rho_h}\) is usually taken as 0.1–0.3, and \(\rho_{h0}\) as 0.2–0.5.

Based on the coherency, a group of generators can be grouped such that generators in one group have small coherency with respect to each other. The algorithm is sketched as follows,

1) calculate the admittance matrix for boundary nodes and generator nodes of external subsystem.

2) mark all the generators not eligible for being aggregated, which are usually big generator units that draw our special interests.

3) calculate \(d_{ij}\) and sort them in a list in ascending order. Create the first empty group \(\{G\}\) and initialize the \(d_{\{G\}}\) as infinity.

4) take the first branch from the top of the list. If the branch’s terminal node is marked non eligible for being aggregated, remove it and repeat this step. If no branch left in the list, end the program.

5) check the criteria of (1.20). If it is fulfilled, add the generators associated with this branch into \(\{G\}\) and update \(d_{\{G\}}\) and move to Step 4). Otherwise, create a new set of \(\{G\}\), initialize its \(d_{\{G\}}\) and move to Step 4).
1.3. Coherency Recognition

Features of Coherency-based
In this section, attention will be drawn into some properties of the Zhukov’s method in aggregating the generators. These properties are summarized as follows with their technical details given after.

1) Zhukov’s method has physical explanations.
2) The aggregation “operator” and linearization “operator” are commutative.
3) If a group of generators are exact coherent, an elegant mathematical modal analysis can be derived, which reveals the deep interact between detail represented internal subsystem and reduced external subsystem.

Zhukov’s method in electrical point of view — De Mello’s model
De Mello’s model is in fact a physical explanation of the Zhukov’s method that is based on mathematical equivalent. Similar to Figure 1.2 in Dimo’s method, but the admittance $y_{ij}$ is replace by ideal transformers as shown Figure 1.4.

The transfer ratio $\theta_i$ is chosen in such way that their secondary side has the same voltage. Thus, an aggregated node $a$s can be added, the transformers are integrated into “reduced network”. Attention should be drawn that the transfer ratios are set of complex value, and

$$v_{primary} = \theta_i v_{primary} \quad i_{secondary} = \theta_i i_{primary}.$$  

Linearization
Firstly, the incremental equations for the original system are

$$\begin{bmatrix} \Delta p_R \\ \Delta p_A \end{bmatrix} = \begin{bmatrix} H_{RR} & H_{RA} \\ H_{AR} & H_{AA} \end{bmatrix} \begin{bmatrix} \Delta \delta' R \\ \Delta \delta' A \end{bmatrix},$$  

where $H = \frac{\partial p}{\partial \delta}$. If the aggregation is based on grouping coherent generators, then, assuming only one group is aggregated, the entities of $\Delta \delta' A$ have approximately the same value and it is equal to $\Delta \delta' A$ in aggregated model. Thus, the
incremental equations for the aggregated model are

\[
\begin{bmatrix}
\Delta p_R \\
\Delta p_A
\end{bmatrix} = \begin{bmatrix}
H_{RR} & H_{RA}1_A \\
1_A^T H_{AR} & 1_A^T H_{AA}1_A
\end{bmatrix} \begin{bmatrix}
\Delta \delta'_R \\
\Delta \delta'_A
\end{bmatrix},
\]

(1.25)

where \(1_A\) is a column vector of length of group \(\{A\}\) and filled with element 1.

One of the important observation for original and aggregated models is that the order of aggregation and linearization will not influence the result of incremental equations.

In fact (1.24) and (1.25) gives a new method for aggregation. For a power network, give incremental equations for retained nodes \(\{R\}\) and aggregated nodes \(\{A\}\); group nodes in \(\{A\}\) such that each node in a group has similar value of \(\Delta \delta'\); then use (1.25) to give reduced model representation. This method is referred to as Di Caprio and Marconato’s method. Compared with Zhukov’s method, the only difference is operation, in that the Caprio and Marconato’s method linearizes system to get the incremental equation before aggregation. If linearizing the system aggregated with Zhukov’s method, one can get the same result.

**Aggregation in the view of modal analysis**

In the case where the a group of generators are exact coherent, it allows a very elegant mathematical expression in modal analysis. With comparison with the modal analysis of original system, it reveals that a disturbance originated from internal subsystem can only trigger one mode of oscillation in the aggregated generators, which is the fundamental of the Zhukov’s aggregation method.

If neglecting all conductance in the system networks, its incremental equations(1.24) can be written as with some linear algebra manipulation,

\[
\begin{bmatrix}
\Delta p_R \\
\Delta \delta_R
\end{bmatrix} = \begin{bmatrix}
H_{AA} - H_{AR}H_{RR}^{-1}H_{RA} & H_{AR}H_{RR}^{-1} \\
-H_{RR}^{-1}H_{RA} & H^{-1}_{RR}
\end{bmatrix} \begin{bmatrix}
\Delta \delta_A \\
\Delta p_A
\end{bmatrix}.
\]

(1.26)

For convenience, denote

\[
H_A = H_{AA} - H_{AR}H_{RR}^{-1}H_{RA},
\]

(1.27)

\[
R_A = H_{AR}H_{RR}^{-1}.
\]

(1.28)

Then looking at the group of generators to be aggregated, there swing equations can be written as,

\[
M_A \Delta \delta'_A = -H_A \Delta \delta'_A - R_A \Delta p_R - D_A \Delta \dot{\delta}'_A,
\]

(1.29)

where \(M_A\) is a diagonal matrix with its diagonal elements as the generator’s inertia, so \(M_A^{-1}\) is well defined. Rewrite the equation as,

\[
\ddot{\delta}'_A = -M_A^{-1}H \delta'_A - M_A^{-1}R_A \Delta p_R - M_A^{-1}D_A \Delta \dot{\delta}'_A
\]

(1.30)
If neglecting the damping factors $D$ in the following discussion, the equation above can be rewritten in matrix form,

$$
\begin{bmatrix}
\delta_A \\
\Delta \dot{\omega}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-M_A^{-1}H & 0
\end{bmatrix}
\begin{bmatrix}
\delta_A \\
\Delta \dot{\omega}
\end{bmatrix} - 
\begin{bmatrix}
0 \\
M_A^{-1}R_A
\end{bmatrix}
\Delta p_R,
$$

(1.31)

where the 0 is matrix (or vector) of appropriate dimension. Now the $\Delta p_R$ can be seen as the system input.

By taking the advantage that the structure of (1.31) that there is a zero block located at the upper left or top of system state matrix ($A$) and input matrix ($B$) and neglecting damping factors, solely the block $-M_A^{-1}H$ (second order dynamic) can give a lot of information about the stability of the aggregated generators [35]. Let vectors, $w_i$ and $u_i$ be the right and left eigenvector of $-M_A^{-1}H$ respectively corresponding to eigenvalue $\mu_i$. Accordingly the eigenvalue of the whole system is $\sqrt{\mu_i}$, and various independent eigenvectors compose the square matrix $W$ and $U$ (assuming the $-M_A^{-1}H$ is semi-simple),

$$
W = [w_1 \ w_2 \ ... \ w_N] \quad U = W^{-1} = \begin{bmatrix}
u_1^T \\
u_2^T \\
\vdots \\
u_N^T
\end{bmatrix}
$$

On (1.24) an important observation is,

$$
\begin{bmatrix}
0_R \\
0_A
\end{bmatrix} = 
\begin{bmatrix}
H_{RR} & H_{RA} \\
H_{AR} & H_{AA}
\end{bmatrix}
\begin{bmatrix}
1_R \\
1_A
\end{bmatrix}.
$$

(1.32)

Then it can be verified that

$$
H_A1_A = (H_{AA} - H_{AR}H_{RR}^{-1}H_{RA})1_A = 0,
$$

thus, after pre-multiplying $M_A'$,

$$
-M_A^{-1}H1_A = 0_A = 01_A.
$$

The vector $1_A$ is an eigenvector of the second order differential system. On the other hand, considering the relation $UW = I$, one can get,

$$
u_1^T1_A = 1
$$

$$
u_2^T1_A = 0
$$

$$
\vdots
$$

$$
u_N^T1_A = 0.
$$

(1.33)
1. Introduction

Changing the basis with new vectors in new basis named $z$

$$z = U \Delta \delta'_A$$

$$= \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_N^T \end{bmatrix} \Delta \delta'_A = \begin{bmatrix} u_1^T \Delta \delta'_A \\ u_2^T \Delta \delta'_A \\ \vdots \\ u_N^T \Delta \delta'_A \end{bmatrix}. \tag{1.35}$$

In the case that the aggregated generators are exact coherent, $\Delta \delta'_{A,1} = \Delta \delta'_{A,2} = \cdots = \Delta \delta'_{A,N} = \Delta \delta'_a$, the new vector $z$ is by invoking the observation (1.33),

$$z = [\Delta \delta'_a \ 0 \ \cdots \ 0]^T.$$

This means that the only one mode of oscillation can be excited from the interaction between aggregated generators and rest of system if the aggregated generators are exact coherent.

The particular characteristic for $B$ in the case of exact coherency can also be derived from new basis from the fact the input $\Delta p_A$ does not excite disturbance in $z$ other than $z_1$. Rewrite the second equation of (1.31) in new coordinate system,

$$\ddot{z} = \Lambda z - R \Delta p_R, \tag{1.36}$$

where

$$\Lambda = -UM^{-1}_AHW \quad R = UM^{-1}_AR_A. \tag{1.37}$$

Taking account (1.28), the expression can be simplified further

$$R = UM^{-1}_AHR_AR^{-1}_{RR} \quad \text{and} \quad WRH_{RR} = M^{-1}_AHR_AR \tag{1.38}$$

If a $\Delta p_A$ only excite one modal variable of $z_1$, it must have such structure like,

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ \vdots \\ \lambda_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} - \begin{bmatrix} r_1^T \\ 0^T \\ \vdots \\ 0^T \end{bmatrix} \Delta p_R \tag{1.39}$$

where assume that the modal variable $z_1$ refers to the modal variable corresponding to the only oscillations that can only be excited by disturbance inside internal subsystem. From the analysis, $1_A$ is one of the system’s eigenvectors, so

$$WR = [1_A \ w_2 \ \cdots \ w_n] \begin{bmatrix} r_1^T \\ 0^T \\ \vdots \\ 0^T \end{bmatrix} = \begin{bmatrix} r_1^T \\ r_1^T \\ \vdots \\ r_1^T \end{bmatrix} \tag{1.40}$$
then

\[ \begin{bmatrix} r_1^T \\ r_1^T \\ \vdots \\ r_1^T \end{bmatrix} H_{RR} = \begin{bmatrix} h_1 \\ h_1 \\ \vdots \\ h_1 \end{bmatrix}. \]  

(1.41)

From the equality of (1.38)

\[ M_A^{-1} H_{AR} = W R H_{RR} = \begin{bmatrix} h_1 \\ h_1 \\ \vdots \\ h_1 \end{bmatrix}, \]  

(1.42)

which means \( \frac{H_{jk}}{M_{j}} \), \( \forall k \in \{R\} \) are identical. This means that the sufficient and necessary condition for group of generators to be exact coherent is, for all \( l \in \{R\} \) it has

\[ \frac{H_{jl}}{M_{j}} = \frac{H_{kl}}{M_{k}}, \quad \forall j, k \in \{A\}. \]  

(1.43)

**Eigenvalues and Eigenvectors of the reduced network**

In the following paragraphs, attention will to drawn into deriving a mathematical expression for the aggregation of generators by Zhukov’s method. Then by using this result for modal analysis including the aggregated generators and the internal subsystem.

Assuming \( \boldsymbol{\delta} \) as a vector of rotor angles of all generators including the ones to be reduced,

\[ \boldsymbol{\delta}' = \begin{bmatrix} \delta'_R \\ \delta'_A \end{bmatrix}^T, \]

where \( \delta_R \) and \( \delta_A \) denote the rotor angles of retained and reduced generators. On the other hand, the state vector after aggregation should have such structure

\[ \boldsymbol{\delta}'_e = \begin{bmatrix} \delta'_R \\ \delta'_a \end{bmatrix}^T, \]

where \( \delta_R \) is a vector of rotor angle corresponding to retained generators, and the \( \delta_a \) can be calculated from,

\[ \delta'_a = \frac{1}{n} \sum_{i \in \{A\}} \delta'_i. \]  

(1.44)

This is different from the Zhukov’s method (1.13) and (1.14). These two methods are equivalent only when the aggregated generators are exact coherent. However, (1.44) can give an insight into the system structure before and after
the aggregation, and will be very helpful to analysis the oscillations for the whole system under such aggregation.

Based on (1.44) one can define a projection matrix projecting the system state vector \( \delta \) onto \( \delta_e \) defined on a space of less dimension than original one, i.e.

\[
\delta_e' = C \delta' 
\]

\[
= \begin{bmatrix}
1 & 0_{RA} \\
0_{RR} & \frac{1}{n} & \cdots & \frac{1}{n}
\end{bmatrix} \delta'
\]

Then fore the whole system, not like the previous case and (1.31) where only consider the aggregated generators. From the swing equations,

\[
\begin{bmatrix}
\Delta \dot{\delta}' \\
\Delta \dot{\omega}
\end{bmatrix} = \begin{bmatrix}
\Delta \omega \\
-M^{-1} \Delta p - M^{-1} D \Delta \omega
\end{bmatrix},
\]

where \( M \) and \( D \) are diagonal matrix with its diagonal entity as the mechanical inertia and damping for each generator. The \( p \) can be calculated from incremental equations

\[
\Delta p = H \delta'.
\]

From (1.46) and (1.47) and neglecting the damping effects, the dynamic can be written as

\[
\begin{bmatrix}
\Delta \dot{\delta}' \\
\Delta \dot{\omega}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-M^{-1} H & 0
\end{bmatrix} \begin{bmatrix}
\Delta \delta' \\
\Delta \omega
\end{bmatrix}.
\]

Only consider the second order system with partition according to generators to be retained or to be aggregated,

\[
\begin{bmatrix}
\Delta \ddot{\delta}_R' \\
\Delta \ddot{\delta}_A'
\end{bmatrix} = - \begin{bmatrix}
M^{-1}_R H_{RR} & M^{-1}_R H_{RA} \\
M^{-1}_A H_{AR} & M^{-1}_A H_{AA}
\end{bmatrix} \begin{bmatrix}
\Delta \delta'_R \\
\Delta \delta'_A
\end{bmatrix}.
\]

Pre-multiplying \( C \) on both sides of the dynamic equation, one can get

\[
C \begin{bmatrix}
\Delta \ddot{\delta}_R' \\
\Delta \ddot{\delta}_A'
\end{bmatrix} = -C \begin{bmatrix}
M^{-1}_R H_{RR} & M^{-1}_R H_{RA} \\
M^{-1}_A H_{AR} & M^{-1}_A H_{AA}
\end{bmatrix} C \begin{bmatrix}
\Delta \delta'_R \\
\Delta \delta'_A
\end{bmatrix}.
\]

The state vector \( \delta' \) has been projected into reduced dimension vector \( \delta'_e \). Next, the state vector \( \delta' \) on the right hand side also needs to be reduced. Pre-multiply \( \delta' \) by \( (C^T (CC^T)^{-1}) C \), which employing the pseudo-inverse matrix, since the \( C \) is not square. Thus,

\[
C \begin{bmatrix}
\Delta \ddot{\delta}_R' \\
\Delta \ddot{\delta}_A'
\end{bmatrix} = -C \begin{bmatrix}
M^{-1}_R H_{RR} & M^{-1}_R H_{RA} \\
M^{-1}_A H_{AR} & M^{-1}_A H_{AA}
\end{bmatrix} (C^T (CC^T)^{-1}) C \begin{bmatrix}
\Delta \delta'_R \\
\Delta \delta'_A
\end{bmatrix}.
\]
1.4. Splitting-based Dynamic Equivalencing in Power Systems

Finally get,

\[
\begin{bmatrix}
\Delta \delta_R' \\
\Delta \delta_a'
\end{bmatrix} = -C \begin{bmatrix}
M_R^{-1} H_{RR} & M_R^{-1} H_{RA} \\
M_A^{-1} H_{AR} & M_A^{-1} H_{AA}
\end{bmatrix} \left(C^T (CC^T)^{-1}\right) \begin{bmatrix}
\Delta \delta_R' \\
\Delta \delta_a'
\end{bmatrix}
\]

(1.52)

\[
= A_r \delta_t',
\]

(1.53)

where \(A_r\) is the state matrix in the reduced system,

\[
A_r = -C \begin{bmatrix}
M_R^{-1} H_{RR} & M_R^{-1} H_{RA} \\
M_A^{-1} H_{AR} & M_A^{-1} H_{AA}
\end{bmatrix} \left(C^T (CC^T)^{-1}\right).
\]

(1.54)

Then, by applying (1.45b)

\[
A_r = - \begin{bmatrix}
M_R^{-1} H_{RR} & M_R^{-1} H_{RA} 1_A \\
\frac{1}{n} M_A^{-1} H_{AR} & \frac{1}{n} M_A^{-1} H_{AA} 1_A
\end{bmatrix}.
\]

(1.55)

Parts of eigenvectors and eigenvalues are retained in the (1.55). If the groups of aggregated generators to be aggregated are exact coherent, the reduced model can accurately retain oscillation of the original system. But in fact there are rarely such cases, so only approximation of the eigenvectors and eigenvalues are expected. Given a disturbance in the internal subsystem, a good model reduction can retain the mostly excited oscillation as precisely as possible.

1.4 Splitting-based Dynamic Equivalencing in Power Systems

This method is outlined from [32] that adopts the concepts of sharing factors generated from Fuzzy theory [42, 7] or principal component analysis [26]. One of the advantages over traditional methods is that this method is able to retain more physical characteristics.

1.4.1 Recitation of traditional methods

Many classical methods and enhancements are proposed in the history. However, they all can be generally divided into three steps

- **Identification**: the task that comes first is to identify coherent generators in *external subsystem* like what mentioned above in Zhukov’s method. Alternatively, *Electromechanical Identity Recognition of Generators* [31] can be used instead to form machine groups sharing similar electromechanical properties.

- **Aggregation**: aggregate generators based on inertial and slow aggregation such that each generator group is represented by an equivalent one. Whenever a disturbance, the equivalence can response in a way similar to the original generator group.
• **Static network reduction**: after aggregation one may furthermore reduce the resulting network by statically eliminating nodes and creating new lines.

Some drawbacks of traditional methods are also pronounced and successfully avoided in the *Splitting-based Dynamic Equivalencing*. With traditional methods, the equivalent generators generated from aggregation procedure are assigned parameters in a much direct method compared to the splitting-based method where each generator in external subsystem has its own contribution to characteristics of final equivalent generators. The inertial and slow aggregation method adopts zero and first order approximations of the two-time-scale model.

On the other hand, this linearly approximated model dispense with the many details in the machine which could enhance the generator equivalence’s nonlinear behavior response; the coherency of generator groups used for aggregation procedure depends on impressed disturbance; the two-time scale model discards non-generator buses, and as a result it changes the network structure; and finally, aggregating one coherent generator group can effect other coherent group electrically.

### 1.4.2 Splitting-based approach

A representative equivalent generator is still needed to be derived but its parameters are assigned based on sharing factor rather than using direct method [11]. The *Splitting-based Approach* splits one generator into fictitious parts, each part contribute to a certain representative equivalent generator, such that the coherency recognition is not longer necessary. The comparison between the classical is illustrated in Figure 1.5.

Though the *coherency recognition* is no longer necessary with this method,
1.4. Splitting-based Dynamic Equivalencing in Power Systems

Sharing parameters are to be defined to represent how much characteristics of a synchronous generator retained in the equivalent. To derive it, we first need to look at the following schematic Figure 1.6 in classical aggregation method (De Mello’s model). Figure 1.6, The aggregation is derived from the generators’ internal buses tie \( a, b \). These two buses are both tied up to a common bus \( E \) through transformers with appropriate complex transfer ratios \( a \) and transient reactances. The active and reactive powers are supplied by the “real” generators behind buses \( a, b \), so are the equivalent generator’s inertia. A new internal bus \( q \) is created by adding a negative reactance. The reactances and transfer ratios involved are chosen in such way to preserve that the same amount of power is extracted from the original buses, i.e. bus \( a \) and \( b \).

The Splitting-based method takes one step further by taking into account splitting factors, which are represented in the following row stochastic matrix \( A \in \mathbb{R}^{M \times N} \),

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1N} \\
a_{21} & a_{22} & \cdots & a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN}
\end{bmatrix}, \quad \sum_j a_{ij} = 1 \quad \forall i = 1 \cdots M, \quad (1.56)
\]

where the \( M \) is number of equivalent generators after aggregation, the \( N \) is the number of generators in original networks. After defining the splitting factors, \( a_{ij} \), one can calculate the internal voltage of the bus \( P \), which is the weighted average value by the product of each generator’s rated apparent power and
1. Introduction

splitting factors,

\[
E_P = \frac{\sum_j S_j a_{ij} E_j}{\sum_j S_j a_{ij}} \quad \forall i = 1 \ldots M.
\] (1.57)

The power quantities can be defined in the following ways,

\[
S_{E_i} = \sum_j S_j a_{ij} \quad \forall i = 1 \ldots M \quad (1.58a)
\]

\[
P_{E_i} = \sum_j P_j a_{ij} \quad \forall i = 1 \ldots M \quad (1.58b)
\]

\[
Q_{E_i} = \sum_j Q_j a_{ij} \quad \forall i = 1 \ldots M, \quad (1.58c)
\]

where the \( S_j, P_j \) and \( Q_j \) are the nominal values for each generator in the external subsystem of original network. Other parameters of the aggregated equivalent generators can also be calculated based on the product of power rating and splitting factor,

\[
R_{E_i} = \frac{\sum_j S_j a_{ij}}{\sum_j \hat{S}_j a_{ij}} \quad \forall i = 1 \ldots M \quad (1.59)
\]

\[
X_{E_i} = \frac{\sum_j S_j a_{ij}}{\sum_j \hat{X}_j S_j a_{ij}} \quad \forall i = 1 \ldots M
\]

and the time constants,

\[
T_{mE_i} = \frac{\sum_j T_{mj} S_j a_{ij}}{\sum_j S_j a_{ij}} \quad \forall i = 1 \ldots M \quad (1.60)
\]

Splitting factor calculation

The splitting factors constituting (1.56) can be determined by fuzzy clustering method [7, 60] and principal component analysis [26] according to [32].

Fuzzy clustering  The process of determining splitting factors are in fact an optimization problem with the cost function \( J_m \) over \( A \) as,

\[
\min_A J_m(X; A, C) = \sum_{j=1}^N \sum_{i=1}^M a_{ij}^n \| x_j - c_i \|^2,
\] (1.61)

where the \( X \) is the data set comprising states in the original generator dynamic model, \( a_{ij} \) is entry of \( A \) (1.56), and \( x_j \) and \( c_i \) are column vectors of \( X \) and \( C \) correspondingly, which are state transient measurement of each generator.
in original subsystem and reference generators defined by cluster (group) center. The minimum is approached in iteration manner, along with the cluster centers. Before introducing the iteration procedure, we look at the necessary condition of minimum points,

$$a^*_{ij} = \frac{1}{\sum_{r=1}^{M} \left( \frac{\|x_j - c^*_i\|}{\|x_j - c^*_r\|} \right)^{2/(m-1)}}$$

where the $a^*_{ij}$ and $c^*_i$ are the optimal value attained and $m$ indicates the "fuzziness". The $a^*_{ij}$ and $c^*_i$ and hence the optimal problem can solved by the following iteration procedure,

- **Initialization** The initial guesses for the $a_{ij}$ are,

$$a_{ij} = \frac{\|x_j - x^*_i\|^{-1}}{\sum_{r=1}^{N} \|x_j - x^*_r\|^{-1}}$$

(1.63)

- **Iteration** this step involves two parts: updating cluster center

$$c_{i(k+1)} = \frac{\sum_{j=1}^{N} a^m_{ij(k)} x_j}{\sum_{j=1}^{N} a^m_{ij(k)}}$$

$\forall i = 1, \ldots, M$, (1.64)

and updating the fitting factors,

$$a_{ij(k+1)} = \frac{1}{\sum_{r=1}^{M} \left( \frac{\|x_j - c^*_i(k)\|}{\|x_j - c^*_r(k)\|} \right)^{2/(m-1)}}$$

$\forall i = 1, \ldots, M$, (1.65)

where the $(k)$ and $(k + 1)$ indicate the number of iteration.

**Principal component analysis** This method is applied on the state measurement matrix, $X$, formed by recording data after a perturbation. Therefore the experiments should carefully prepared [34].

The *time behavior matrix* is formed by the following way,

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_N(t) \end{bmatrix}^T,$$

(1.66)

where $x_i(t) (i = 1, \ldots, N)$ are some measurements of generators in the external subsystem at time $t$. Some statistic data can be derived from this matrix,

$$\mu_x = E_t\{x(t)\}.$$  

(1.67)
1. Introduction

This is the expected vector of $x$ over the whole measurement period. Another statistic data—covariance matrix $\Sigma_x$ can be defined as,

$$
\Sigma_x = E \left\{ \left( x(t) - \mu_x \right) \left( x(t) - \mu_x \right)^T \right\}.
$$

Assuming the $X$ is diagonalizable, applying eigen-decomposition,

$$
\Sigma_x v_i = \lambda_i v_i \quad i = 1, \cdots, N
$$

where the $\Sigma_x$ is symmetric matrix, the $\lambda_i$ is the variance, and the $N$ is the number of generators in external subsystem. Reorder the eigenvalues, number them and their corresponding eigenvectors in descending order, and get

$$
\Lambda = \text{diag} \{ \lambda_1, \lambda_2, \cdots, \lambda_N \}.
$$

The vectors $v_i (i = 1, \cdots, N)$ form a set of basis. The eigenvectors of the first few cardinals indicate the direct in space $\mathbb{R}^N$ where the dynamic has more energy and should be retained. Taking out these eigenvectors, they then form a projection matrix $C \in \mathbb{R}^{M \times N}$ transferring the original $N$ dimension space into reduced space of dimension $M$, where the $v_i$ are rows of $C$.

1.5 Border and Synchrony method

There are many reduction methods developed in the history all based on their similarity (synchrony identification) as the step, and then one representative generator is chosen from each group while keeping the remaining ones by assigning correction factors. This structure is proved to be reliable in many cases, while more improvements or extensions can be made to develop better reduced models and, at the same time, keeping desired physical structure.

The following summarizes the [36] work, which revisits the synchrony in the view of controllability and observability.

1.5.1 Revisit the synchrony from in the view of uncontrollability and unobservability

Synchrony

In classical methods, a group of generators is similar if the generators’ internal rotor angles will undergo the approximately the same transients in response to a particular disturbance, whereas the slow-coherency is another similarity criterion identified by the slowest mode of the system rather than a particular perturbation. If generators exhibit similar responsive behaviors due to a perturbation that sufficiently excites the slowest oscillation mode, then they are called slowly-coherent.
1.5. Border and Synchrony method

On the other hands, another difficulty might arise from situation where the information of the whole system is not available but it is necessary for the method mentioned above. As a solution, the notion of border synchrony is defined based on measures of controllability and observability. \[54\] and \[52\] generalize these coherency under a single oscillation mode to the a class of modes, \(\Lambda\), of our interests, namely the chord. We say two generators are \(\Lambda\)-synchronous if two generators’ rotor angle transients make following constraint hold after any arbitrary perturbation that excites at least one of oscillation modes in \(\Lambda\),

\[
|\delta_i - k\delta_j| < \varepsilon
\]  

(1.71)

where \(i\) and \(j\) denote the different generators in the \(\Lambda\)-synchronous group, and \(k \in \mathbb{R}\) can be negative constant. \(\varepsilon \in \mathbb{R}_+\) is a non-negative threshold value.

Lacks of controllability and observability

The coherency methods based on synchrony are closely connected with the lack of controllability or observability. This point is better illustrated with the two-generator system as in Figure 1.7, where the two generators are represented by their classical model \[29\] and no load is assumed in this system. Its dynamic is merely swing equations of the two generators,

\[
M_i \ddot{\delta}_i = P_m - P_e \quad \forall i \in \{1, 2\},
\]  

(1.72)

where \(M_i = \frac{2H_i}{\omega_0}\), \(H_i\) and \(P_m\) are the generator’s inertia, mechanical power input respectively. The electrical power outputs \(P_e\) can be calculated as,

\[
P_{e_1} = P_s(\Delta \delta_1 - \Delta \delta_2)
\]  

(1.73a)

\[
P_{e_2} = P_s(\Delta \delta_2 - \Delta \delta_1)
\]  

(1.73b)

where the \(P_s = \frac{E_0' \delta_0'}{X}\) \(\cos(\delta_i' - \delta_j')\). The \(E_0'\) is the constant internal voltage of generators in classical model, and the \(\delta_0\) is the rotor angle of initial state (steady state). To highlight that (1.73b) is obtained from linearization, \(\Delta\) is added before \(\delta\) to indicate that \(\Delta \delta\) is the angular distance from the initial value \(\delta_0\). Moreover, the assumption that no load exists in the system implies that the mechanical power inputs to these two generators have the following relation,

\[
P_{m_1} + P_{m_2} = 0.
\]  

(1.74)

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Use (1.73b) and (1.74), swing equation (1.72) can be written in matrix form by choosing state variables as 
\[ x = [\delta_1 \dot{\delta}_1 \delta_2 \dot{\delta}_2]^{T} \] and input as \( u = P_{m_i} \)

\[
\dot{x} = Ax + Bu
\] \hspace{1cm} (1.75a)

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{P_s M_1}{M_1} & 0 & \frac{P_s}{M_1} & 0 \\
0 & 0 & 0 & 1 \\
\frac{P_s}{M_2} & 0 & -\frac{P_s}{M_2} & 0
\end{bmatrix}
\] \hspace{1cm} (1.75b)

\[
B = \begin{bmatrix}
0 \\
\frac{1}{M_1} \\
0 \\
-\frac{1}{M_2}
\end{bmatrix}
\] \hspace{1cm} (1.75c)

and

\[
y = C x
\] \hspace{1cm} (1.76a)

\[
C = \begin{bmatrix}
1 & 0 & 1 & 0
\end{bmatrix}
\] \hspace{1cm} (1.76b)

There are some important observations to be made at this point regarding (1.75). Firstly, the matrix \( A \) is not invertible and its null space is spanned by 
\( v_1 = [1 0 1 0]^{T} \). \( v_1 \) is one of the right eigenvectors corresponding to \( \lambda_1 = 0 \). Secondly, the matrix \( A \) is not semi-simple which has only three independent right eigenvectors; and especially one eigenvector \( v_1 \) corresponds to double eigenvalues \( \lambda_1 = \lambda_2 = 0 \). From these there are two observations, one can conclude two out of 4 (the rank of the matrix \( A \)) modes are coupled. Indeed, by multiplying the second row of the system (or augmented matrix) by \( M_1 \), the fourth row by \( M_2 \), and add them together, one can get

\[
M_1 \Delta \dot{\delta}_1 + M_2 \Delta \dot{\delta}_2 = 0.
\] \hspace{1cm} (1.77)

If view \( \xi = \Delta \delta_1 + \Delta \delta_2 \) as a new variable, solve for it, its solution is,

\[
\xi(t) = M_1 \Delta \delta_1(t) + M_2 \Delta \delta_2(t) = c_1 t + c_2,
\] \hspace{1cm} (1.78)

where \( c_1 \) and \( c_2 \) are constant. Keep in mind that all conclusions got so far are from the linearized system (1.75), so in real system the \( \xi = M_1 \Delta \delta_1 + M_2 \Delta \delta_2 \) cannot become infinite large, i.e. \( c_1 \) must equal 0. And the \( \Delta \delta_1 \) and \( \Delta \delta_2 \) denote the deviation with respect to initial point \( \delta_{20} \) and \( \delta_{20} \), so \( \xi(0) = c_2 = 0 \). Therefore, \( \xi = M_1 \Delta \delta_1 + M_2 \Delta \delta_2 = 0 \) is a coupling constrain in this linearized system. The dynamic (1.77) has no term of control input (if in modal canonical form, the corresponding entry of transformed \( B \) is zero), and
therefore it is uncontrollable. This conclusion can also be drawn from the range of controllability matrix \( \mathcal{P} = [B \ AB \ A^2B \ A^3B] \),

\[
\text{range}\{ \mathcal{P} \} = \text{span}\left\{ \begin{bmatrix} M_2 \\ 0 \\ M_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ M_2 \\ 0 \\ M_1 \end{bmatrix} \right\}.
\] (1.79)

For a linear time-invariant system, the image of controllable matrix is reachable subspace. It can be seen that the system states are confined by

\[
M_1 \Delta \delta_1 + M_2 \Delta \delta_2 = 0 \tag{1.80}
\]

\[
M_1 \Delta \dot{\delta}_1 + M_2 \Delta \dot{\delta}_2 = 0 \tag{1.81}
\]

or

\[
M_1 \Delta \delta_1 + M_2 \Delta \delta_2 = 0 \tag{1.82}
\]

\[
s(M_1 \Delta \delta_1 + M_2 \Delta \delta_2) = 0. \tag{1.83}
\]

For the second observation, taking into account the matrix \( C \) in the (1.76), the controllability matrix \( \mathcal{O} = [C^T \ A^T C^T (A^T)^2 C^T (A^T)^3 C^T]^T \) has rank of two. The null subspace of the observability matrix is

\[
\text{null}\{ \mathcal{O} \} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \tag{1.84}
\]

For a linear time-invariant system, the null space of the observability matrix is the subspace that cannot be observable. In other word, the states subject to

\[
\Delta \delta_1 = \Delta \delta_2 \tag{1.85}
\]

\[
\Delta \dot{\delta}_1 = \Delta \dot{\delta}_2 \tag{1.86}
\]

or

\[
\Delta \delta_1 - \Delta \delta_2 = 0 \tag{1.87}
\]

\[
s(\Delta \delta_1 - \Delta \delta_2) = 0. \tag{1.88}
\]

can not be observable.

Both the uncontrollable mode \( M_2 \Delta \delta_1 + M_1 \Delta \delta_2 = 0 \) and the unobservable mode \( \Delta \delta_1 - \Delta \delta_2 = 0 \) are associated with the double poles of \( s = 0 \). Then, try
to isolated the controllable/observable parts from the uncontrollable/unobservable parts in the dynamic system. From (1.82) and (1.83), define

$$\sigma = \Delta \delta_1 = -\frac{M_1}{M_2}\Delta \delta_2$$  \hspace{1cm} (1.89)

$$\dot{\sigma} = \Delta \dot{\delta}_1 = -\frac{M_1}{M_2}\Delta \dot{\delta}_2$$  \hspace{1cm} (1.90)

as new states, i.e. $\dot{x} = [\sigma \ \dot{\sigma}]^T$. The dynamic of (1.75) can be written as

$$\dot{x} = \tilde{A}x + \tilde{B}u$$  \hspace{1cm} (1.91)

$$y = \tilde{C}\dot{x},$$  \hspace{1cm} (1.92)

where

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ -\frac{M_1+M_2}{M_1 M_2} \rho_s & 0 \end{bmatrix} \hspace{1cm} \tilde{B} = \begin{bmatrix} 0 \\ \frac{1}{\rho_s} \end{bmatrix}$$  \hspace{1cm} (1.93)

and

$$\tilde{C} = \begin{bmatrix} \frac{M_1+M_2}{M_2} & 0 \end{bmatrix}.$$  \hspace{1cm} (1.94)

The controllability matrix and observability matrix are,

$$\tilde{\Phi} = \begin{bmatrix} 0 & \frac{1}{\rho_s} \\ \rho_s & 0 \end{bmatrix}$$  \hspace{1cm} (1.95)

$$\tilde{O} = \begin{bmatrix} \frac{M_1+M_2}{M_2} & 0 \\ 0 & \frac{M_1+M_2}{M_2} \end{bmatrix}.$$  \hspace{1cm} (1.96)

Now that controllability matrix and observability matrix both have full rank, the new system (1.93), the minimum realization of original system, is successfully separated from uncontrollable and unobservable subspace, with only one non-zero oscillation mode $\lambda_1 = \lambda_2 = \pm j\sqrt{\frac{M_1+M_2}{M_1 M_2}}$. This observation has significant meaning, in that the double poles $s = 0$ in original system are solely associated with the uncontrollable and unobservable subspace. This double zero poles are sometimes referred to as input-output decoupling zero. The first zero is from the reference angle to calculate $\Delta \delta_1$ and $\Delta \delta_2$ since the $\delta_1$ and $\delta_2$ are not independent. The second zero mode (uncontrollable and unobservable mode) is so-called the frequency mode because of the constraints of (1.88). Right after a disturbance, the speeds of all generators in a synchronous zone decrease or increase at the same rate, and will come back at the same speed if they have enough damping. Moreover, this oscillation mode is in fact called exact synchrony.
1.5. Border and Synchrony method

1.5.2 Generalization

In real case, the system setup is much more complicated. [36] furthermore generalizes the concept of synchrony to some cases that they do not exactly have the double input-output decoupling zeros, and help to identify an effective class of synchrony based on poorly controlled and observed modes. In linear system theory [8, 43], the controllability gramian $W_r$ and observability gramian $W_o$ serves as good indications for how easily one mode can be controlled or observed. They are,

$$W_r(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(t) B(t) \Phi(t_1, \tau) \, d\tau \quad (1.97)$$

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) C(t) C(t) \Phi(\tau, t_0) \, d\tau. \quad (1.98)$$

where the controllability gramian and observability gramian are generally valid for all linear system and $\Phi(t_0, t_1)$ is the transition matrix from time $t_0$ to $t_1$. For this linearized system like (1.75) and (1.93), its time-invariant property however implies that $B(t) = B$, $C(t) = C$ and $A(t) = A$. The transition matrix has analytical expression $\Phi(t_0, t_1) = e^{A(t_1-t_0)}$. As a consequence of these particular features and talking the time to the infinity,

$$W_r = \int_{0}^{\infty} e^{A(t-0)} BB^T e^{A(t-0)} \, dt$$

$$W_o = \int_{0}^{\infty} e^{A(t-0)} C^T C e^{A(t-0)} \, dt. \quad (1.100)$$

The gramians are also the solution of the following Lyapunov’s equations [36, 56],

$$A W_r + W_r A^T = -BB^T \quad (1.101)$$

$$A^T W_o + W_o A = -C^T C. \quad (1.102)$$

Thus, these gramians can be calculated in a very efficient way.

A direct application of these controllability gramian and observability gramian is the balanced reduction [43]. Assume there are many $T$:s such that

$$\tilde{W}_r = T W_c T^T \quad \tilde{W}_o = T^{-1} W_o T^{-1}. \quad (1.103)$$

Compute the one which can make,

$$\tilde{W}_r = \tilde{W}_o = \text{diag} \{ \sigma_1, \sigma_2, \cdots, \sigma_n \}$$

and $\{ \sigma_1, \sigma_2, \cdots, \sigma_n \}$ are in descending order, where $\sigma_i$, usually called as Hankel singular values, is the controllability/observability gramian in transformed
basis. Accordingly, the dynamic system in transformed basis $\hat{x} = Tx$

$$\dot{\hat{x}} = T A \hat{x} + T Bu \quad (1.104)$$

$$y = C \hat{x} + Du. \quad (1.105)$$

The balanced reduction truncates states associated with smaller Hankel singular values. However, such method usually results in a rather mathematical abstract model losing much of physical structure.

### 1.5.3 Balanced model reduction preserving physical structure

To reconcile the drawback of balanced reduction method, [36] gives a suggestion which is to view the synchrony as redundancy of in power system. One can delicately select inputs and outputs to remove such redundancy till only the oscillation modes in the chord remain.

The framework is also based on the the division between study subsystem (or internal subsystem) and the external subsystem (sometimes the “subsystem” is also called subnetwork or zone). The external zone is to be reduced such that the overall system (detailed internal zone plus reduced external zone) has oscillation modes similar to the original system. Whenever possible, the boundary between these two subsystems is also subject to optimization to ensure the synchrony constrains will not be broken. Based on different situations, different reduction methods are given.

**Standard synchrony** Suppose the disturbance are initialized at the generators and load buses. However, disturbances originated from generators are sufficient to excite all oscillation modes, so generator mechanical power inputs $[P_{m1}, P_{m2}, \ldots, P_{mn}]^T$ (including generators in study subsystem and external subsystem) are taken as the input vector for the reduced system. And as shown above, generators in synchronous group have the same rotor angle deviations $\Delta \delta_i$ if neglecting generator’s damping power, so the outputs are set as $\Delta \delta_i - \Delta \delta_j$. If one special mode $\lambda$ is weekly observed in output $\Delta \delta_i - \Delta \delta_j$ and $\Delta \delta_i - \Delta \delta_k$, then this mode is also weekly observed in $\Delta \delta_k - \Delta \delta_j$. Thus, the number of system outputs are significantly shrunk by only considering $[(\Delta \delta_i - \Delta \delta_1) (\Delta \delta_i - \Delta \delta_2) \cdots (\Delta \delta_i - \Delta \delta_k) \cdots (\Delta \delta_i - \Delta \delta_m)]^T$, where $k \neq i$ and $i$ can be any number in $\{1, 2, \ldots, m\}$. In the next step, we need to formulate the overall system in balanced realization, the weekly controlled and observed modes are subject to reduction (truncation) if they are only relevant to external subsystem. The reduction details are postponed into section section 1.5.3.

**External-zone Equivalent** In this situation, in contrast to the case of Standard Synchrony, where the borders separating different power network zones, are predetermined and a priori imposed. The borders might even traverse across a synchronous zone and, as a result, the real reduction procedure is not
1.5. Border and Synchrony method

Figure 1.8: External subsystem connected with infinite buses

exactly a case of Standard Synchrony. Nevertheless, the data of the external subsystem is still assumed available when making such reduction.

The external zone synchrony is considered in this case. Similar to the Standard Synchrony, the mechanical power inputs \([P_{m₁}, P_{m₂}, \ldots, P_{mₙ}]^T\) are the system inputs. But the rotor angles \([\Delta \delta₁, \Delta \delta₂, \ldots, \Delta \deltaₙ]^T\) are regarded as system outputs. The index set \(\{1, \ldots, n\}\) numbers generators in external subsystem. The system is then subject to the same of reduction procedures.

Border Equivalent Consider cases, where persons outside the internal subsystem may only have interests on what influence the internal subsystem (study subsystem) can bring to the outside world, for the purposes like, as an example, studying inter-area oscillation mode. Indeed, inter-area oscillations are usually indicated by the oscillation of active power exchanges through the tie-lines.

As the result of these observations, the external system is connected with infinite buses at the end of cross-area tie-lines to represent the unknown study subsystem, Figure 1.8. Then, the system is defined by selecting the following inputs/outputs: \(u = [P_{m₁}, P_{m₂}, \ldots, P_{mₙ-n}]^T\) and \(y = [P_{fr₁}, P_{fr₂}, \ldots, P_{frₙ}]\), where the \(n\) and \(m\) have the same meaning as before, and the \(l\) is the number of tie-lines connecting internal and external subsystems. Such choice of output and input variables only permits observation of inter-area oscillations. Thus, the reduced external subsystem ideally only contains states relevant to the inter-area mode.

Reduction procedure

Finally, when modes to be reduced have been identified, the reduction procedures follow. The classical balanced reduction results in an abstract mathematical model without a clear physical meaning. The following aggregation based on [4] can gives a good reduced model while retaining much the physical structure. This method exploits the linear relation of \(x_u = K x_r\), where the original system states are the disjoint union of remaining states \(\{x_r\}\) and \(\{x_a\}\), \(\{x\} = \{x_r\} \cup \{x_a\}\). For the first, we need to look at the balanced realization of the original system and partition the state space corresponding to states to
be aggregated or to be eliminated,

\[ \dot{x} = \hat{T} \hat{A} \hat{T}^{-1} \dot{x} + \hat{T} Bu \]  

\[ = \begin{bmatrix} \hat{A}_{rr} & \hat{A}_{ra} \\ \hat{A}_{ar} & \hat{A}_{aa} \end{bmatrix} \dot{x} + \begin{bmatrix} \hat{B}_r \\ B_a \end{bmatrix} \bar{u}. \]  

Since the states \( x_a \) is negligible, compared with the remaining states \( x_r \), we assume \( \dot{x}_a = 0 \). We also set \( u = 0 \), then

\[ \dot{x}_a = \dot{K} \dot{x}_r, \]  

where,

\[ \dot{K} = \hat{A}_{aa}^{-1} \dot{A}_{ar} \dot{x}_r. \]  

The \( T \) transferring original system to balanced realization is

\[ T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \]  

and the change of basis matrix \( L \) transfers state variables from the balanced realization into this new set of state variables \( \ddot{x} = [x_r^T x_a^T]^T \) is a permutation matrix

\[ L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \]  

where each row is composed of zeros except one entry being number 1. From above, \( \hat{T} = LT \). The choice of matrix \( L \) is subject to optimization such that the \( \hat{T}_{11} = L_{11}T_{11} + L_{12}T_{21} \) has larger determinant. One simple way [36] to find this optimal is summarized in the following steps: 1) determine a set of oscillation modes by their eigenvalues \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_p \} \), which should be kept in the reduced system, 2) let \( V \) be angle components of right eigen-matrix, i.e. its entry \( v_{ij} \) is the angle of the \( i \)-th entry of eigenvector corresponding to \( \lambda_j \) in the set \( \Lambda \) (other entries in the eigenvectors are just discard). The \( p \) chosen machines whose rotor angles appear in the diagonal of ordered \( V \) are called reference machines. 3) In the last step, reorder matrix \( V \) using Gaussian pivot technique, register the new ordering in the permutation matrix \( L \).

The reference machines including their auxiliary dynamics of regular, governors, etc. are kept in detailed model, whereas other generators in the same synchronous generator group with the reference machine are modeled as current injections in the form \( i_k = i_{0.k} + E_k u_e + K_r x_r \) [53, 55]. The \( i_{0.k} \) is the steady-state current of the generator bus; \( E_k \) is dynamic correction factor, \( E_k = E_{\text{old}} + E_{\text{ward}} + E_{\text{adjust}} \), where the \( E_{\text{old}} \) is the electromagnetic force of the reduced generator in steady states, \( E_{\text{ward}} \) is the correction terms for Ward-type technique, and the \( E_{\text{adjust}} \) is for adjusting the effect of the buses directly connecting the study zone.
In summary, the reference [36] gives a novel theoretical view of the synchrony concept based on controllability and observability and applied in many reduction techniques. Based on this new explanation of synchrony, a reduction is derived from classical balanced reduction techniques and it can preserve the system’s physical structure with little sacrifices on the reduction quality. A new concept, border synchrony is also introduced for dealing with some practical situations where not all information is available, especially the mutual exclusion between different TSOs for the reason of commercial safety.

1.6 Preliminary Studies on the transient behavior of Distribution Network with large renewable energy penetration

As mentioned at the beginning part, the modern power systems, especially the distribution networks, have large renewable power energy penetration, which might post a major difficulty on the network reduction for the modern power network. More studies are necessary for modeling different types of renewable generations to be used in simulations, [40] classified and analyzed the transient responses of different renewable source in transient stability study.

1.6.1 Type of power sources

Different types of power sources are consider in [40]:

1) Synchronous generators (SG), like gas fuel generators, are modeled as GAST type (refer to [71] for more gas turbine modeling). All of synchronous generators are usually equipped with AVR (Automatic Voltage Controller) which can be described in IEEE model [29]. Some SG’s are even controlled to regulate voltage level;

2) Doubly fed induction generators (DFIG) is a primary type of wind generation. In [40], their pitch control, two-mass model of shaft and rotor angle/current controllers (with limit boundaries) are considered. More details concerning their modeling and simplification can be found in [27];

3) Fixed speed induction generators (FSIG) like the DFIG is another primary type of wind power generator.

4) Converter connected generators (CCG) characterizes many kinds of power sources or storage that are interfaced with a power electronics device. Some examples include wide-range wind turbines, fuel cells, photovoltaic panels, flywheel storage, etc. interfaced into power networks by power electronics converter. They are modeled together as a current source in parallel with a dc-link capacitor in the [27].
1. Introduction

1.6.2 Test system

The test system considered in [27] of different generation and load patterns are on the 11kV distribution network Figure 1.9. Readers are referred to [40] for more details of the system. The system loads are considered lumped and distributed evenly under each feeder.

To study the transient responses, two different fault types are assumed. One locates inside the distribution system at Bus 4, feeder 2; another one is in the external network at the connecting point, i.e. Bus 1. Symmetric three-phase faults are assumed, and both faults are cleared after 500ms. The fault impedances are chosen in a way such that the voltage drops to about 50% of the rated value.

The loads under considerations include static load (SL), modeled as constant power consumption in the voltage range between 0.7 and 0.8 p.u. and as constant impedance load beyond such range. The dynamic loads (IM) are modeled as induction motors, which mainly represent the industrial loads. The simulations were conducted under different load compositions, namely “100% SL”, “60% SL + 40% IM”, “40% SL + 60% IM” and “100% IM”.

The generation patterns considered in the simulations are summarized in Table 1.1. The “X” in the table indicates the generation type, and should be replaced by “DFIG”, “FSIG” or “CCG”. The output of DG of generation type X is increased by the amounts indicated in the last column while keeping the total generation being the same at 11 MW.
1.6. Preliminary Studies on the transient behavior of Distribution Network with large renewable energy penetration

### Table 1.1: GENERATION MIXTURES [27]

<table>
<thead>
<tr>
<th>Case</th>
<th>SG1 (MW)</th>
<th>SG2 (MW)</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>6</td>
<td>1.5</td>
<td>Unit rated</td>
</tr>
<tr>
<td>X1</td>
<td>4.5</td>
<td>0</td>
<td>+3 MW</td>
</tr>
<tr>
<td>X2</td>
<td>0</td>
<td>1.5</td>
<td>+6 MW</td>
</tr>
<tr>
<td>X3</td>
<td>0</td>
<td>0</td>
<td>+7.5 MW</td>
</tr>
</tbody>
</table>

#### 1.6.3 Simulation results and comments

**The influence of load type and size**

Firstly, attention are given to the influence of varying load size on the transient responses. Meanwhile, the transformers are adjusted to make sure the 120% overload limits would not be broken. The Figure 1.10 shows the responses of different load size for “100% SL” (top-half) and “60% SL + 40% IM” (bottom-half). The arrows in the graphics indicate the increasing of load sizes. It can be seen that, in the “100% SL” case, increasing load size does not have significant influence on the transient responses in contrast to “60% SL + 40% IM” case, where the inclusion of inductive loads makes the transient responses sensitive to load size variation. The increase in the reactive power consumption reduces the voltage stability. As a result, the voltage takes longer time to recover after the same fault.

Next, consider the influence of different load type mixture on voltage stability of distribution network as shown in Figure 1.10. As seen from the graphics, large IM load component decreases voltage instability. However, the rate of voltage drop during the fault is smaller when induction load takes higher share due to the fact that induction motors cannot be demagnetized abruptly. When the loads have very large IM share, the induction motor’s operating point will go beyond the stable operating region, eventually motor will stall, high current and reactive power consumption entail. In practice, the motor’s under-speed protection or transmission line’s thermal protection will be triggered before the worst situation happens.

**The influence of generation type and size**

Different generation patterns (Table 1.1) are considered in the following simulations. Starting from the influence of generation types in Figure 1.12. The Figure 1.12 (A) illustrates the worst case X3 while the load is kept at 100% level of pure induction motor. It can be seen that the FSIG gives the worst performance since it absorbs much reactive power after a fault happened. Some improvements can be made for such system, like increasing the pitch angle, rate of change and reducing the pit controller speed reference. Additionally, reactive power sources can be installed to enhance the voltage stability. In
1. Introduction

Figure 1.10: Influence of load size on transient response for different load types: top-row — 100% SL SL load, bottom-row — 60% SL + 40% SL; the arrows indicate increasing of load from 50% to 200%

Figure 1.11: Influence of load type on transient response for different load types for 125% load size: solid — 100% SL; dash-dot — 60% SL + 40% IM; dotted — 40% SL + 60% IM; dashed — 100% IM

The CCG case, though large amount of active power lost in the distribution network, the system eventually recovered, but took longer time. The active power mismatch is supplied by external networks. However, if the distribution network is under island operation or the external networks are too week to supply enough active/reactive power, the network might have serious frequency/voltage instability issues that probably lead to overall system collapse. The DFIG case gives the best performance due to its excellence in active and reactive power control. Figure 1.12 (B) shows the same generation mixture but having only half of nominal load. In such case, three different generation patterns do not have much difference. As a comparison, the \(X_1\) case with 100% load level is juxtaposed along with other two cases in Figure 1.12 (C).

The transient stability with CCG generation strongly depends on the loading
1.6. Preliminary Studies on the transient behavior of Distribution Network with large renewable energy penetration

Figure 1.12: different generation types: solid — DFIG; dash — CCG; dash-dot — FSIG

Figure 1.13: influence of disconnecting sensitive CCG for cases 50% and 150% of load size

Figure 1.14: Influence of fault location on transient stability in FSIG3 case: load size increases from 50% to 150% in step of 25% along the direction indicated in the arrow

Figure 1.15: Influence of fault location on transient stability in different generation: solid — basecase; Dash-dot — DFIG3; dashed – CCG3

level in the distribution network. Figure 1.13 shows the influence of triggering a sensitive CCG. When the network is overloaded (more loads than generation in the distribution network), the voltage profile eventually becomes stable but lower than the initial value. This is because the active power mismatch after tripping a sensitive CCG is compensated by the external generation. The power transmission from external network through longer electrical distance means larger reactive power consumption, therefore the voltage level in the new steady state is lower.

The influence of fault location

The faults are chosen at two different locations, one is inside the network (Bus 4) and another one locates outside (Bus 1 at 33 kV), Figure 1.14. In both cases, the static load size had no influence on the transient performance when the distributed generation was purely DFIG as shown in the top half of graphics. If the load was purely inductive, system collapsed in highly loaded cases for faults happened both inside and outside the distribution system. The internal fault are more likely to make the system vulnerable since it locates much closer to induction motors.
Figure 1.15 shows the transient response for different types of generation (all cases use exclusively one type of generation i.e. X3 type). The active power consumption of the distribution network increased in the case of pure CCG generation since it lost some active power sources. Another remarkable feature is the spike in the current through in the case of pure DFIG generation.

1.7 Synchronic Modal Equilalencing method

In 1990s several papers [53] were published introducing a novel method in power system equivalencing at that time. Though decades passed, it can still give a good result in many practical studies. More importantly, it can give much clear definition of generator clusters and the physical meanings in the resulting equivalent. The following content in this section recaps the main procedures and features of this method.

In face, many methods presented above have origin on this method. In the first step, the method divides the power system networks into study area and several external areas based on the concept of synchrony rather than chord-coherency or slow-synchrony. The generators in study area are kept as detailed model, along with reference generators picked out for representing each different external area. Other synchronous generators are reduced via algebraic manipulation; or, equivalently, based on its physical structure. The resulting equivalent is modeled by a voltage controlled linear multi-port “admittance”.

In the following subsection, the main procedure is summarized in two parts, generator clustering and modal equivalencing.

1.7.1 Generator clustering

The generator clustering method suggested in [53] is based on chord-synchrony. The synchrony is strongly associated with a delicately selected subset, the so-called chord, of modes of linearized models. Two generators, $i$ and $j$, are said exactly in a synchronic of a chord, $\Lambda$, if their angular variance for any perturbation that only excites oscillation modes in the selected chord has the following equality,

$$\Delta\delta_i = k\Delta\delta_j,$$

where the $k \in \mathbb{R}$. Moreover, the maximal set of such generators is the a synchronic group, or synchronic “area”. The concept is usually implemented with modal analysis but not necessarily based on classic, undamped, swing-equation model for small signal stability analysis. By using $\Delta\delta_i$ for rotor angle deviation, $\mathbf{J} = \text{diag}\{j_i\}$ the synchronous power (synchronizing coefficient), and $\mathbf{M} = \text{diag}\{M_i\}$ the inertia of generator, the swing equation (only the second
order part) is
\[ \Delta \ddot{\delta} = -M^{-1} f \Delta \delta. \]  
(1.112)

The eigenvalues of \(-M^{-1} f\) are all real number and
\[ \sigma_{n-1} \leq \ldots \leq \sigma_1 \leq \sigma_0 = 0. \]

Keeping in mind that eigenvalues \(\lambda\) of the original full-order system is the square root of \(\sigma\), which should be clearly distinguished. The synchronic chord is then chosen from from these \(\sigma_i\) with the methods suggested in [54], the resulting clustering usually differs from using slow-coherency approach. Once the chord is chosen, the generators in each area (cluster) can be naturally determined from their associated eigenvectors of matrix \(-M^{-1} f\). The details about how to decide generator members in each cluster is referred to [54, 40].

1.7.2 Dynamic Model Equivalent

Once got the synchronic areas, we can process to construct dynamic equivalent. In this method, the borders between external areas and study area that is much relevant to the planned power system study, and external cannot be defined arbitrarily, but are subject to the constraints of synchronic area, i.e. the borders between study and external areas cannot violate the borders due to power facility ownership or geographic boundaries.

Next, we define relevant modes which include all modes in the chord \(\Lambda\) that were basically inter-area modes and used search for synchronic areas and the intra-area modes (local modes) of the study area. The reference generators, each one is picked from different external areas and together can represent the inter-area oscillation, and all generators from study area are termed as relevant generators and retained in detail models. Their state variables is referred to as relevant variables. All other generators are reduced to multi-port voltage-controlled admittance as what will be shown later in this subsection.

The whole reduction procedure keeps the system description in differential algebraic equations (DAE) setups, which is suitable to be applied in typical power system analysis software. The DAE description of the subsystem at \(j\)-th bus is,
\[
\dot{x}_j = A_j x_j + B_j u_j + \mathcal{B}_j w_j 
\]
(1.113a)
\[
y_j = C_j x_j + D_j u_j + \mathcal{D}_j w_j. 
\]
(1.113b)

where the \(w_j\) is the perturbation in the external inputs to the subsystem, \(u_j\) is the perturbation of bus voltage, and the \(y_j\) is the current injection to this bus. If the bus is purely load bus (all loads are assumed to be static load), the (1.113a) is omitted. The power system network is then described by
\[
y = Y u. 
\]
(1.114)
where $Y$ is network admittance matrix.

We furthermore assume $B_j$ for all non-reference generator buses. With these assumptions $w_j$ is totally isolated out as the system inputs, i.e. the $w_j$ is viewed as the system input into whole power network which is a black box and subject to reduction. Use (1.113) and (1.114) to eliminate current vector $v_j$, get

$$
\begin{bmatrix}
\dot{x}_r \\
0 \\
0 \\
\dot{x}_z
\end{bmatrix} =
\begin{bmatrix}
A_{dr} & B_{dr} & 0 & 0 \\
C_{dr} & J_{rr} & J_{rz} & 0 \\
0 & J_{zz} & J_{zz} & C_{dr} \\
0 & 0 & B_{dz} & A_{dz}
\end{bmatrix}
\begin{bmatrix}
x_r \\
u_r \\
0 \\
x_z
\end{bmatrix},
$$

(1.115)

where $A_d$, $B_d$, $D_d$ is block diagonal matrix, like $A_d = \text{diag}(A_j)$, and the $J = D_d - Y$. The subscript $r$ denotes the portion to be retained, whereas the $z$ marks the part to be reduced. The loads here are not reduced to facilitate the reduction of non-reference generators, but it can be reduced by simply using Ward-type reduction technique later on.

**The situation when only considering retaining single eigenvalue** In such situation, assume only one mode, $\lambda_i$ for example, of oscillation can be incited in the power system. It can be shown that,

$$
\dot{x} = \lambda_i x,
$$

(1.116)

which implies that, notice $x = [x_r^T \ u_r^T \ u_z^T \ x_z^T]^T$,

$$
\dot{x}_z = \lambda_i x_z. 
$$

(1.117)

Substitute the (1.117) into the (1.115), then get the selective modal analysis description of the power system,

$$
\begin{bmatrix}
\dot{x}_r \\
0 \\
0 \\
\dot{x}_z
\end{bmatrix} =
\begin{bmatrix}
A_{dr} & B_{dr} & 0 & 0 \\
C_{dr} & J_{rr} & J_{rz} & 0 \\
0 & J_{zz} & J_{zz} & C_{dz} \\
0 & 0 & B_{dz} & A_{dz} - \lambda_i I
\end{bmatrix}
\begin{bmatrix}
x_r \\
u_r \\
0 \\
x_z
\end{bmatrix}.
$$

(1.118)

From the last line of the last equation, we can get

$$
B_{dz} u_z = (A_{dz} - \lambda_i I) x_z
$$

(1.119a)

$$
\Rightarrow \quad x_z = (A_{dz} - \lambda_i I)^{-1} B_{dz} u_z
$$

(1.119b)

$$
\Rightarrow \quad \hat{J}_{zz}(\lambda_i)
$$

(1.119c)

where $\hat{J}_{zz}(\lambda_i) = C_{dz}(A_{dz} - \lambda_i I)^{-1} B$. The last equation along with the (1.119c), we can derive it by simply using Gaussian Elimination technique,

$$
\begin{bmatrix}
\dot{x}_r \\
0 \\
0 \\
\dot{x}_z
\end{bmatrix} =
\begin{bmatrix}
A_{dr} & B_{dr} & 0 & 0 \\
C_{dr} & J_{rr} & J_{rz} & 0 \\
0 & J_{zz} & J_{zz} + \hat{J}_{zz}(\lambda_i) & C_{dz} \\
0 & 0 & B_{dz} & A_{dz}
\end{bmatrix}
\begin{bmatrix}
x_r \\
u_r \\
0 \\
x_z
\end{bmatrix}.
$$

(1.120)
1.7. Synchronic Modal Equivalent method

Notice the term \( \tilde{J}_{zz} = C_{dz}(A_{dz} - \lambda_i I)^{-1}B \) is added in the SMA representation. It is a diagonal matrix and has clear physical meaning in that each diagonal entity can be viewed as voltage controlled current sources added between each bus and ground.

**Multi-Mode Equivalent** To derive a condition suitable for such application where multiple oscillation modes are excited, firstly we need to inspect the single oscillation case again. The voltage evolution at \( u_z \) is

\[
u_z = \mu_{zi} e^{\lambda_i t}
\]

(1.121)

where the magnitude value \( \mu_{zi} \) is the solution of \( u_z \) in (1.115) by making substitutions, \( \dot{x}_r = \lambda_i v_{ri} \) and \( \dot{x}_{zi} = \lambda_i v_{zi} \). The \( v_{ri} \) and \( v_{zi} \) are the portions of right eigenvectors \( v_i \) corresponding to the eigenvalues of \( \lambda_i \) to be reduced and the remaining ones respectively. Notice that the substitution is only meaningful under an assumption of single eigenvalue of the “state matrix” of (1.115), and the the solution \( \mu_z \) is not time dependent because of the elimination of differentiation, i.e. the differential algebraic equations equation (1.115) becomes pure algebraic by replacing \( \dot{x}_r \) and \( \dot{x}_{zi} \) with \( \lambda_i v_{ri} \) and \( \lambda_i v_{zi} \). In the same fashion, the multiple mode equivalent has an “universal” \( \tilde{J}_{zz} \) that makes the following equality hold,

\[
\tilde{J}_{zz} = \tilde{J}_{zz}(\lambda)\mu_{zi}
\]

(1.122)

for all \( \lambda_i \) and \( v_i \) corresponding to the modes in the selected chord. Typically, a large network has much more generators than the modes to be retained in the equivalent. Thus, many solutions to (1.122) are usually permitted. Then the equivalent of (1.120) for multiple modes case is

\[
\begin{bmatrix}
\dot{x}_r \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
A_{dr} & B_{dr} & 0 \\
C_{dr} & J_{rr} & J_{rz} \\
0 & J_{xz} & J_{zz} + \tilde{J}_{zz}
\end{bmatrix}
\begin{bmatrix}
x_r \\
u_r \\
u_z
\end{bmatrix} .
\]

(1.123)

The \( \tilde{J}_{zz} \) is independent of any \( \lambda \). Though it does not have a diagonal structure in general, it can be modelled as multi-port voltage control “admittance.” The current injection at each bus is linearly determined not only by the voltage of the single bus but other \( z \)-buses as well. In the end, resume the input terms,

\[
\dot{x}_r = A_{dr} x_r + B_{dr} u_r + B_r w_r
\]

(1.124)

\[
0 = C_{dr} x_r + J_{rr} u_r + J_{rz} u_z
\]

(1.125)

\[
0 = J_{xz} u_r + (J_{zz} + \tilde{J}_{zz}) u_z
\]

(1.126)

To compare different methods discussed in the introduction part, some discussions concerning this method are compiled in the end of the [53]. Those
1. INTRODUCTION

methods were proposed in the history sharing similar structure, where a co-
herency identification procedure is performed before aggregation to give a
better delimitation between parts of area we have interests and parts we do
want remove. The aggregation methods are more or less limited to the aggre-
gation of synchronous generators. However, in modern power system, there
are new challenges. The renewable power generations are preferably modeled
as active loads, which are not discussed in previous studies. Moreover, the co-
herency identification methods may successfully group synchronous generators
but fail to specify how to group load buses, which should also be addressed
considering that the distributed power generation gains the popularity in the
modern power system.
Chapter 2

Coherency and coherent area identification

Synchronization is one of many properties of alternating current power system that for a long time have been the focus of power system research and recently stimulated studies on complex network, like low-dimensional dynamics of heterogeneous populations. The roots of many power system failures reside deeply in the loss of such synchronization, and it helped developing various kinds of controllers, where the most famous example might be power system stabilizer (PSS) [29, 2]. On the other hand, synchronization can be used for facilitating system reduction. The synchronization phenomenon provides invaluable information for decomposing large power system into several communities or coherent areas, in which each generators and load buses tend to have very similar behaviors during disturbances, i.e. during the lose of synchrony.

In a power network comprising of \( n \) synchronous generators. The (strict) synchronization phenomenon, or (strict) coherency can be defined by

\[
\dot{\delta}_1 = \dot{\delta}_2 = \cdots = \dot{\delta}_n = \omega \approx \omega_0
\]  

(2.1)

where \( \omega_0 \) is the reference value of generator rotor speed that is usually 50 Hz in Europa and 60 Hz in USA. Based on power generation and consumption, \( \omega \) is close but not necessarily equal to \( \omega_0 \). Synchronous generator in a coherent group can still maintain the equality of (2.1), that is, for generators \( \#1, 2, \ldots, m \) of a single (strict) coherent area and for another (strict) coherent groups of generators \( \#m+1, m+2, \ldots, m+l \), there exists the following relations during disturbances,

\[
\dot{\delta}_1 = \cdots = \dot{\delta}_n \neq \dot{\delta}_{m+1} = \cdots = \dot{\delta}_{m+l}.
\]  

(2.2)

Such synchronization and coherent properties have been a topic in power system studies [59, 10] and some in-depth properties have also been researched.
In nonlinear system field under non-uniform Kuramoto oscillator models [58, 16, 14].

In real systems, the strict coherency (2.2) hardly exists. Instead, the synchronization within a coherent group is overlaid by another layer of intra-area oscillation. Due to structure properties of the power network such as the inertia distribution, strength of network connections, the deviations of \( \delta \) between generators of different coherent areas are much larger than generators within the same coherent area during a short time after system disturbance [29]. Later on, we will formulate it mathematically. The objective of this chapter is to decompose the whole power system network into several coherent area comprising of coherent generators as well as transmission links, load buses, etc. The coherent synchronous generator grouping services as a first step toward generator aggregations that will be discussed in chapter 3, and the grouping of load buses helps aggregating dynamic loads (c.f. chapter 4) and renewable energy sources.

In the section 2.1 of this chapter, we will introduce power system model especially the swing equations that characterize in detail how the power system loses synchronization and the how the coherent areas emerge from it. Then section 2.2 will introduce the tolerance-based aggregation algorithm developed by [10]. Finally, the last section presents my creative method of grouping other power network components.

### 2.1 Rotor angle dynamics and power system networks

For an alternating current power system network, as mentioned earlier, synchronization of rotor angle speed is the most conspicuous quantity that has direct impact on the power system frequency. For the \( i \)-th generator, the simplest model of rotor angular speed dynamics is the *swing equation* [2, 29],

\[
\frac{2H_i}{\omega_0} \ddot{\delta} = P_{m,i} - P_{e,i}
\]  

(2.3)

where \( P_{m,i} \) is the mechanical power provided by the generator prime mover, \( P_{e,i} \) is the power demanded by the power system network, and \( \omega_0 \) is the reference angular speed or reference system frequency. Usually, in an analogous manner to Newton’s low on rotating object, we define \( m = \frac{2H_i}{\omega_0} \) as the generator’s moment of inertia. Moreover, since \( \delta \) is a second-order derivative, we can augment the dynamics into an equivalent description of two first-order differential equations,

\[
\dot{\delta} = \omega,
\]  

(2.4a)

\[
\frac{2H_i}{\omega_0} \dot{\omega} = P_{m,i} - P_{e,i}.
\]  

(2.4b)
Static components of the power network can be described by admittance matrix $Y = [y_{ij}]$ [2, 15]. It is defined in such way that, if $i \neq j$ the $y_{ij} = -\frac{1}{Z_{ij}} = -g_{ij} - jb_{ij}$, otherwise for $j = i$ the $y_{ii}$ is defined as $y_{ii} = -\sum_j y_{ij} + y_{i0}$. The $Z_{ij}$ is the impedance of the branch between node $i$ and $j$, and the $y_{i0}$ is the shunt connection at bus $i$. The power flow is the solution of active and reactive power balance at each nodes. For active power,

$$0 = P_{e_i} - \sum_{j, j \neq i} \Re \{V_i e^{j\theta_i}(V_j e^{-j\theta_j} - V_j e^{-j\theta_j})\} - g_{i0} V_i^2 \tag{2.5a}$$

and for reactive power,

$$0 = Q_{e_i} - \sum_{j, j \neq i} \Im \{V_i e^{j\theta_i}(V_j e^{-j\theta_j} - V_j e^{-j\theta_j})\} + b_{i0} V_i^2 \tag{2.5b}$$

where $y_{ij}$ is the entry of admittance matrix $Y$, and the overline represents the complex conjugation, like $\bar{x} = \Re\{x\} - j\Im\{x\}$.

**Linearization** By combining rotor angle dynamic and power flow balance, the whole power system is described by the mixture of differential and algebraic equations,

$$\dot{x} = f(x, y) \tag{2.6a}$$

$$0 = g(x, y), \tag{2.6b}$$

where $x$ is the dynamic variables $(\delta, \omega)$ and $y$ is $(\theta, V)$. Solve for $y$ from the second equation and substitute the results in the first equation, we get

$$\dot{x} = f(x).$$

[44] Generally speaking, it is quite hard to get the solution to a non-linear differential system, especially when considering a typical real power system contains several hundreds to several thousands of nodes and links. For short term stability study and small disturbance, we can linearize it near a stable equilibrium,

$$P_{m_i} = P^*_{m_i} \quad \delta_i = \delta^*_i \quad \omega_i = \omega^*_i.$$

According to [44], the linearization gives

$$\begin{bmatrix} \Delta\delta \\ \Delta\omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ K & -B \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\omega \end{bmatrix} \tag{2.7}$$

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where

\[ [K]_{ij} = \kappa_{ij} = \begin{cases} \sum_{j \neq i} \kappa_{ij} & \text{if } i \neq j \\ \frac{-\omega_0 E_i E_j}{2H_i} (G_{ij} \sin(\delta_i - \delta_j) - B_{ij} \cos(\delta_i - \delta_j)) & \text{if } i = j \end{cases} \]  

(2.8)

\[ B = \text{diag} \left\{ \frac{\omega_0}{2H_i} \right\} \text{diag} \left\{ \frac{1}{d_i} + \frac{1}{R_j} \right\}. \]  

(2.9)

where \( G_{ij} \) and \( B_{ij} \) are the effective admittance between the internal nodes of generator \( i \) and \( j \). The equation (2.7) can also be rewritten for the \( i \)-th generator in two separate components,

\[ \begin{bmatrix} \Delta \dot{\delta}_i \\ \Delta \dot{\omega}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} \Delta \delta_i \\ \Delta \omega_i \end{bmatrix} + \sum_{j=1}^{n} \kappa_{ij} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta_j \\ \Delta \omega_j \end{bmatrix} =: P_1 + \sum_{j=1}^{n} \kappa_{ij} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta_j \\ \Delta \omega_j \end{bmatrix} =: P_2. \]  

(2.10)

The \( P_1 \) characterizes damping factors of a generator, which also reflects artificial damping introduced by system controller, while \( P_2 \) represents the influence of network structure. For coherent area identification, much consideration is given into the second part \( P_2 \).

### 2.2 Tolerance based generator grouping

As introduced at the starting of this chapter, generators in a complex power system exhibit a clearly community features during a short interval after a system disturbance, in which generators from a coherent area (community) tend to share a similar behavior during the post-fault transient. This section dedicates to introducing widely accepted method the Tolerance-based coherency grouping algorithm \[10\] to identify such such coherent community. Based on the standards of to which extent the (2.2) holds true, this algorithm results in a loosely and tightly defined coherent generator grouping.

For the first step, we need to inspect the \( K \) matrix in equation Equation 2.7. Recall, in the last section, we characterize \( K \) as a matrix describing network properties, \( B \) indicates generator damping and control. If we neglect \( B \) matrix, the power system can be described by a set of second-order differential equations,

\[ \Delta \ddot{\delta} = K \Delta \delta. \]  

(2.11)

where \( K_{ij} \) is defined by (2.9). If denote \( m_i = \frac{2H_i}{\omega_0} \), an usual entry of \( K \), \( \kappa_{ij} \), \( i \neq j \)

\[ \kappa_{ij} = \frac{E_i E_j}{m_i} \left( B_{ij} \cos(\delta_i - \delta_j) - G_{ij} \sin(\delta_i - \delta_j) \right). \]
2.2. Tolerance based generator grouping

The \( K \) can be decomposed into symmetric and skew symmetric parts,

\[
K = \frac{K + K^T}{2} + \frac{K - K^T}{2}
\]

\[
= \left[ \frac{E_i E_j}{m_i} B_{ij} \cos(\delta_i - \delta_j) \right] := K_{\text{sym}}
\]

\[
- \left[ \frac{E_i E_j}{m_i} G_{ij} \sin(\delta_i - \delta_j) \right] := K_{\text{asym}}
\]

For a power transmission network, it has large \( \frac{|B_{ij}|}{|G_{ij}|} \) ratio, and \( \delta_i - \delta_j \) usually has a smaller value \( \frac{\cos(\delta_i - \delta_j)}{\sin(\delta_i - \delta_j)} \), then

\[
\begin{bmatrix}
\frac{E_i E_j}{m_i} B_{ij} \cos(\delta_i - \delta_j) \\
\frac{E_i E_j}{m_i} G_{ij} \sin(\delta_i - \delta_j)
\end{bmatrix}
\gg 1.
\]

Therefore, \( K \) is approximately a symmetric matrix with real eigenvalues. The symmetric part of \( K \), \( K_{\text{sym}} \), is named as dynamic connectivity matrix \([5]\). If a system \((2.11)\) is interconnected, i.e. the whole system itself is the only disconnected component, \( K \) can have only one null eigenvalue with eigenvector \( v_1 \in \text{span}\{1\} \) corresponding to a uniform shift of rotor angles that gives a freedom of how to choose reference point for angular measurement. If the power system is stable, all other eigenvalues of \( K \) are strictly negative real numbers,

\[
\max \{ \lambda(K) \} \leq 0,
\]

\[
\ker(K) = \text{span}\{1\}.
\]

Suppose \( \lambda \) is eigenvalue of \( K \), power system oscillation frequency \( f \) is approximated from

\[
\sqrt{\lambda} = -j2\pi f.
\]

Let \( \Lambda = \{0, \lambda_2, \ldots, \lambda_n\} \), sorted in ascending order, be a set of eigenvalues of \( K \), eigen-analysis of \( K \) gives

\[
K = PJP^{-1}, \quad (2.12)
\]

in which \( J \) is the Jordan matrix whose diagonal entries are \( 0, \lambda_2, \ldots, \lambda_n \). \( P = [v_1 \ v_2 \ldots \ v_n] \), \( v_i \) is the (generalized) eigenvector corresponding to \( i \)-th oscillation mode (if not all eigenvalue are distinct, count them by their algebraic multiplicity). \( P \) is invertible and it can be used to define a coordinates \( z \in \mathbb{R}^n \) for the same space of \( \delta \) such that \( z = P^{-1} \delta \). Under new set of variables \( z \), the original dynamics system, can be described by

\[
\dot{z} = \Lambda z, \quad (2.13)
\]
2. Coherency and coherent area identification

Each element of $z, z_2, z_3, \ldots, z_n$ is now uncoupled, the equation above can be equivalently described by a set of one-dimensional differential equations,

$$\dot{z}_i = \lambda_i z_i,$$

for $i = 1, \ldots, n$. The $z_i$ represents the $i$-th oscillation mode. One the other hand, $\delta = Pz$ can be expanded like

$$\delta_k = v_{k1} z_1 + v_{k2} z_2 + \cdots + v_{kn} z_n,$$

where $(v_{k1}, \ldots, v_{kn})$ is called modal shape representing how much the $i$-th mode observed in $\delta_k$. Based on choices of oscillation modes and the modal shape, we can get a grouping of coherent generators in an interconnected power system. Usually when developing reduced order equivalent, the slow-modes (low frequency oscillation modes) are often of the main concern since it is usually associated with large inertia (c.f. subsection 3.2.2), a “collective” motion. By contrast with many literatures, which often exclude the oscillation mode of $\lambda = 0$ usually for the purpose of controller design, in which case the oscillation mode associated with $\lambda = 0$ is uncontrollable, we would rather keep the oscillation mode with $\lambda = 0$ along with other $m$ slowest oscillation modes in order to maintain a proper structure of reduced order equivalent, that is, to make $K_{\text{red}}$ (the counterpart of $K$ matrix in reduced system model) still be a Laplacian matrix. The choices of interested oscillation modes can be more flexible. We can handpick these modes or by Singular Value Decomposition etc.

Let $S$ be

$$S = \begin{bmatrix} v_1 & v_{i2} & \cdots & v_{im} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

(2.14)

where $v_1, v_{i2}, \cdots, v_{im} \in \{v_1, v_2, \ldots, v_n\}$ are eigenvectors we want to keep in the reduced system. The eigenvector $v_1 \in \{a1 : a \in \mathbb{R}_+\}$ corresponds to eigenvalue of $\lambda_1 = 0$. Let $w_1, w_2, \ldots, w_m$ be the rows of $S$, $\gamma \in (0, 1]$ the tolerance factor, we can define the coherent factor as follows

$$c_{ij} = \begin{cases} \frac{\langle w_i, w_j \rangle}{|w_i||w_j|} - \gamma & \text{if } i \neq j \\ 0 & \text{if } i = j, \end{cases}$$

(2.15)

which constitute the coherency matrix,

$$C = [c_{ij}].$$

(2.16)
2.2. Tolerance based generator grouping

Figure 2.1: Clusters of four normalized vectors \( \mathbf{w} \) on the model shape plot

The \( C \in M_n(\mathbb{R}) \) is symmetric matrix. Usually if coherent generators \( i \) and \( j \) are coherent, they exhibit similar oscillation modes shortly after initial perturbation, i.e. the angle between \( \langle \mathbf{w}_i, \mathbf{w}_j \rangle \) is much smaller (see Figure 2.1 for the case of two oscillation modes). Based on this observation, the *Loosely Coherent Area Identification* method can be summarized in following rules

i. If \( c_{ij} > 0 \), then we deem generator-\( i \) and generator-\( j \) are coherent. For convenience, we denote this as \( i \sim j \).

ii. (Transitivity) If \( i \sim j \) and \( j \sim k \), then \( i \sim k \).

The coherent areas identified by this method is called *Loosely Coherent Area Identification Method*, resulting in several disjoint sets of loosely coherent generators. If we denote the \( i \)-th set of loosely coherent synchronous generators as \( \mathcal{G}_i \), the full set of synchronous generators as \( \mathcal{U} \), we have

\[
\mathcal{U} = \bigsqcup_a \mathcal{G}_a \quad a \in 1, 2, \cdots m. \tag{2.17}
\]

However, one drawback of such method is that not all pairs of generators from a grouping set \( \mathcal{G}_a \) have \( c_{ij} \geq 0 \), \( i, j \in \mathcal{G}_a \). And each time you may even get different generator groupings based on how you index generators. To counter with such situation, *Tightly Coherent Area Identification* method moves generators that have the most different mode shape into another group and, if necessary, setup up a new group of their own such that generators of each coherent area have stronger coherency. Iterator through such steps till the generator coherency of each area cannot be stronger. These rules are summarized below and shall be performed after finishing *Loosely Coherent Area Identification Method*,

2. Coherency and coherent area identification

i. Define \( C_a = C[G_a, G_a] \) as a submatrix of \( C \), that is, \( C_a = [c_{ij}] \) \( \forall i, j \in G_a \). If any column sum of \( G_a \) is negative, we move out of the original set the generator corresponding to the negative-sum column to build up a new generator group of its own.

ii. A partition \( \mathcal{P} \) of set \( \mathcal{G} \) is a set of many nonempty subsets of \( \mathcal{G} \) such that \( \mathcal{G} \) is a disjoint union of these subsets. We say a partition \( \mathcal{P}_a \) is tighter than another partition \( \mathcal{P}_b \) over the same set \( \mathcal{G} \) if

\[
1^T(G_{a1} \oplus G_{a2} \oplus \ldots \oplus G_{a|a|})1 > 1^T(G_{b1} \oplus G_{b2} \oplus \ldots \oplus G_{b|b|})1 \quad (2.18)
\]

where \(|a|\) denotes the cardinality of \( \mathcal{P}_a \), i.e. the number of areas in the first partition, whereas \(|b|\) denotes the cardinality of \( \mathcal{P}_b \).

Iteratively apply step i) and ii) to improve the coherent area partitions until no more improvement can be made. The above process is called Tight Coherent Area Identification Method.

2.3 Load bus aggregation

The loose coherent area identification and tight coherent area identification methods result in a grouping of coherent generator, but not load buses. For the reduction of generator dynamics, as in the traditional background of system reduction that focus on transient stability analysis (issue in the first swing of oscillation) of a power system with no little renewable power sources, getting a grouping of coherent generators is enough, and the load buses can be eliminated by Kron reduction (some may call it as Ward reduction). System loads can be converted into constant impedance equivalents and will eventually be aggregated as well. However, for longer time stability analysis the dynamics of load and renewable power sources is no longer appropriate. The aggregation of ZIP loads will be treated in chapter 4. This section dedicates into the question like which load bus shall be associate to which coherent generator group? This problem along with clustering of the synchronous generator are sometimes categorized together being called community detection in the study of complex network \([48, 5]\), and have already been widely applied in many applications such as sociology, parallel computing, biology etc.

Unlike the problem of clustering synchronous generators, in which the system network is effectively trimmed into “skeleton” of interconnected generator terminal nodes whose dynamics linearization is captured in matrix \( K \) of (2.7), clustering load buses should be done in original network and each node except for generator buses, load buses and other buses with active controller is usually associated with no dynamics. Their community structure is decided
2.3. Load bus aggregation

Figure 2.2: a typical transmission line of π model

purely by admittance matrix. Moreover, the resulting load clusters should be subordinate to generator clustering since we hope these “external” areas, comprising of less important generators and load buses in its vicinity, will eventually collapse into several equivalent dynamic elements at the area border, and the controllers applied on some synchronous generators may force some load buses that are separated by a long electrical distance into the similar pattern of behavior. As an example of such influence, we may consider the wide-area damping controller (WADC). It can artificially relate two distant generators together such that they and the load buses in their vicinity tend to have similar behavior pattern during system transients.

In what follows, we will be presented two methods of load bus clustering. The first one is my original algorithm based on some facts of algebraic theory, and the second one is adapted from community detection methods in [48].

2.3.1 The first method: based on generator influence factors

Prior to our discussion, we need to make some necessary assumptions about the power system under our consideration. Firstly, all transmission lines have high $\frac{X}{R}$ ratio; secondly, all synchronous generators are in II-order model; all the loads are in ZIP-like; and finally, no Flexible AC Transmission System (FACTS) compensators, like Static Synchronous Series Compensator (SSSC), Static VAR Compensator (SVC), etc. are assumed in our network. However, the following algorithm is possible to be extended in more complicate cases. For example, we can modify our admittance matrix to appropriately reflect the changes towards “electrical distances” introduced by some FACTS compensators.

First of all, we consider a typical power transmission line like Figure 2.2. We can easily derive the complex power transmission as

$$S_{12} = V_1 \bar{T}_{12} = \bar{y}_{12} |V_1| e^{j\theta_1} (|V_1| e^{-j\theta_1} - |V_2| e^{-j\theta_2}) - y_{sh} |V_1|^2$$ (2.19a)

$$S_{21} = V_1 \bar{T}_{21} = \bar{y}_{21} |V_1| e^{j\theta_1} (|V_1| e^{-j\theta_1} - |V_2| e^{-j\theta_2}) - y_{sh} |V_2|^2$$ (2.19b)

expanded in sin, cos, and neglect any term about $g_{12}$ and $g_{sh}$ due to their
### 2. Coherency and Coherent Area Identification

Insignificance,

\[
P_{12} = |V_1^*||V_2|b_{12}\sin(\theta_1 - \theta_2) \tag{2.20a}
\]

\[
Q_{12} = |V_1^*||V_2|b_{12}\cos(\theta_1 - \theta_2) \tag{2.20b}
\]

If we express the equations above in Taylor series around a stable working point \((V_1^*, V_2^*, \theta_1^*, \theta_2^*)\), while assuming the \(|V_1|\) and \(|V_2|\) do not change very much,

\[
P_{12} = |V_1^*||V_2^*|b_{12}\cos(\theta_1^* - \theta_2^*) (\theta_1 - \theta_2) + \cdots
\]

\[
Q_{12} = |V_1^*||V_2^*|b_{12}\sin(\theta_1^* - \theta_2^*) (\theta_1 - \theta_2) + \cdots.
\]

Since \(|V_1^*||V_2^*|b_{12}\cos(\theta_1^* - \theta_2^*) > 0\) in most usual cases, we conclude more active power flow of \(P_{12}\), more angle difference of \((\theta_1 - \theta_2)\). Moreover, we have \(|V_1^*||V_2^*|b_{12}\sin(\theta_1^* - \theta_2^*) > |V_1^*||V_2^*|b_{12}\sin(\theta_1 - \theta_2)\), such that the \((\theta_1 - \theta_2)\) is more sensitive to active power \(P_{12}\) than to \(Q_{12}\). Based on these observations, the load bus can be clustered based on the relative voltage phase angle deviation during system disturbance under the influences of fluctuation of active injection, which eventually originate from generators. Another advantage of using voltage phase angle as the criterion rather than voltage magnitude is that the voltage magnitude is more easily to be changed due to relatively more reactive power sources supplied by reactive power compensator or transformer with tap changer.

Since we have already got a generator grouping, a nature idea is to extend the generator grouping from their terminal buses into other load buses whose voltage phase angles are much more likely being influenced by the these generators. Another layer of complexity added onto the problem is that there is a distinct direction of power flow from generators (the sources) to the load nodes (targets), or equivalently speaking, the graph induced by power system network is directed. For this reason, I suggest marking the direction of each network branch such that energy goes from generators to loads or to other generators. Take the Kundur’s system (Figure 2.3) for an example. We have already got a tight groupings for coherent synchronous generators, in which Gen 1 and Gen 2 are clustered as one coherent group while Gen 3 and Gen 4 constitute another coherent group.

Starting from Gen 3 and Gen 4, mark the direction of each network branch like the red arrows along each network branch. Along the marked direction, we can construct power flow path from energy sources of Gen 3 and Gen 4 into sinks that include loads and other synchronous generators. The loads are obvious power consumers, and synchronous generators can absorb some energy and convert into kinetic power. If one network branch is passed by more than one flow path and traversed in opposite directions, then we need to mark it in both direction, like the the branch between Bus 11 and Bus 4.
2.3. Load bus aggregation

Figure 2.3: (modified) Kundur’s 2-area-4-machine system, different areas (or equivalent called as communities, clusters) is marked with different color and is known a priori

(a) System with marked circular direction  
(b) Digraph of the cyclic network

Based on the marking and network topology, we can define a weighted digraph with an adjacency matrix like

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}, \]  

(2.21)

where

\[ a_{ij} = \max \left\{ |V^*_i||V^*_j|b_{ij}\cos(\theta^*_i - \theta^*_j), 0 \right\} \]

if there exists a branch between vertex (or bus) \( i \) and \( j \) and marked with orientation \( j \rightarrow i \). Possibly there may exist some cycles in the directed graph like the one shown in Figure 2.4a. This is allowed since there is persistent energy abstract from the network and there is nearly no energy stored in the power network. Assuming a constant non-zero energy injection at \( s_1 \), and no energy injected at \( s_2 \), if one circulates along the circular network and subtract the energy consumed by loads or absorbed by other generators, he will find no energy left when coming back at \( s_1 \). Reflected on the matrix \( A \), if we assume the energy source is \#1 on Figure 2.4a, other nodes are sinks, when we walk along \#1 \( \rightarrow \) \#2 \( \rightarrow \) \( \cdots \) \( \rightarrow \) \#6 we will at some point find the the entry \( a_{ij} \) cease to be non-zeros depending on the sign of \( \theta_i - \theta_j \). So the digraph associated with \( A \), like Figure 2.4b, might be somewhat different from Figure 2.4a. In this case, the branch between \#2 and \#3 is omitted because \( \theta_{\#3} \) and \( \theta_{\#4} \) happened to be the same, i.e. \( a_{34} = 0 \) \( a_{43} = 0 \). This step is in fact a path finding problem that can be done by breadth-first search and edge betweenness[48], which is an important component of the second method of load bus clustering that will
be explained later. For another thing, it is worth mentioning the adjacency matrix $A$ can be viewed as selected part of power flow sensitivity matrix.

Then we define a normalized adjacency matrix

$$A' = \frac{A}{\|A\|_\infty}.$$  \hfill (2.22)

The remaining “portion” of energy at node $i$ after taking $n$ steps traversing from source node $j$ to node $i$ is $A' e_j$, where $e_j$ is a column vector whose entries are all zero except for the $i$-th element, it is 1. Summing over all the length of $n$, the $i$-th element of

$$\sum_{n=0}^{\infty} A' e_i$$  \hfill (2.23)

gives a measurement about how much of energy coming from the source node $j$ to node $i$, which can be used as influence factor. Based on these discussion, we have the following algorithm

\begin{enumerate}
  \item Finish Power flow calculation and initialize power system at certain working point.
  \item Complete the Tightly or Loosely Coherent Area Identification and get a partition of generators, $\mathfrak{P} = \{ \{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_m}\}, \{G_{\beta_1}, G_{\beta_2}, \ldots, G_{\beta_m}\}, \ldots \} \{G_{|\mathfrak{P}|_1}, G_{|\mathfrak{P}|_2}, \ldots, G_{|\mathfrak{P}|_m}\}$ (where the “$| \cdot |$” denotes the cardinality of $\mathfrak{P}$).
  \item If the power simulation software has power flow sensitivity analysis, we can construct adjacency matrix $A$ by reading appropriate data. If not, we can construct it by ourselves.
  \item Starting from generator buses of an arbitrary coherent area, mark the orientation of each edge (branch) such that all elements of a signed path vector corresponding to path starting from one generator node (generator terminal bus) of the chosen area and ending at any load nodes, or any another generator node of another coherent area are all positive. If certain edges (branches) are traversed in both directions, then mark it twice in both directions. Iterate this step until no more unmarked paths traveling out from generators in the chosen area and ending at generators in another area, or ending at any load nodes are marked.
\end{enumerate}

Some remarks:

- The following fact be helpful to finding aforementioned path
2.3. Load bus aggregation

For an undirected graph with associated adjacency matrix \( A \), a path from \( i \) to \( j \) of length \( m \) exists if and only if the element \( [A^m]_{ij} \neq 0 \).

v. Construct matrix \( A' \) by \( [A']_{ij} = \frac{1}{\|A\|_{\infty}} [A]_{ij} \).

vi. Calculate

\[
v = \sum_{k=0}^{\infty} A'^k v^{(0)} = (I - A')^{-1} v^{(0)}.
\] (2.24)

where \( v^{(0)} \) is a column vector with all zero entries except for the \( i \)-th element being 1 corresponding to the generator at \( i \)-th node. Calculating the inversion of \( n \)-dimensional matrix needs \( O(n^3) \) in the worst case. If the longest path length \( l \) is known, we only need to calculate the summation up to \( k = l \). When \( k = l + 1, l + 2, \ldots, \), \( A'^k = 0 \). A conservative choice is setting \( l = n - 1 \) where \( n \) is the total number of nodes in a system.

vii. Define an appropriate threshold \( \gamma \). If \( |v_i| > \gamma \), then the \( i \)-th node is deemed closely related with the chosen group of coherent generators.

As an example, the algorithm is applied on a simple 2-area-4-machine test system with the results of influence factor calculation being shown in Figure 2.5. Each entry of vector \( |v| \) is plotted against buses index Figure 2.5. Then it can be seen which generators have great influence on a load bus. From Figure 2.5a where \( v_3 = v_4 = 1 \), that is, the 3-th and 4-th elements of \( v \) are 1. If we pick \( \gamma = 0.45 \) as the tolerance threshold, then Bus-\#9 – \#11 having influence factors greater than 0.45 are deemed strongly connected with generator buses Bus-\#3 and Bus-\#4, and should be categorized together with generator buses to complete the augmented coherent area. Because there are only two coherent...
groups for the simple Kundur’s system, the other load buses join Bus-#1 and Bus-#2 to compose another coherent group.

On the other hand, we notice the difference of $|v_9 - v_8|$ in Figure 2.5a has the largest value and $(9, 8) \in \mathcal{E}$ ($\mathcal{E}$ denotes the set comprising all possible branches in the power system). Thus, the edge $(9, 8)$ divides the total network into two separate parts. We can conclude that $(9, 8)$ is the weakest line or “bottleneck” that is the connection that should be the boundary of a bus cluster including Gen 3 and Gen 4. For improving the overall power system stability, we may strengthen this connection by introducing another parallel transmission line or adding ancillary controllers. Furthermore, I plotted influence factors of Gen 3 and Gen 4 in Figure 2.5b and Figure 2.5c separately.

2.3.2 The second method: based on shortest-path betweenness

Recent years with increasing activity in the study of network, there is increasing improvement on solving problems like graph partitioning, hierarchy clustering, etc. It turns out they share some kinds of similarity with the load clustering. Thus, we can modify some algorithms developed for clustering general network (usually undirected) for clustering load buses.

Community detection in general cases

In [48], prof. Newman introduced a community finding algorithm based on an technical report by Anthonisse at Stichting Mathematisch Centrum [3], in which a center piece is the concept of edge betweenness (or rush as called in Anthonisse’s original document). Here, we adapt the method in [45, 48, 3] for our own problem. A measurement qualifying the goodness of our clustering and some connections compared to previous method will be addressed afterwards.

Divisive method, dendrogram and other concepts In the aspects of general community detection and clustering, there are two kinds of method, agglom-
erative methods and divisive methods, whose concepts and differences can be explained in an artificially generated general network Figure 2.6 that is a kind of network proposed in [48], where it has 24 nodes evenly arranged into 4 compartments. Nodes within the same compartment have denser connections, whereas inter-compartment has relatively weaker connections. The relative “strength” of connections is indicated by the edge thickness. In the agglomerative methods, all edges are removed at the beginning; then, reconstruct link one by one based on “distance” (speaking of the edges) or “similarity” (speaking of the vertices) between nodes. Here the “distance” might be geographic (Euclidean) measurement between two locations, or allowable flow through a pipe system. The “similarity” might be coauthor relationship in the network of collaboration. From another point of view, at the beginning, the network has as many communities as its nodes since it is assumed having no links among them. Then we calculate and sort the “distance” or “similarity” among them. Finally, we reconnected the graph links with shorter distance or link the nodes with higher similarity at first then links with longer distance and less similar. Such method can be represented in Figure 2.7 if we start the clustering from the bottom level up till the highest level at which it is the original graph.

For the divisive methods, the process is reverse going down from the top level of Figure 2.7 by removing the most “center” at each step. The “center” edge is the one that most flow with fixed source and target goes through. Taking for an example the small tonnage seaborne shipment from London to Bombay, we have many possible routes, 1) through Strait of Gibraltar, Suez Canal and Bab-el-Mandeb, 2) through Cape of Good Hope. Obviously route-1 is more “center” than the route-2 in the sense it the first route has shorter distance. Later on, we will formulate the concept of betweenness, which is the rigorous measurement of “center” concept. Equivalently, that is to say we have a complete graph at the beginning, then we remove one by one the most “center” edge, hopefully it will partition the graph into appropriate communities.

Community evaluation in unweighted graph and betweenness For now, we assume the graph in question is undirected, and then move onto a weighted case, which is a little more complicate in some sense. In the beginning, we consider another problem. We known how a graph is divided into multiple communities, but how we measure the community betweenness between one community and another. Starting from two communities comprising of only one nodes (say \(v_1\) and \(v_2\) in this case), the obvious choice of such community betweenness is calculated in the following way. In the first step, we find all cut sets,

\[
\mathcal{S} = \{(v_i, v_j) \in E(\mathcal{G}) : (v_i, v_j) \text{ separates the two communities}\},
\]
then we count the short paths that go from one community to another while also passing through every edges in the cut, finally the smallest number of all short paths that route through any of the cut set can be defined as **betweenness**. For the normal cases, each community has more than one vertices and many edges, the calculation of **community betweenness** can be analogically applied except the replacing the single target and source nodes of the shortest path to multiple single and target nodes in the communities having many nodes and edges. The definition of **community betweenness** is justified. We cannot replace shortest paths by any paths because, like we con not definition of metric in Euclidean space, any paths other than shortest path has no meaning of closeness. On the other hand, we must count all shortest paths passing through *every* edges in a cut set because removing only one edge in a cut set that has multiple edges cannot successfully separate two communities, which will eventually underestimate the true **community betweenness**. By making use of the **community** one may be able to find a minimal cut set that is most “center” to connection between two communities in question. However, a more important concept is the **edge betweenness** that characterizes how center an edge is, and should be used in a **divisive** method of community detection. A fast algorithm of edge betweenness calculation based on Anthonisse’s definition * is introduced in [48], where the graph is assumed to be unweighted and weighted version is certainly possible and has been discussed in [47]. The following content is basically adapted from these two sources, with some consideration given for our specific application in clustering load buses. At first, we look at the algorithm for unweighted graph. The whole algorithm is divided into two subroutines. The first is in fact breadth-in shortest path searching, shortest path searching and edge betweenness calculation. For the first subroutine, it is in fact a breadth-in algorithm. It is carried out by the following steps,

**i.** Arbitrarily choose one vertex as initial vertex, assign distance \(d_s = 0\) and repetitiveness** \(r_s = 1\)**

Then we are about to find all shortest paths from the chosen initial vertex to every other vertex in the graph.

**ii.** assign \(d_i = d_s + 1\) to every vertex adjacent to the initial vertex \(i\), and repetitiveness \(r = r_s\).

**iii.** For every vertex \(j\) adjacent to vertex \(i\), depending on different cases

\*In his original document, it is called rush. But the author of [48] argues “**edge betweenness**” is a better word due to its similarity to **vertex betweenness** defined in [19].

\†In original algorithm[48], the name for \(r\) is “weight” representing how many times this edge is included in any short path. Here, for distinguish with the weight of weighted graph, we rename it as “repetitiveness”.
2.3. Load bus aggregation

![Diagram of a test system for edge betweenness analysis](image)

Figure 2.8: A test system for edge betweenness analysis

![Additional diagrams for edge betweenness analysis](image)

Figure 2.9: The betweenness calculation for a directed simple graph (the numbers besides each edge are not edge weight but edge betweenness.)

\[
\begin{align*}
\text{If } d_j \text{ is not defined for vertex } j, & \text{ then assign } d_j = d_i + 1 \text{ and } r_j = r_i, \\
\text{If } d_j = d_i + 1, & \text{ then update } r_j = r_j + r_i, \\
\text{If } d_j < d_i + 1, & \text{ this edge is not part of shortest path, do nothing.}
\end{align*}
\]

iv. repeat the previous steps until there is no remaining vertex who is assigned distance \( d \) but its neighbors are not.

After finding out all the shortest paths starting from the chosen vertex, we then calculate the betweenness contributed by these paths.

v. Find every “leaf”, \( t \), in the graph that is the end vertex of all paths previously derived.

vi. For each vertex \( i \) adjacent to \( t \), start assigning the first set of \( (i, t) \) the edge betweenness \( b_{(i,t)}^{(s)} = \frac{r_i}{r_j} \). The superscript \( (s) \) indicates the edge betweenness
is contributed by shortest paths from vertex $s$ as source.

**vii.** Starting from $i$ whose adjacent edges $(i, t)$ are assigned edge betweenness, assign upwards from the edge most far away from the chosen source $s$ to the source $s$ along the identified shortest path (as can be seen from the distance variable $d$) the edge betweenness. The betweenness of next upward edge is the sum of all betweennesses of edge immediately downwards along any short path, then multiply it by $\frac{r_{ij}}{r_{ii}}$. For example, the next edge upwards from $(i, k)$ is $(j, i)$, and $(j, i)$ is also involved in many shortest paths. Then, the betweenness should be assigned as $b_{(j,i)}^{(o)} = \frac{r_{ij}}{r_{ii}}(1 + b_{(i,k)}^{(o)} + b_{(i,l)}^{(o)} + \cdots)$, where the $\cdots, (j, i), (i, k), \cdots, (j, i), (i, l), \cdots$ are parts of shortest paths.

**viii.** Repeat the process until the source node $s$ is reached.

Right now, we have find all betweenness contributed from all shortest paths from one chosen source $s$ to every other reachable nodes. Finally, we should repeat the process on every node in the graph, that is the final steps,

**ix.** Repeat the steps item i. to item viii. for every node as initial node.

For an example, let us consider the example of Figure 2.8, which is a directed graph such that the shortest path must follow the indicated arrows, but undirected version is certainly possible by allowing the edges to be traversed in both directions. At the beginning, we choose $x_1$ as the initial node, and calculate the edge betweenness by following the algorithm above. The calculation result in shown in Figure 2.9a, where “$d_i | r_i$” labels besides a vertex indicate the distance and repetitiveness, and the label beside an edge indicates the edge betweenness. Since every vertex is reachable by following paths from $x_1$, all vertices and edges in the graph are labeled. Betweennesses calculated from other initial vertex, except those from which the shortest paths have only one edge, are also included in Figure 2.9. By adding betweenness calculated from all possible initial vertices together, we complete the betweenness calculation of the test system. For the directed graph, edge $(x_1, x_2)$ has the largest betweenness. It is plausible because this edge is essential for the whole graph connection, whose removal losses two paths from $x_1$ to $x_7$. More interesting, edge $(x_3, x_4)$ has the largest betweenness value along the path $x_1, (x_1, x_3), x_3, (x_3, x_6), x_6, (x_6, x_7), x_7$. Normally, one might argue the three edges along the shortest path should have the same importance, but $(x_3, x_6)$ is unarguably more center than $(x_1, x_3)$ and $(x_6, x_7)$ along the path. Edge betweennesses results are shown in Figure 2.11 for undirected graph. The edge betweennesses results changed a lot in comparison with directed version. Edge edge $(x_1, x_6)$ and $(x_6, x_7)$ have the largest largest edge betweennesses and other edges symmetric along the axis $x_3$–$x_4$ also have same values. The symmetricity arises from the symmetry graph, and the largest edge betweennesses of $(x_1, x_2)$ and $(x_6, x_7)$ separate the subgraph $G({x_1})$ and subgraph $G({x_2, x_4, x_5, x_7})$, two of the most distinct subgraphs.
Modularity While we have an edge betweenness value for each edge in a graph, we can remove the most “center” edge (having highest betweenness) to separate a graph into several induced subgraphs. But how should we define the strength of interconnection in a graph? This is answered in the concept of modularity [48]. Usually we expect the communities we identify should have higher value of modularity [46] in the respective of the original graph without any removal of edge or vertex. First, we define a matrix $B \in M_k$ for a graph clustered into $k$ communities, such that the $B_{ij}$ is the fraction of all edges from community-$i$ to community-$j$. Owning to the fact that an edge of an undirected graph can be traversed in both directions in the shortest path problem—a cornerstone in the construction of our algorithm, we need to regard a directed edge with one direction (foreword or backward) as one edge when constructing $B$, but ought to count undirected edge as two directed edges. We also define a vector $t \in [0, 1]^n$ where $t_i = \sum_j B_{ij}$. The modularity is defined as,

$$Q = \sum_i (B_{ii} - \bar{t}_i^2) = \text{Tr} \{ B \} - 1^n B^2 1^n$$

(2.25)

where $\bar{t}_i^2$ calculates the change of a connection starting from and ending at community-$i$ for the same set of vertices assuming having same sets of communities but totally random connections between them. If the each community have same amount of connections to other communities and within the community, then the modularity is $Q = 0$. If having more internal connections than outgoing connections, the modularity measurement is large positive value, indicating a strong community components.

For an example, revisit the graph Figure 2.11, the undirected version of Figure 2.8. If regarding the subgraph induced by $\{x_2, x_4, x_5, x_7\}$ as one community and the subgraph induced by remaining vertices as the other community, we will identify two communities. The modularity of this community classifica-
2. Coherency and coherent area identification

Then the modularity for the assumed community division can be calculated as

\[ Q = \frac{1}{2} + \frac{1}{4} - \left( \left( \frac{1}{2} + \frac{1}{8} \right)^2 + \left( \frac{1}{8} + \frac{1}{4} \right)^2 \right) = 0.21875. \]

**Break down the graph**  When finishing edge betweenness calculation for every edge, we do get a good judgment on their relative importance in the graph, but the graph is still connected (if the original graph is connected). The next step is to divide the connected graph (or connected components) into several communities by making use of our just calculated edge betweennesses. This task can be achieved by the following algorithm[48],

1. Calculate the edge betweenness (in Newman & Anthonisse’s definition [48, 3]) by following the steps above or using other algorithms,
2. remove edges with highest edge betweenness,
3. calculate modularity for the different communities,
4. repeat Step i. to Step iii., stop the algorithm when modularity of current community division is large enough, or reaching desired number of communities.

**Weighted graph**  For the discussion of community detection algorithm up to now, we assume the graph is unweighted. However, for our purpose of clustering load buses in a power system where the network is modeled as weighted digraph (2.21), we need to extend the algorithm to include weighted graph. If giving close look into the problem, we would find such extension is not trivial. For a weighted graph, we can always find two factors that determines the community structure within a network. For the first, a group of interconnected vertices having more connections within group than connections to other group tends to become community; for the second, some vertices having strong connections between each other also tends to become community. One such extension [47] starts from a simple idea that is to convert the weighted graph into unweighted one with multiple edges. Without going into the math-
emathematical details, we make the following assumptions that is necessary to carry
on previous conclusions into weighted graph,

**Assumption 1.1** For a given directed graph $G$ or a undirected graph seen
as directed one where each edge is equivalent to bidirected edge,

a) The weight of edges can be infinitely small but larger than zero, i.e. the
edge weight $w$ is allowed to be $w \downarrow 0$.

b) The number of edges between two vertices is allowed to be infinitely
many, i.e. $|\{(x_i, x_j) \in E(G) : x_i, x_j \in V(G)\}| \to \infty$ is allowable, where
the $| \cdot |$ denotes the cardinality of set.

c) Under the first two assumptions, there is no qualitative change to the
properties of finite graph.

d) The edge weight reflects the strength of connection between two end
vertices.

Given a weighted directed graph and its weight $\{w_1, w_2, \ldots\}$, if we are lucky
to to find the “greatest common divisor”\footnote{These assumptions are also implicitly included in \cite{47}} $g$ then we can replace an edge
having weight $w_{ij} = mg$, for all $(i, j) \in E(G)$ with a batch of $m \in \mathbb{N}$ edges
connecting vertex $i$ and $j$ with unified weight of $w = g$. In the case when
finding a real common divisor is not possible, we can take the advantage
of analysis technique by setting $g \downarrow 0$ for $g \in \mathbb{R}$, then assume each edge in
original graph can be replaced by infinite many edges, $m \to \infty$ for $m \in \mathbb{N},$
of unified weight of $g$ such that $w_{ij} = mg$. Due to our assumptions above,
we can apply the original edge betweenness calculation algorithm onto the
newly developed infinite multigraph. Assuming the betweenness for edge $(i, j)$
in original weighted graph is $b_{ij}$, it can be verified that the edge betweenness
of one of possibly many edges connecting vertex $i$ and $j$ in the multigraph
is $b_{ij}/w_{ij}$. For procedures of removing edges with high betweenness value and
modularity calculation, the algorithms are similarly applied. However, some
cautions should be taken that we should remove multiple edges connecting the
same two vertices all together at the same time because the original graph is
not a multigraph, but we just represent it in the form of multigraph for easing
our discussion.

Another thing that deserves attention is about the definition of weights. Taking
the power system for an example, we may model the power system network
by using line impedance to construct adjacency matrix, which generates

\footnote{The greatest common divisor is for integers. Here, we mean a greatest common divisor
analogically defined in such way: $g \in \mathbb{R}_+$ is greatest common divisor (real number) of $x$ and
$y$ if and only if $g$ is the largest real number such that $\frac{x}{g}$ and $\frac{y}{g}$ are integer. However, it can
be proved (c.f. Appendix B.1) that pairs of real numbers do not have great common divisor
if they are not commensurable.}
Coherency and coherent area identification

2. Coherency and coherent area identification

A proper undirected weighted graph, or we can use admittances or admittance matrix for our network graph. Obviously, these two choices will result in dramatically different community divisions. Based on our Assumption 1.1, the weight should reflect the strength of edge, the admittances should be used for defining edge weight rather than impedances. From another point of view, the equivalent impedance of $m$ parallel lines of equal impedance $z_{ij}$ is $\frac{z_{ij}}{m}$, so the conversion from weighted graph to unweighted multigraph by above method is not valid anymore.

In summary, the algorithm for evaluating community in a weighted directed or undirected graph is as follows,

1. Assume the graph is unweighted, calculate the edge betweenness by following the algorithm steps i. to ix. on page 54–56.
2. Divide each edge betweenness for unweighted graph by the its weight as the edge betweenness for the weighted graph.
3. Remove the edge with highest betweenness such that two or more disconnected subgraphs emerge. Calculate the modularity under the community structure. Stop removing edges if the modularity is high enough or the desired number of communities is reached. Dendrogram plot might be built during this process to help the determination of communities.

Applied onto power system

Back to our goal of clustering load buses, the power system network resembles the weighted system to a great extent. However, there exist many differences between them. For the first, unlike the normal weighed graph the buses (or vertices in the parlance of graph theory) in a power network has different roles. Much like “master-slave” relations, the active and reactive power is injected into generator buses (it may also have load at the generator buses, so we may define the power injection as net injection) whereas the active power can be extracted from load buses or no extraction. The reactive power can be extracted at load buses by some reactive power compensators. Disregarding the reactive power flow as in defining the network weights (2.21), we then have a graph with predefined sources and targets which are the generator buses and load buses respectively. For the second difference, clustering load buses must be subordinate to clustering generator buses that is handled by generator grouping techniques as discussed in the Section 2.2. Therefore, we need to modify the algorithm above to take into account these particularities arising from the power system.

Since the power system has distinct sources and targets we need to modify the
betweenness calculation algorithm, which involves finding out shortest path starting from every vertex as sources and to every reachable vertex as targets. Here, we adopt the same techniques as before. When finding the shortest paths (calculating the $d$) and calculating the repetitivenesses (calculating the $r$), we mark the generator bus from one of coherent area as a source and other load buses as targets, particularly we need to mark as targets other generator buses as well. For the leaves, we choose them to be the vertices corresponding to reachable load buses with net active power extraction and other reachable generator buses other than the one marked as source. Then we calculate the contributions to the edge betweenness by going backwards from marked as leaves to the single source, which is calculated as

$$b_{(i,j)} = b_{(i,j)}^{(s_1)} + b_{(i,j)}^{(s_2)} + \cdots$$

(2.26)

where the $b_{(i,j)}^{(s_k)}$ is calculated by

$$b_{(i,j)}^{(s_k)} = \frac{r_i}{r_j}$$

for edges immediately adjacent to target $t$, and

$$b_{(i,j)}^{(s_k)} = \frac{r_i}{r_j} (1 + b_{(i,k)}^{(s_k)} + b_{(i,j)}^{(s_k)} + \cdots)$$

for normal edges, in which the edges $(i,k)$, $(i,l)$, etc. are adjacent to edge $(i,j)$ and have already been assigned $b_{(i,k)}^{(s_k)}$ or $b_{(i,l)}^{(s_k)}$. Repeat the process until the edge betweenness contributions when taking every generator bus (or vertex) as source have been calculation. Summing up the contributions to each edge together as in (2.26), we complete the edge betweenness calculation. For the rest parts, just like community identification in general graph, we remove the edges with high betweenness until we have satisfied number of bus clusters or vertex communities or modularity value is greater than a threshold.

As an example, we demonstrate the algorithm in original Kundur’s system Figure 2.3 (without branch Bus 3–Bus 4, and not yet marked any direction), which is represented in the form of graph Figure 2.12, to give a clear view. In the graph expression, severely parallel transmission lines are represented as
2. Coherency and coherent area identification

Table 2.1 The edge betweenness of Kundur’s 4 machine system

<table>
<thead>
<tr>
<th>Edge</th>
<th>Edge betweenness</th>
<th>Edge</th>
<th>Edge betweenness</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 5)</td>
<td>20.0</td>
<td>(6, 7)</td>
<td>56.0</td>
</tr>
<tr>
<td>(2, 6)</td>
<td>20.0</td>
<td>(7, 8)</td>
<td>60.0</td>
</tr>
<tr>
<td>(3, 11)</td>
<td>20.0</td>
<td>(8, 9)</td>
<td>60.0</td>
</tr>
<tr>
<td>(4, 10)</td>
<td>20.0</td>
<td>(9, 10)</td>
<td>56.0</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>36.0</td>
<td>(10, 11)</td>
<td>36.0</td>
</tr>
</tbody>
</table>

Table 2.2 Number of inner and external connections

<table>
<thead>
<tr>
<th>Communities</th>
<th>No. of inner edges</th>
<th>No. of external edges</th>
<th>$B_{i*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[2, 4, 5, 7]</td>
<td>8</td>
<td>2</td>
<td>$\frac{5}{10}$</td>
</tr>
<tr>
<td>[1, 3, 6]</td>
<td>2</td>
<td>10</td>
<td>$\frac{1}{10}$</td>
</tr>
</tbody>
</table>

| total number of edge | 20 |

one edge. The weighted of this graph is defined by adjacency matrix of (2.21). For the edge betweenness calculation, we can get the edge betweenness result in Table 2.1. From the results, one can find the highest edge betweenness emerges in edge (7, 8) and (8, 9). Removing any one of the edges will separate the system in two, perfectly matching the two real areas (see the color of Figure 2.12). The modularity under such load buses clustering is calculated as, $Q = \frac{2}{5} + \frac{1}{2} - \left(\frac{2}{5} + \frac{4}{10}\right)^2 - \left(\frac{1}{10} + \frac{1}{2}\right)^2 = 0.29$. Due to the algorithm property, the community division comes from removing edge rather than vertex, we included the boundary vertex Bus 7 in the first community.

In practical situations, the boundaries between various coherent areas are largely constrained by geographic limitations and power system facility ownerships. For system buses that has no associated dynamics, their voltage phasors are mainly decided by “algebraic” equation $0 = g(x, y)$. Arranging one load bus into a different area incorrectly may not have significant degradation of final results. For the generator buses, however, we should try to keep tightly coherent generators in a single area or break down the area into several different subareas, but we should avoid classifying a large generator into other coherent areas to which it doesn’t actually belong. The load bus clustering method makes it possible to aggregate renewable generation buses (nodes) together with traditional synchronous generation buses, serving as the first step to aggregate renewable generations.
Chapter 3

Synchronous generator aggregation

After finishing Coherent Area Identification, we usually get a partition of the whole power system into several strongly connected sub-systems, the coherent areas. One thing deserving a particular notice is that such identification method is based on generator’s rotor angle and angular speeds—two of the most conspicuous properties of AC power system. Its basic properties are epitomized in Kuramoto model \[1, 30\]. My extensions outlined in section 2.3 of previous chapter, extend the such coherent aggregation into other load buses and branches. From the system points of view, we get a clearly demarcation between each closely related areas, where synchronous generators and load buses all share similar behavior during rotor angle and angular speed transient. However, the actual order of dynamics has yet reduced. This chapter dedicates to the task of reducing dynamic orders of power system in each area, which amounts to reducing the number of synchronous generators. By contrast, the next chapter, chapter 4, focuses on reducing “algebraic” variables by eliminating load buses and aggregating loads.

In subsequent sections of this chapter, two aggregation method are introduced, the first one is the so called Podmore method, which is basically an averaging method and the second one is Two-time-scale method profound having theoretical supports in Center Manifold Theory, Singular Perturbation Theory.

3.1 Two-time-scale based method

To begin with, we first inspect the Initial Value Problem (IVP) of a nonlinear, generally non-autonomous dynamics,

\[ \dot{x} = f(t, x), \]  

subject to the boundary condition \( x(t_0) = x^0 \). Note the superscript at \( x^0 \) denotes initial value, so as \( x^0, y^0, \) etc. in the following of this section. If
3. Synchronous generator aggregation

$U$ is an open subset of $\mathbb{R}^{n+1}$, we suppose $f \in C(U,\mathbb{R}^n)$, i.e. $f$ is continuous differentiable, though $f$ can be relaxed to be uniformly continuous with respect to the first argument and at least locally Lipschitz with respect to the second argument. These assumptions guarantee existence of a unique solution [61], $\Phi : U \times \mathbb{R} \to U$, which is called flow or orbit. $U$ is open neighbor of $(x^0,t_0)$, usually we only consider $t \in [t_0,T]$ for $T < \infty$ which has the properties

$$\Phi(x,0) = x$$
$$\Phi(\Phi(x,t),s) = \Phi(x,t+s).$$

Very often $\Phi(x,t)$ is denoted as $\Phi_t(x)$ to be regards as a one-parameter group where the “group law” is $\Phi_{t,s} = \Phi_t \circ \Phi_s = \Phi_s \circ \Phi_t$ and the identity is $\Phi_0$. Sometimes, $\Phi(x,t)$ is simply denoted as $x(t)$ without explicitly specifying the initial values as in this text. A lemma of Picard-Lindelöf Theorem states that the solution $x \in C^{k+1}([t_0,T],\mathbb{R}^n)$ if $f \in C^k(U,\mathbb{R}^n)$[61].

Usually, in contrast to Equation 3.1, the power system has the structure of so-called “differential algebraic equations”\(\dagger\)

$$\dot{x} = f(t,x,y), \quad (3.2a)$$
$$0 = g(t,x,y). \quad (3.2b)$$

These equations can be easily converted into Equation 3.1 form by solving for $y$ in the second equation and substitute for $y$ in the first equation. Furthermore, $f$’s and $g$’s do not depend on $t$ explicitly if not considering shift of operating condition or faults, so it is autonomous system. With the aforementioned observations in mind, we are ready to embark on the discussion of Singular Perturbation and Center Manifold, which turn out to be the foundation of our reduction method.

In what follows of this section, the subsection subsection 3.1.1 will introduce Singular Perturbation Method followed by some discussion on Center Manifold in subsection 3.1.2 and subsection 3.1.3 before applying these techniques to reducing power system as discussed in section 6.1 of chapter 6.

3.1.1 Introduction to singular perturbation methods

The Perturbation Methods were first proposed in prominent pioneering works by Lagrange and Laplace in between 18th and 19th century. At first, it were to solve the motions of planets in the solar system when considering the influence between each planets — the Three Body Problem. Since then such methods have drawn great amount of attentions in 20th century [63, 64, 50]

\(\dagger\)Remark: one-parameter group is not a group

\(\dagger\)The $g$’s in Equation 3.2b are in fact transcendental equations because the presence of sin and cos terms.
3.1. Two-time-scale based method

and eventually became the first model used in control and system theory and had played significant roles in quantum mechanics till being replaced by much more advanced models. Some of contents of this subsection is based on or greatly influenced by [28].

We assume a small parameter \( \varepsilon \in [0, \varepsilon^*] \), characterizing how large the system is “disturbed”. Generally speaking, the choice of \( \varepsilon \) is the most difficult part, even “it is a art rather than a science”[24]. However, we only need to look at the following special case, which is particularly pertinent to the reduction of power system where the disturbance term multiplies the highest derivative.

If such disturbance \( \varepsilon \) is added into IVP Equation 3.1, the disturbed system can be assumed having the following structure

\[
\begin{align*}
\dot{x} &= f(x, z, \varepsilon, t) = \bar{f}(x, z, t) + \mathcal{O}(\varepsilon), \\
\varepsilon \dot{z} &= g(x, z, \varepsilon, t) = \bar{g}(x, z, t) + \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \to 0
\end{align*}
\] (3.3a) (3.3b)

subject to the initial condition

\[
x(t_0) = x^0 \quad \text{and} \quad z(t_0) = z^0
\]

where \( x \in \mathbb{R}^n \), \( z \in \mathbb{R}^m \), \( t \in [t_0, T] \), \( T < \infty \) or \( t \in [t_0, \infty) \), i.e. \( t \) should be in a compact set. \( \bar{f}(x, z, t) = f(x, z, \varepsilon, t) \) and \( \bar{f}(x, z, t) = f(x, z, \varepsilon, t) \). \( f \), \( g \), \( \bar{f} \), \( \bar{g} \) are requested at least locally Lipschitz continues in \( x \) and \( z \), uniformly in \( t \), though the original Tikhonov theorem requires \( f, g \in C^\infty \)[66, example 2.1].

Furthermore, we assume the solution to this disturbed system is denoted by \( (t, \varepsilon), z(t, \varepsilon) = \Phi ((x^0, z^0), t) \). The \( \mathcal{O}(\varepsilon) \) is the Bachmann-Landau notation describing limitation behavior:

Let \( u(x) \) and \( v(x) \) be functions defined on same subset real line and co-domain with proper metrics, let \( a \in [-\infty, \infty] \), we note

\[
v(x) = \mathcal{O}(u(x)) \quad \text{as } x \to a
\]

if and only if \( \exists M : 0 < M < \infty \) such that

\[
\|v(x)\| \leq M\|u(x)\| \quad \forall x \geq x_0,
\]

Because \( \mathcal{O}(\varepsilon) \) occurs in the definition of a vector fields that should be able to permit existence and uniqueness of integral curve, we request at least \( \mathcal{O}(\varepsilon) \), like \( g(x, z, \varepsilon, t) \), is at least locally Lipschitz continues in \( x \) and \( z \), uniformly in \( t \). Similarly, \( \bar{f}(x, z, 0, t) = \mathcal{O}(1) \), \( \bar{g}(x, z, 0, t) = \mathcal{O}(1) \). Sometimes, \( x(t, \varepsilon) \) and \( x(t, \varepsilon) \) are denoted as \( x_\varepsilon(t) \) and \( z_\varepsilon(t) \). Apart from the assumed perturbed dynamic system, we further make the following assumption (Tikhonov) [63, 28], which eventually leads to the renowned Tikhonov’s theorem.

---

* Any second or higher order dynamic can be trivially expressed in a set of first derivative form. This is to say, the disturbance term should be (pre-)multiplied onto the derivative of highest order.
3. Synchronous generator aggregation

Assumption 3.1 (Tikhonov on dynamic system)

a) Under the same domain of Equation 3.1, the implicit equality of $0 = g(x, z, 0, t)$ or $0 = f(x, z, t)$ has $k \geq 1$ isolated solutions,

$$\bar{z} = \bar{h}_i(\bar{x}, t), \quad i = 1, 2, \ldots, k. \quad (3.4)$$

The “isolated solutions” is understood in such a way that $\bigcap_{i=1}^{k} W_i = \emptyset$ where $W_i = \{(\bar{x}, \bar{z}) : \bar{h}_i(\bar{x}, t) - \bar{z} = 0\}$.

b) Restrict the domain into the distinct set $W_i$, let $x$ be a parameter in the domain of interest and $z^0 = z(t_0) = \bar{\phi}(x^0, t_0)$ be the initial condition. Consider the “partial” system

$$\dot{z} = g(x, z, 0, t) = \bar{g}(x, z, t). \quad (3.5)$$

Note trajectory decided by Equation 3.4 is a set of equilibrium of the system (3.5). We then assume the trajectory $\bar{z}(t)$ is uniformly asymptotically stable with regards to $x^0$ and $t_0$.

c) The eigenvalues of the linearization $D_z \bar{g}$ along the trajectory of $(x(t), \bar{h}(x, t))$ all has strictly negative real part, that is,

$$\Re \text{spec } \{D_z \bar{g}(x, z, 0, t)|_{z=\Phi(x, t)}\} \leq -\gamma < 0. \quad (3.6)$$

These three assumptions have a intimate connections with the concept of center manifold and slow manifold, which will be postponed into the next subsection for a in-depth discussion. In fact, the assumption b) and c) are quite strong stability conditions. Particularly, assumption c) implies assumption b) by Hartman-Grobman theorem [23, 21, 63, 64].

Now we are ready to introducing the prominent Tikhonov Theorem without presenting its proof.

Theorem 3.1 (Tikhonov on dynamic system) Consider the perturbed initial value problem (3.3). If the Assumption 3.1 are satisfied, we have

$$\lim_{\varepsilon \downarrow 0} x_\varepsilon(t) = \lim_{\varepsilon \downarrow 0} \Phi_\varepsilon((x^0, z^0), t) = \bar{x}(t), \quad \forall t \in [t_0, T] \quad (3.7a)$$

$$\lim_{\varepsilon \downarrow 0} z_\varepsilon(t) = \lim_{\varepsilon \downarrow 0} \Phi_\varepsilon((x^0, z^0), t) = \bar{z}(t), \quad \forall t \in [t_1, T], \quad (3.7b)$$

for some $t_0, T \in \mathbb{R}, t_0 < t_1 \leq L$. Note the limitation operandi, that is $\lim_{\varepsilon \downarrow 0}$ rather than $\lim_{\varepsilon \to 0}$ because of the assumption $\varepsilon \in [0, \varepsilon^*]$.

From Tikhonov’s Theorem, we can draw a quite important conclusion: if there is isolated trajectory decided by $0 = g(x, z, 0, t)$ and the system $\dot{z} = g(x, z, \varepsilon, t)$ of the only assumed variable $z$ is asymptotically stable uniformly in $x$ and $t_0$, then we can get a set of approximation to $\bar{x}(t)$ and $\bar{z}(t)$ for some interval. But how the solutions are approximated is answered by O’Malley-Vasil’eva Expansion.

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Theorem 3.2 (O’Malley-Vasil’eva) Consider the IVP problem,

\[ \dot{x} = f(x, z, \varepsilon, t), \]
\[ \varepsilon \dot{z} = g(x, z, \varepsilon, t) \]

for \( x(t_0) = x^0 \) and \( z(t_0) = z^0 \) satisfies Assumption 3.1. Assume \( t_0 = 0 \). If \( f \) and \( g \) can be expressed in asymptotical expansion up to order \( (m + 1) \), then

a) the system solutions can also be expressed in the asymptotical expansions of the form

\[ x_\varepsilon(t) = \sum_{n=0}^{m} \varepsilon^n a_n(t) + \sum_{n=1}^{m} \varepsilon^n a_n \left( \frac{t}{\varepsilon} \right) + O(\varepsilon^{m+1}), \]
\[ z_\varepsilon(t) = \sum_{n=0}^{m} \varepsilon^n b_n(t) + \sum_{n=0}^{m} \varepsilon^n b_n \left( \frac{t}{\varepsilon} \right) + O(\varepsilon^{m+1}). \]

b) The \( T \) of \([t_0, T]\) and \([t_1, T]\) on which Equation 3.7 are valid is a \( O(1) \) term.

c) The \( z_\varepsilon(t) \) can be expressed by \( \tilde{z}(t) = \tilde{h}(x, t) \) and a series of asymptotical expansion on \( \varepsilon \),

\[ z_\varepsilon(x, t) = \tilde{h}(x, t) + \varepsilon z_1(x, t) + \varepsilon^2 z_2(x, t) + O(\varepsilon^3). \]

3.1.2 Two time scales and boundary layer system

The Tikhonov’s Theorem manifests the attraction of solutions \( x_\varepsilon \) and \( z_\varepsilon \) to the unperturbed intergal curve \( \tilde{x} \) and \( \tilde{z} \) as \( \varepsilon \downarrow 0 \). O’Malley-Vasil’eva Expansion further Tikhonov’s argument into an asymptotic expansion. However, we may ask how the differences \( |x_\varepsilon - \tilde{x}| \) and \( |z_\varepsilon - \tilde{z}| \) evolves. The pursuit of answer to this equation leads to a in-depth arguments about multiple time scale structure mainly developed by Russian Mathematicians A. N. Tikhonov and A. B. Vasil’eva[62, 63]. From the point of view in geometric perturbation theory, N. Fenichel’s researches [17, 18] provide a geometric explanation of such attraction properties in the vicinity of \( \tilde{x}(t) \) and \( \tilde{z}(t) \). But first, we need to specify the task of our system approximation problem.
3. Synchronous generator aggregation

If \( \varepsilon \) is assumed to be 0 in the (3.3), \( f \) and \( g \) satisfy Assumption 3.1, we can define the “quasi-steady-state” system by the following mixed differential functions

\[
\dot{x} = f(\tilde{x}, \tilde{h}(\tilde{x}, t), 0, t) = \tilde{f}(\tilde{x}, t),
\]

\[
\dot{z} = \tilde{h}(\tilde{x}, t).
\]

We set \( \tilde{x}(t_0) = x^0 \), then \( \tilde{z}(t_0) = \tilde{h}(x^0, t) \) rather than \( z(0) = z^0 \), which is different from \( \tilde{h}(x^0, t) \) generally. Thus, we can get a uniform approximation of \( x \) by

\[
x(t) = \tilde{x}(t) + O(\varepsilon) \quad \forall t \in [t_0, T].
\]

However, we cannot get approximation for \( z \) uniformly in \( t \). The best we can do is

\[
z(t) = \tilde{x}(t) + O(\varepsilon) \quad \forall t \in [t_1, T]
\]

where \( t_1 > t_0 \). In the initial interval \( t \in [t_0, t_1] \), that is also called “Boundary Layer”, there is discrepancy between \( z(t) \) and \( \tilde{z}(t) \), but it will decrease with increasing \( t \) due to the assumptions \( \text{ReSpec} \left( D_z \tilde{g}(x, z, 0, t)|_{z=\tilde{h}(x, t)} \right) \leq -\gamma < 0 \). When \( t_1 < t \leq T \) the discrepancy of \( |z(t) - \tilde{z}(t)| \) remains a \( O(\varepsilon) \) value. To give a detailed description on the dynamics in initial interval, we assume a correct term \( \hat{z}(t) \) such that

\[
z(t) = \tilde{z}(t) + \hat{z}(t) + O(\varepsilon)
\]

is an approximation of real dynamics \( z(t) \) uniformly in \( t \in [t_0, T] \). Now we determine the dynamic property of \( \hat{z} \).

To simplify our argument, we assume \( t_0 = 0 \). For \( t_0 \neq 0 \), we can simply set a new time variable \( t' = t - t_0 \) and \( x(t') = \Phi(x(t_0), t') = \Phi(x^0, t) \). Substitute O’malley-Vasil’eva expansions for \( x \), the slow variables, and \( z \), the fast variables,

\[
\dot{a}_0 + \varepsilon \dot{a}_1 + \varepsilon^2 a_2 = f(a_0, a_1 + \varepsilon a_1, b_0 + \varepsilon b_1 + \beta_0 + \varepsilon \beta_1, t, \varepsilon) + O(\varepsilon^2)
\]

\[
\varepsilon \dot{b}_0 + \varepsilon \beta_0 = g(a_0, a_1 + \varepsilon a_1, b_0 + \varepsilon b_1 + \beta_0 + \varepsilon \beta_1, t, \varepsilon) + O(\varepsilon^2).
\]

If we assume

\[
\tilde{x}(x) = a_0(t) + \varepsilon a_1(t) + \varepsilon^2 a_2(t) + \cdots.
\]

i.e. the parts of \( x(t) \) that can be expressed in a regular expansion of \( \varepsilon \). We have

\[
\dot{a}_0 + \varepsilon \dot{a}_1 = f(a_0 + \varepsilon a_1 + \cdots, b_0 + \varepsilon b_1 + \cdots, t, \varepsilon).
\]

\[
\varepsilon \dot{b}_0 + \varepsilon^2 \beta = g(a_0 + \varepsilon a_1 + \cdots, b_0 + \varepsilon b_1 + \cdots, t, \varepsilon).
\]
3.1. Two-time-scale based method

Then, we need to identify the coefficients of \( \varepsilon^n \) for \( n = 1, 2, \ldots \). Using the analytic property of \( f \) and Taylor series*:

\[
\begin{align*}
&f(a_0 + \varepsilon a_1 + \cdots, b_0 + \varepsilon b_1 + \cdots, t, \varepsilon) = f(a_0, b_0, t, 0) \\
&+ f_x(a_0, b_0, t, 0)a_1 \varepsilon + f_y(a_0, b_0, t, 0)b_1 \varepsilon + f_\varepsilon(a_0, b_0, t, 0)\varepsilon + \cdots.
\end{align*}
\]

(3.18)

Then equate both sides of quality, we can find out the coefficient of corresponding \( \varepsilon^n \) term. For \( \varepsilon^0 \),

\[
\begin{align*}
&\dot{a}_0 = f(a_0, b_0, t, \varepsilon) + O(\varepsilon), \quad \text{(3.19a)} \\
&0 = g(a_0, b_0, t, \varepsilon) + O(\varepsilon). \quad \text{(3.19b)}
\end{align*}
\]

We have as \( \varepsilon \downarrow 0 \)

\[
\begin{align*}
&\dot{a}_0 = f(a_0, b_0, t, 0), \\
&0 = g(a_0, b_0, t, 0). \quad \text{(3.20a)} \quad \text{(3.20b)}
\end{align*}
\]

For the initial conditions, we set \( t = 0 \), as \( \varepsilon \downarrow 0 \),

\[
\lim_{\varepsilon \downarrow 0} x(0) = \sum_{n=0}^{\infty} \varepsilon^n a_n(0) + \sum_{n=1}^{\infty} \varepsilon \alpha_n \left( \frac{0}{\varepsilon} \right) + \cdots = a_0(0) = x^0.
\]

We get the initial condition for \( a_0 \). The second equation has degenerated into an equality, the \( b_0(t) \) is no longer decided by any differential equations but by

\[
0 = g(a_0(0), b(0), t, 0), \quad \text{i.e.}
\]

\[
b_0(0) = \tilde{h}_1(x^0, 0).
\]

That is exactly the quasi-steady-state system (3.11). For \( \varepsilon^1 \), similarly,

\[
\begin{align*}
&\dot{a}_1 = f_x(a_0, b_0, t, 0)a_1 + f_y(a_0, b_0, t, 0)b_1 + f_\varepsilon(a_0, b_0, t, 0), \quad \text{(3.21a)} \\
&\dot{b}_1 = g_x(a_0, b_0, t, 0)a_1 + g_y(a_0, b_0, t, 0)b_1 + g_\varepsilon(a_0, b_0, t, 0). \quad \text{(3.21b)}
\end{align*}
\]

From system (3.15) deduct (3.19) and truncate any term of \( O(\varepsilon) \) or higher, we get

\[
\begin{align*}
&\varepsilon \dot{a}_1 (t/\varepsilon) = f \left( a_0(t), b_0(t) + \beta_0(t/\varepsilon), t, \varepsilon \right) - f \left( a_0(t), b_0(t), t, \varepsilon \right), \quad \text{(3.22a)} \\
&\varepsilon \dot{\beta}_0(t/\varepsilon) = g \left( a_0(t), b_0(t) + \beta_0(t/\varepsilon), t, \varepsilon \right) - g \left( a_0(t), b_0(t), t, \varepsilon \right). \quad \text{(3.22b)}
\end{align*}
\]

If define another time scale \( \tau = \frac{t}{\varepsilon} \), the equations above can be rewritten by the chain rule as

\[
\begin{align*}
&\frac{d}{d\tau} a_1(\tau) = f \left( a_0(\tau), b_0(\tau) + \beta_0(\tau), \tau, \varepsilon \right) - f \left( a_0(\tau), b_0(\tau), \tau, \varepsilon \right), \quad \text{(3.23a)} \\
&\frac{d}{d\tau} \beta_0(\tau) = g \left( a_0(\tau), b_0(\tau) + \beta_0(\tau), \tau, \varepsilon \right) - g \left( a_0(\tau), b_0(\tau), \tau, \varepsilon \right). \quad \text{(3.23b)}
\end{align*}
\]

* Here, for the purpose of applying Taylor series expansion, we assume \( f \) defined on \( \varepsilon \) on a tight open neighborhood of \( \varepsilon = 0 \). After arriving at our expansion (3.17), we adhere to our original assumption \( \varepsilon \in [0, t^*] \).

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We define (3.23) as “extended” boundary layer system. Qualitatively, we can see that the existence of $\beta_0$ has a direct effect only on $\alpha_1$ — the coefficient of a $\epsilon^1$ term, and an indirect effect on $\alpha_i$ for $i = 2, 3, \ldots$ through higher order of dynamics on $\epsilon$. That is why we can use expansion of $x$ up to $a_0$ to uniformly approximate evolution of $x$, but hardly can we use only $b_0(t)$ to give an approximation uniformly in $[0, T]$. Furthermore, let $\bar{z}_0 = b_0$, $\hat{z}_0 = \beta_0$, $\bar{x}_0 = a_0$, and neglect any dynamics of $\epsilon^1$, i.e. (3.23a), we have

$$
\frac{d\hat{z}_0}{dr}(r) = g(\bar{x}_0, \bar{z}_0 + \hat{z}_0, \epsilon, t) - g(\bar{x}_0, \bar{z}_0, \epsilon, t),
$$

s.t. $\hat{z}_0(0) = z_0^0 - \bar{z}_0(0)$, $\bar{z} = \bar{h}(x_0^0, 0)$.

The dynamic (3.1.2) is called General Boundary Layer in [28].

### 3.1.3 Center manifold theorem and slow manifold

To complete the arguments of multiple scale property and bring in some kinds of “rigor”, we need to inspect multiple-scale problem in the sense of geometry, there exists a complete theory called Geometric Singular Perturbation Theory dedicating to this topic. In the following paragraphs, however, we will be introduced the most minimal knowledge on Slow and Fast manifolds and finally Reduction Principle. Interested readers are referred to [69, 17, 18].

First let us inspect an autonomous ordinary differential equation

$$
\dot{x} = f(x)
$$

with $x \in \mathbb{R}^n$ and $f \in C^1$. Assume there exists an equilibrium $x_0 = f(x_0) = 0$. Following the usual practice, we linearize the system we get

$$
\dot{x} = Ax
$$

where $A = Df(x_0)$ is the Jacobian matrix. Suppose $A$ has $k_s$ eigenvalues with real parts strictly less than 0, $k_u$ eigenvalues with real parts strictly larger than 0, and $k_c$ with real parts exactly equal to 0. We have generalized eigenvectors, $w_i = u_i + jv_i$, associated with the eigenvalue $\lambda_i$ for each $i = 1, \ldots, n$.

For the linearized system, we can decompose the $\mathbb{R}^n$ into three categories,

\textbf{a) Stable subspace: } $E_s = \{\text{span}\{u_i, v_i\} : \Re\{\lambda_i\} < 0\}$, \hspace{1cm} (3.27a)*

* For the situation when $v_i$ exists, obviously $w_i = u_i + jv_i \not\in \mathbb{R}^n$. However we know for any matrix on real numbers, a complex eigenvector must has its twin—if there is eigenvector $u_i + jv_i$, there must exist $w_{i+1} = u_i - jv_i$. Moreover, we should notice $u_i \neq kv_i$, $k \in \mathbb{R}$, otherwise, for $A \in M(\mathbb{R})$ and $k \in \mathbb{R}$, $A(v_i + jk v_i) = \lambda(v_i + jk v_i) = \lambda(1 + jk)v_i$, that is to say $\lambda(1 + jk)$ should be an eigenvalue rather than $\lambda$. Thus, we can define two independent real vectors $u_i = (w_i + w_{i+1})/2$ and $v_i = -(u_i - w_{i+1})/2 \in \mathbb{R}^n$ for the eigenspace corresponding to the $i$-th eigenvalue.
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b) **Unstable subspace**: \( E_u = \{ \text{span}\{u_i, v_i\} : \Re\{\lambda_i\} > 0 \} \), \hspace{1cm} (3.27b)

c) **Center subspace**: \( E_c = \{ \text{span}\{u_i, v_i\} : \Re\{\lambda_i\} = 0 \} \). \hspace{1cm} (3.27c)

Since we know a set of generalized eigenvectors form a basis for \( \mathbb{R}^n \), it follows that \( \mathbb{R}^n = E_s \oplus E_u \oplus E_c \) where \( E_s \), \( E_u \) and \( E_c \) has dimension of \( k_s \), \( k_u \) and \( k_c \) respectively. A manifold \( W \in \mathbb{R}^n \) is invariant manifold under the flow \( \Phi_t(x) : W \to W, x \mapsto \Phi_t(x) \), generated by a smooth vector field \( f(x) \). An equilibrium point \( x^* \) is hyperbolic if no eigenvalues of its linearization \( Df(x^*) \) has real part equal to 0, i.e. \( k_c = 0 \). The following theorem states the relations between various category of invariant manifolds and subspaces.

**Theorem 3.3** Suppose \( f \in C^r \) is a smooth vector field, and \( x^* \) is a saddle point or \( x^*(t) \) travels along a periodic orbits. Then we have,

a) **Locally** unique \( k_s \)-manifold \( W_s(x^*) \) containing \( x^* \) tangent to subspace \( E_s \) at \( x^* \),

b) **Locally** unique \( k_u \)-manifold \( W_u(x^*) \) containing \( x^* \) tangent to subspace \( E_u \) at \( x^* \),

c) If \( x^0 = x \in W_g(x^*) \) is the initial point for the autonomous system \( \dot{x} = f(x) \), then \( x(t) \in C^r \) and \( x(t) \to x^* \) as \( t \to \infty \),

d) If \( x^0 = x \in W_g(x^*) \) is the initial point for the autonomous system \( \dot{x} = f(x) \), then \( x(t) \in C^r \) and \( x(t) \to x^* \) as \( t \to -\infty \).

e) A non-unique \( k_c \)-manifold \( W_c \).

Then the following (Local) Center Manifold Theorem in its simplest form bring us the following conclusions that is particularly pertinent to dynamic reduction [38]. More details can be found in [39, 51]

**Theorem 3.4 (Local Manifold Theorem for Flows)** Let \( f \in C^r (k \geq 1) \) be a vector field in \( \mathbb{R}^n \), assume \( f(0) = 0 \). Let \( \Phi_t(x) \) be the flow generated by the autonomous system \( \dot{x} = f(x) \). Separate the spectra of \( Df \) into two disjoint sets \( \text{Spec}\{Df(0)\} = \{\lambda_c \cup \lambda_{nc}\} \) where \( \Re\{\lambda_c\} = 0 \) and \( \Re\{\lambda_{nc}\} \neq 0 \). Then we can divide finite dimensional space \( \mathbb{R}^n \) into \( \mathbb{R}^n = E_c \oplus E_{nc} \), where \( E_{nc} = E_u \oplus E_s \).

a) Then there exist open neighborhood \( U \in \mathbb{R}^n \) around origin and an invariant manifold \( W_c \) containing the origin point (c.f. Theorem 3.3e)

b) The manifold \( W_c \) can be expressed as graph \( (x, h(x)) \) with \( h(0) \) and \( Dh(0) = 0 \).

Theorem 3.4b implies that all globally defined flow \( \Phi_t(x) \) while contained in a neighborhood \( U \) around a equilibrium points or period orbits for all \( t \) are actually contained in \( W_c \).

*can be extended to global uniqueness[6]
Theorem 3.4 is also called *Approximation Theorem*. For an solution on the neighborhood of the equilibrium points or periodic orbits of our concern, the solution converge to or diverge from the center manifold at an exponential rate. This will be much clear if we compare the Center Manifold Theorem with the asymptotic expansion (3.9) on the *autonomous* two-time-scale system modified from (3.8) which is generally a non-autonomous system,

\[
\begin{align*}
\dot{x} &= f(x, z, \varepsilon), \\
\varepsilon \dot{z} &= g(x, z, \varepsilon),
\end{align*}
\]

for \( x \in D \subseteq \mathbb{R}^n \) and \( z \in G \subseteq \mathbb{R}^m \). Substitute \( t = \varepsilon \tau \) for \( t \),

\[
\begin{align*}
\frac{d}{d\tau} x &= \varepsilon f(x, z, \varepsilon), \\
\frac{d}{d\tau} z &= g(x, z, \varepsilon).
\end{align*}
\]

Assume \( x \) and \( y \) are independent of \( \varepsilon \). If \( f \) and \( g \) are regular enough with respect to all arguments near \( x = x^*, z = z^* \) and \( \varepsilon = 0^* \), we can expand them in Taylor expansion, and we focus ourselves on \( O(1) \) and \( O(\varepsilon) \) terms. Replace the \( f \) and \( g \) on (3.29) by their Taylor expansion with respect to \( \varepsilon \), neglecting terms higher than \( O(\varepsilon) \),

\[
\begin{align*}
\frac{d}{d\tau} x &= f(x, z, 0)\varepsilon + D_{\varepsilon} f(x, z, 0)\varepsilon^2 + O(\varepsilon^3), \\
\frac{d}{d\tau} z &= g(x, z, 0) + D_{\varepsilon} g(x, z, 0)\varepsilon + O(\varepsilon^2).
\end{align*}
\]

Notice the right hand side of first equations starts from term of order \( \varepsilon^1 \) rather than \( \varepsilon^0 \) because we already have \( \varepsilon \) at (3.29a). It is obvious that \( x \) is a slow variable, \( z \) is a fast variable. Since the \( \varepsilon \) only exists on the right sides of equations, we can safely take \( \varepsilon \to 0 \). At the limitation,

\[
\begin{align*}
\frac{d}{d\tau} x &= 0, \\
\frac{d}{d\tau} z &= g(x, z, 0).
\end{align*}
\]

Obviously, the equilibrium (orbit) is decided by \( g(x, z, 0) = 0 \). Suppose it has isolated solution\(^1\),

\[
z = h(x),
\]

\(^1\)If we require the the domain of \( x, y \) to be compact, we can always find such solution and it is unique\(^6\).
3.1. Two-time-scale based method

with $Dh(x) = 0$ at equilibrium points $x = x^*$ or along the periodic trajectory decided by (3.31). According to Theorem 3.4, the $h$-induced graph $(x, h(x))$ defines a center manifold. To verify this, we check the local behavior near the solution (3.32), linearize the system at equilibrium and replace $x$ by the solution $x^* = h(z^*)$, we get

$$
\frac{d\Delta x}{d\tau} = \epsilon D_x f (x, h(x), \epsilon) \Delta x + \epsilon f (x, h(x), \epsilon) + \cdots
$$

$$
\frac{d\Delta z}{d\tau} = D_z g (x, h(x), \epsilon) \Delta z + g (x, h(z), \epsilon) + \cdots
$$

Obviously, at the limitation of $\epsilon \to 0$,

$$
\frac{d\Delta x}{d\tau} = 0,
$$

$$
\frac{d\Delta z}{d\tau} = D_z g (x, h(x), 0) \Delta z.
$$

Let $u = [x^T, z^T]^T$. The differential equations above are equivalent to

$$
\frac{du}{d\tau} = \begin{bmatrix} 0 & 0 \\ 0 & D_z g (x, h(x), 0) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \end{bmatrix}.
$$

Here we assume $D_z g (x, h(x), 0)$ has no eigenvalues on imaginary axis (hyperbolic), i.e. all eigenvalues are hyperbolic. Hence, $\begin{bmatrix} 0 & 0 \\ 0 & D_z g (x, h(x), 0) \end{bmatrix}$ has $n$ eigenvalues of $\lambda = 0$ with center subspace $\{[\Delta x^T, \Delta z^T]^T : \Delta x \in D \subseteq \mathbb{R}^n, \Delta z = 0_m\}$. The manifold $M_0$ is indeed tangent to the center subspace defined above, we are assured $M_0$ is the center manifold.

Compare dynamics (3.30) to (3.31), it is known that the flow generated by $\dot{x} = f(x, h(x), \epsilon)$ on center manifold $M_0$ is an approximation of error $O(\epsilon)$ to the manifold $M_\epsilon$ that is set of $(x, z)$ decided by the full dynamic (3.28) for every possible initial conditions $x^0$ and $z^0$ near $M_0$. If assume all the eigenvalues of $D_z g (x, h(x), 0)$ have negative real parts, then the stable manifold $M_s$ might bring the system near $M_\epsilon$ and keep staying there for the rest of time.

Compared with Tikhonov’s theorem and O’malley asymptotic expansion, the arguments based on center manifold does not require $D_z g (x, h(x), 0)$ only has eigenvalue with negative real parts but they should only be hyperbolic. The slow manifold usually exists when contraction or expansion along one direction (fast manifold) is larger than the other direction (slow manifold) within a neighborhood of critical points, either they are attracting or repelling critical points. If the slow manifold is “slow” enough, then the whole dynamic can be approximated by the slow manifold, like $\dot{x} = f(x, h(x), \epsilon)$ with $(x, h(x)) \in M_0$ from the previous example. However, Tikhonov’s theorem is quite general and can be applied in non-autonomous system.
3. Synchronous generator aggregation

3.2 Aggregation methods

Now we are ready to embark on our task of aggregating synchronous generators. Here we mainly focus on discussions about how to convert a group of generators into a nicely looking two-time scale model like the one in (3.8), how to apply Tikhonov’s theorem and O’Malley’s asymptotic expansion into its linearized model, and finally how to achieve a nonlinear approximation. Due to the non-linearity of power system, the main task concerning reduction is based on linearized model, while the nonlinear approximation will be developed later to approximate the reduced linearized system based on the same working points for creating reduced linearized model.

To give a broader picture, we consider the simple system that omits any dynamics of $e''_q$, $e''''_q$, etc., i.e. consider only second order model,

\[
\begin{align*}
M \ddot{\delta} &= f(\dot{\delta}, \delta, v, \theta) \tag{3.33a} \\
0 &= g(\dot{\delta}, \delta, v, \theta). \tag{3.33b}
\end{align*}
\]

where $M$ is a diagonal matrix of the generator inertia, $v$ is a vector of load bus voltage magnitudes, and the $\theta$ is a vector of voltage phase angle $\theta$. If see the (3.33) as constrains, we can have

\[
M \ddot{\delta} = F(\dot{\delta}, \delta, v(\delta, \dot{\delta}), \theta(\delta, \dot{\delta})). \tag{3.34}
\]

Based on previous discussion, especially the parts about coherent area identification, we know some groups of $\delta$’s in a coherent area tend to oscillate against another group, usually at a low frequency—slow dynamics; on the other hand, generators in a single coherent group oscillate against each other usually at higher frequency that corresponds to fast dynamics. Therefore, we need to define a projection matrix to separate these fast/slow dynamics, then the previous arguments about slow and center manifolds, O’Malley and Vasil’eva’s, and Tikhonov’s theorem can be applied for developing reduced order system.

In fact, we can develop aggregation algorithm either in (3.33) or in the (3.34). For preserving as much system structure as possible, which is valuable for reconstructing non-linear model after getting a reduced linearized model, the algorithm to be introduced below is based on (3.33). The cases under our consideration are little bit complicated that may draw our special attentions. First, not all generators are in second order model and simply neglect too many dynamic variables may sacrifice some degree of accuracy. Second, how to relate the abstract model we have derived from linearized model to real modeling requires special attention. The whole algorithm for handling with the reduction problem is arrange in three-layer structure. The first layer is aggregating generator models such that generators of various kinds of model can be converted into third-order modeling. Then second handles system level aggregation where each coherent area is represented by a single artificial
3.2. Aggregation methods

generator, an abstract mathematical model, others are eliminated. In the third and also the last layer, real non-linear generator is reconstructed from our artificial mathematical model from previous layer. In this layer, we also answer the surrounding questions, like how to design a terminal bus which the newly constructed generator can sit on and feed into the power system of our concern. The following contents are organized according to these three layers.

3.2.1 Layer 1: Aggregation on Generator Level

Generator models and linearization

Usually a synchronous generator’s dynamic is encapsulated in several ordinary differential equations, the most common one is VI-order model and III-order model. The following text lists the VI-order model

\[
\begin{align*}
\dot{\delta} &= 2\pi f_0(\omega - 1) \\
\dot{\omega} &= \frac{1}{m} \left( P_m - P_{gen} - D(\omega - 1) - R_a (i_d^2 + i_q^2) \right) \\
\dot{e}_q &= \frac{1}{T_{d0}'} \left( v_{f} - f_s(e'_q) - (x_d - x'_d) - \frac{T_{d0}''}{T_{d0}'} (x_d - x'_d) \right) i_d \\
\dot{e}_d &= \frac{1}{T_{q0}'} \left( -e'_d + \left( x_q - x'_q + \frac{T_{q0}''}{T_{q0}'} (x_q - x'_q) \right) i_q \right) \\
\dot{e}_q'' &= \frac{1}{T_{d0}''} \left( -e''_q + e'_d - \left( x'_d - x''_d + \frac{T_{d0}''}{T_{d0}'} (x_d - x'_d) \right) i_d \right) \\
\dot{e}_d'' &= \frac{1}{T_{q0}''} \left( -e''_d + e'_q + \left( x'_q - x''_q + \frac{T_{d0}''}{T_{d0}'} (x_q - x'_q) \right) i_q \right),
\end{align*}
\]

and the III-order model,

\[
\begin{align*}
\dot{\delta} &= 2\pi f_0(\omega - 1) \\
\dot{\omega} &= \frac{1}{m} \left( P_m - P_{gen} - D(\omega - 1) \right) \\
\dot{e}_q &= \frac{1}{T_{d0}'} \left( v_{f} - f_s(e'_q) - (x_d - x'_d) \right) i_d \\
\dot{e}_d &= \frac{1}{T_{q0}'} \left( -e'_d + \left( x_q - x'_q + \frac{T_{d0}''}{T_{d0}'} (x_q - x'_q) \right) \right) i_q,
\end{align*}
\]

where the generated power \( P_{gen} \),

\[ P_{gen} = v_d l_d + v_q l_q \]

and the \( i_d \) and \( i_q \) are undesired variables that can be eliminated by power flow calculation.

\[
\begin{align*}
i_d &= \frac{r_a (e''_d - v_g \sin(\delta - \theta)) + x''_q (e''_q - v_g \cos(\delta - \theta))}{r_a^2 + x''_a x''_d} \quad (3.35a) \\
i_q &= \frac{r_a (e''_q - v_g \cos(\delta - \theta)) - x''_a (e''_d - v_g \sin(\delta - \theta))}{r_a^2 + x''_a x''_d} \quad (3.35b)
\end{align*}
\]
3. Synchronous generator aggregation

After eliminating undesired variables, the complete equations with dependence only on power flow parameters $v$, $\theta$ are included in Appendix A.1.1. We may notice the transcendental terms $\sin(\delta - \theta)$ and $\cos(\delta - \theta)$ which should be kept as independent identities in the aggregation procedure due to its unique features*. Hence, we define,

\[
\begin{align*}
    s &= \sin(\delta - \theta) \\
    c &= \cos(\delta - \theta).
\end{align*}
\]

Here, we assume some general situations. The generators in a coherent area are all in $VI$-order model, while on the other hand the aggregated generator as the final result is in $III$-order model. If generators have more detailed model, we can simply neglect some of its dynamics without losing much accuracy. If the original generators are in $II$-order, the conclusions in this subsection can be analogically applied.

Linearize the generator model at a predefined operating point,

\[
\begin{align*}
    \omega &= \omega^* \\
    v &= v^* \\
    \theta &= \theta^* \\
    e'_q &= e'^*_q \\
    e'_d &= e'^*_d \\
    e''_q &= e''_q \\
    e''_d &= e''_d
\end{align*}
\]

Then we have the linearized system

\[
\begin{bmatrix}
    \Delta \ddot{\delta} \\
    \Delta \ddot{\omega} \\
    \Delta \dot{e}'_q \\
    \Delta \dot{e}'_d \\
    \Delta \dot{e}''_q \\
    \Delta \dot{e}''_d
\end{bmatrix} =
\begin{bmatrix}
    \frac{\partial f_2}{\partial \delta} & \frac{\partial f_3}{\partial \delta} & 0 & 0 & \frac{\partial f_2}{\partial e'_q} & \frac{\partial f_3}{\partial e'_q} \\
    \frac{\partial f_1}{\partial \delta} & 0 & \frac{\partial f_2}{\partial e'_q} & \frac{\partial f_3}{\partial e'_q} & 0 & 0 \\
    \frac{\partial f_1}{\partial \delta} & 0 & \frac{\partial f_2}{\partial e'_q} & \frac{\partial f_3}{\partial e'_q} & 0 & 0 \\
    \frac{\partial f_1}{\partial \delta} & 0 & \frac{\partial f_2}{\partial e'_q} & \frac{\partial f_3}{\partial e'_q} & 0 & 0 \\
    \frac{\partial f_1}{\partial \delta} & 0 & \frac{\partial f_2}{\partial e'_q} & \frac{\partial f_3}{\partial e'_q} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    \Delta \ddot{\delta} \\
    \Delta \ddot{\omega} \\
    \Delta \dot{e}'_q \\
    \Delta \dot{e}'_d \\
    \Delta \dot{e}''_q \\
    \Delta \dot{e}''_d
\end{bmatrix}.
\]

\( (3.36) \)

The \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) is the Jacobian matrix, denoted as $A_{aa}$, the detailed expression for each entry is included in Appendix A.1.2. Thus, the linearized system is,

\[
\Delta \dot{x} = A_{aa} \Delta x + A_{ab} \Delta y, \quad (3.37)
\]

\[
0 = A_{ba} \Delta x + A_{ba} \Delta y, \quad (3.38)
\]

where $\Delta x = [\Delta \delta \Delta \omega \cdots \Delta e''_q]$ is a vector of deviations of dynamic variables near the $x^*$, and $\Delta y = [\Delta v \Delta \theta]$ a vector of deviation of power flow calculation variables near $y^*$.

* We will not be able to directly approximate these terms cannot even be approximated as an asymptotic expansion in time interval longer than $O(1)$, or even lose the periodicity (see Ch. 10, Ch. 11 of [65]. This is the main reason why we aggregate generators based on its linearized model rather than aggregate them directly on nonlinear model.
3.2. Aggregation methods

Similarly, the linearization of III-order model has the following form,

\[
\begin{bmatrix}
\Delta \delta \\
\Delta \omega \\
\Delta e' \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
\frac{\partial f_2}{\partial \delta} & \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial e'} \\
\frac{\partial f_2}{\partial \delta} & 0 & \frac{\partial f_2}{\partial e'} \\
\end{bmatrix}
\begin{bmatrix}
\Delta \delta \\
\Delta \omega \\
\Delta e' \\
\end{bmatrix}
\]

(3.39)

Its entries are listed at Appendix A.2.2.

As the first step to derive an approximation, we need to separate the slow variables and fast variables. Usually generators in a single coherent area have similar parameters and set-ups, so the slow and fast dynamic variables are not so obvious. J. H. Chow suggested a projection matrix in [10] and an reduction algorithm based also based on slow/fast dynamics. In what follows, we will be introduced a similar projection matrix, and a modified two-stage reduction algorithm. Compared with Chow’s method, the algorithm explained below generate III-order model generator for each coherent area rather than II-order model; two-stage reduction scheme handles dynamic reduction on generator-level and on system-level separately providing more flexibility.

**Generator Model Reduction**

For a single generator in VI-order model, there are 6 dynamic variables. They can be clearly separated as fast and slow variables. For example, in following dynamics associated with \(\omega, e'_q\) and \(e''_q\),

\[
\begin{align*}
m \dot{\omega} &= f_2(\delta, \dot{\delta}, v, \theta) \\
T'_d e'_q &= f_3(\delta, \dot{\delta}, v, \theta) \\
T''_d e''_q &= f_4(\delta, \dot{\delta}, v, \theta).
\end{align*}
\]

we have \(T'_d\), the time constant associated with \(e'_q\), that is usually larger than \(T''_d\) associated with \(e''_q\). The inertia \(m\) of torque balance equation is even larger. In the Kundur’s model[29], we have \(T'_d = 8\) (p.u.) and \(m = 111.15\) (p.u.) in contrast to \(T''_d = 0.03\) (p.u.). As various parameter has already been converted to per-unit measurements, except the time \(t, f_1(...), f_2(...), f_3(...)\) have similar magnitude, i.e. they all belong to \(O(1)\). Naturally, the difference between \(m, T'_d, \) and \(T''_d\) can be used to construct \(\varepsilon\). Taking \(\varepsilon_1 = \frac{T'_d}{m}\) and \(\varepsilon_2 = \frac{T''_d}{m}\), we can find

\[
\begin{align*}
\dot{\omega} &= \frac{1}{m} f_2(\delta, \dot{\delta}, v, \theta) \quad (3.40a) \\
\varepsilon_1 e'_q &= \frac{1}{m} f_3(\delta, \dot{\delta}, v, \theta) \quad (3.40b) \\
\varepsilon_2 e''_q &= \frac{1}{m} f_4(\delta, \dot{\delta}, v, \theta). \quad (3.40c)
\end{align*}
\]

which are in the form of (3.8). It can be easily seen that \(\omega\) is the slow variable, \(e'_q\) is fast variable, and \(e''_q\) is even much faster. The objective of the first stage
is to eliminate those fast variables in a single machine. For $VI$-order model, it is to eliminate $e'_d, e''_q$ and $e''_d$.

Now we are ready to apply O’Malley-Vasil’eva’s asymptotic expansion into our linearized system. Using the same naming convention as before, $x$ includes slow dynamic variables, that is, $x = [\delta, \omega, e'_q]^{T}$, and $z$ is a vector of faster variables, $z = [e'_d, e''_q, e''_d]$. We can eliminate all “algebraic variables” by expressing $y$ in the terms of $x$, resulting in

$$\dot{x} = A_{11}x + A_{12}z \quad (3.41a)$$

$$\varepsilon \dot{z} = A_{21}x + A_{22}z. \quad (3.41b)$$

Assume the real part of all eigenvalues of $A_{22}$ are strictly negative. If the graph $(x, h(x, \varepsilon))$ denotes points in $M_\varepsilon$ that satisfies the constrain,

$$\varepsilon (D_x h) \frac{dh}{d\tau} = \varepsilon \frac{dz}{d\tau} \quad (3.42)$$

that is,

$$\varepsilon (D_x h) (A_{11}x + A_{12}h) = A_{21}x + A_{22}h. \quad (3.43)$$

Based on previous discussion, we neglect any term higher than $O(\varepsilon^1)$ and express in (3.9), two-time scale expansion,

$$x_\varepsilon = a_0(t) \quad (3.44a)$$

$$z_\varepsilon = b_0(t) + \beta_0 \left(\frac{t}{\varepsilon}\right) \quad (3.44b)$$

Substitute these two approximations for $x$ and $z$ in the linearized system (3.43), we define the approximation system,

$$\dot{a}_0 = A_{11}a_0 + A_{12}(b_0 + \beta_0) \quad (3.45a)$$

$$\varepsilon(b_0 + \frac{1}{\varepsilon} \beta'_0) = A_{21}a_0 + A_{22}(b_0 + \beta_0). \quad (3.45b)$$

To eliminate slow dynamics of $z$, we first take $b_0(t) = h(a_0(t), 0) = -A_{22}^{-1} A_{21}a_0(t)$, the solution on center manifold,

$$\dot{a}_0 = (A_{11} - A_{12}A_{22}^{-1} A_{21}) a_0 + A_{12} \beta_0 \quad (3.46a)$$

$$-\varepsilon A_{22}^{-1} A_{21} \dot{a}_0 + \beta'_0 = A_{22} \beta_0, \quad (3.46b)$$

solve for $\beta'_0$ from the last equation,

$$\beta'_0 = A_{22} \beta_0 + \varepsilon A_{22}^{-1} A_{21} \dot{a}_0(t) \quad \Rightarrow \quad \beta'_0 = A_{22} \beta_0 + \varepsilon A_{22}^{-1} A_{21} \left( (A_{11}a_0 + A_{12}(b_0 + \beta_0)) \right).$$

Neglect the “boundary layer” dynamics by setting $\beta'_0 = 0$,

$$\beta_0 = -\varepsilon A_{22}^{-2} A_{21}(A_{11}a_0 - A_{12}A_{22}^{-1} A_{21}a_0) - \varepsilon A_{22}^{-2} A_{21} A_{12} \beta_0. \quad (3.47)$$
3.2. Aggregation methods

Because we have $\beta_0$ on the left and $\epsilon \beta_0$ on the right, this is an iterative equation. The final expression of $\beta_0$ will be the summation of $\epsilon$ terms with increasing order on $\epsilon$ starting from $\epsilon^1$. Furthermore, we neglect any term of $\epsilon^2$ or higher, we get

$$\beta_0 = -\epsilon A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21}) a_0.$$  (3.48)

$\beta_0$ is usually classified as $O(\epsilon)$ value because of the multiplier of $\epsilon$. Hence, the reduced model is

$$\dot{x} = A_s x$$  (3.49)

where

$$A_s = (I - \epsilon A_{12} A_{22}^{-1} A_{21}) A_0.$$  

Now we finally eliminated unwanted dynamic variables, i.e. $z = [e' d, e'' q, e'' d]$.

In practice, the aforementioned aggregation can be applied on the “differential-algebraic” (3.38) directly, without damaging the structure of “algebraic equations”. In this case,

\[
\begin{align*}
\Delta \dot{x} &= A_{ba} \Delta x + A_{bb} \Delta z + A_{bc} \theta + A_{bd} \Delta v \\
\epsilon \Delta \dot{z} &= A_{ba} \Delta x + A_{bb} \Delta z + A_{bc} \theta + A_{bd} \Delta v \\
0 &= A_{cc} \Delta x + A_{cd} \Delta z + A_{ce} \theta + A_{ce} \Delta v \\
0 &= A_{da} \Delta x + A_{db} \Delta z + A_{dc} \theta + A_{de} \Delta v
\end{align*}
\]  (3.50a-d)

and we want to keep the third equation (3.50c) and (3.50d). We can do the trick by setting slow dynamic as $[x, \theta, v]$, and apply the same procedure again to eliminate $z$. In the first stage aggregation, the synchronous generator (assume they are modeled in VI-order) have been reduced as III-order with a set of clearly meaningful dynamics variables as $[\delta, \omega, e'_q]$. If there is no strict requirement on the accuracy, we can simply neglect these higher order dynamic variables, like $e'_d, e''_q, \text{etc.}$.

3.2.2 Layer 2: System level aggregation

Then it comes to the second stage aggregation. It is quite common all generators in a single area have similar inertia and time constants. Thus, unlike aggregation in a single generator level, there is no natural distinction between slow and fast dynamic. However, from the model analysis, we know that inter-area oscillation is most related to low frequency that should be retained in the aggregated modeling, while the high frequency oscillations are between various generators in a single coherent area. If we find a way to separate these low/high frequency modes we can again apply the approximation algorithm. [10] proposed a transformation matrix that can be easily applied to derive a multiple-timescale system model in a new coordinate basis.
3. SYNCHRONOUS GENERATOR AGGREGATION

Assume we have a total of $v$ generator in a single coherent areas, the $i$-th generator can be linearized like,

$$
T_i \Delta \dot{x}_i = K_{aa} \Delta x_i + K_{ab} \Delta y_i
$$

(3.51a)

$$
0_i = K_{ba} \Delta x_i + K_{bb} \Delta y_i.
$$

(3.51b)

Here, for compactness, $y$ encapsulates all algebraic variables including $v$, the magnitude of load bus voltage, and $\theta$, the phase angles.

$$
T_i = \text{diag} \left\{ m_i, T_{d0}, \ldots \right\}
$$

(3.52)

is a diagonal matrix of time constants and inretia. For each coherent area, we can write the whole generator dynamics together by setting $\Delta x_{\text{sum}} = [\Delta x_1^T, \Delta x_2^T, \ldots, \Delta x_N^T]^T$, $\Delta y_{\text{sum}} = [\Delta y_1^T, \Delta y_2^T, \ldots, \Delta y_N^T]^T$. where $\Delta x$ and $\Delta y$ is the deviations of retained “differential” variables and “algebraic” variables respectively from the previous reduction stage. Then the whole group of generator dynamics in a single area can be expressed as

$$
T \dot{x}_{\text{sum}} = K_{aa\text{Sum}} x_{\text{sum}} + K_{ab\text{Sum}} y_{\text{sum}}
$$

(3.53a)

$$
0 = K_{ba\text{Sum}} x_{\text{sum}} + K_{bb\text{Sum}} y_{\text{sum}},
$$

(3.53b)

in which

$$
K_{aa\text{Sum}} = K_{aas1} \oplus \cdots \oplus K_{aasv},
$$

$$
K_{ab\text{Sum}} = K_{abs1} \oplus \cdots \oplus K_{absv},
$$

$$
K_{ba\text{Sum}} = K_{bas1} \oplus \cdots \oplus K_{basv},
$$

$$
K_{bb\text{Sum}} = K_{bbs1} \oplus \cdots \oplus K_{bbsv}.
$$

As mentioned before, the (3.53) does not have a naturally defined slow and fast dynamic separation. A projection matrix[10, 11] can be used to efficiently separate slow dynamic variables that usually represents the effect of the whole coherent area on inter-area oscillation, and fast dynamic variables that usually represents intra-area oscillation.

**Projection matrix**

The projection matrix is per-area based. We focus ourself into the first coherent area. An aggregated rotor angle that represents the “common-mode” motion can be defined as inertia weighted rotor angle average,

$$
\delta_{\text{agg}} = \sum_{i=1}^{N} m_i \Delta \delta_i
$$

(3.54)

where $m_i$ is the inertia of the $i$-th generator, $\Delta \delta_i$ is the rotor angle of the $i$-th generator, and

$$
m_{\text{sum}} = \sum_{i=1}^{N} m_i
$$

(3.55),

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the total inertia. The “differential-mode” motion $\delta_{\text{diff}}$ can be defined like,

$$
\Delta \delta_{\text{diff}} = G \begin{bmatrix}
\Delta \delta_{\text{diff}1} \\
\Delta \delta_{\text{diff}2} \\
\vdots \\
\Delta \delta_{\text{diff}N}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & 0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
\Delta \delta_{1} \\
\Delta \delta_{2} \\
\vdots \\
\Delta \delta_{N}
\end{bmatrix}.
$$

where

$$
G = \begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & 1
\end{bmatrix}.
$$

In matrix form, they can be written as

$$
\begin{bmatrix}
\Delta \delta_{\text{agg}} \\
\Delta \delta_{\text{diff}}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sum_{i=1}^{N}} T_{d0i}' \text{diag}(m_{1}, m_{2}, \ldots, m_{N}) \\
G
\end{bmatrix} \Delta \delta
$$

Similarly $e_{q}'$ which is omitted in Chow’s algorithm[10] can be aggregated by time constants $T_{d0}'$.

$$
T_{d0}'_{\text{agg}} = \frac{1}{T_{d0}'_{\text{sum}}} \sum_{i=1}^{N} T_{d0i}' e_{d}'
$$

where

$$
T_{d0}'_{\text{sum}} = \sum_{i=1}^{N} T_{d0i}'.
$$

Though it does not have significant meaning to define “differential-mode” transient voltage $e_{q}'$. Nevertheless, for a completeness we define the transformation matrix as

$$
\begin{bmatrix}
e_{q}_{\text{agg}}' \\
e_{q}_{\text{diff}}'
\end{bmatrix} = \begin{bmatrix}
\frac{1}{T_{d0}'_{\text{sum}}} T_{d0}'_{\text{agg}} \text{diag}(T_{d0}'_{1}, T_{d0}'_{2}, \ldots, T_{d0}'_{N}) \\
G
\end{bmatrix} e_{q}'
$$

The system (3.53) can be expressed equivalently as,

$$
T \begin{bmatrix} P_{1} & P_{2} \end{bmatrix}^{-1} \begin{bmatrix}
\Delta \delta_{\text{agg}} \\
\Delta \delta_{\text{diff}} \\
\Delta e_{q_{\text{agg}}} \\
\Delta e_{q_{\text{diff}}}
\end{bmatrix} = K_{\text{aSum}} \begin{bmatrix} P_{1} & P_{2} \end{bmatrix}^{-1} \begin{bmatrix}
\Delta \delta_{\text{agg}} \\
\Delta \delta_{\text{diff}} \\
\Delta e_{q_{\text{agg}}} \\
\Delta e_{q_{\text{diff}}}
\end{bmatrix} + K_{\text{abSum}} \Delta y_{\text{sum}}
$$

$$
0 = A_{\text{aSum}} \begin{bmatrix} P_{1} & P_{2} \end{bmatrix}^{-1} \begin{bmatrix}
\Delta \delta_{\text{agg}} \\
\Delta \delta_{\text{diff}} \\
\Delta e_{q_{\text{agg}}} \\
\Delta e_{q_{\text{diff}}}
\end{bmatrix} + A_{\text{bSum}} \Delta y_{\text{sum}}
$$

where $T = \text{diag}(m_{1}, \cdots) \oplus \text{diag}(T_{d0}'_{1}, \cdots)$. Any we omit the $\omega$, or equivalently $\delta$ term for compactness and for the reason that will be described later of this
subsection. On the first $N$ rows, the left side of coefficient matrix can be calculated as

\[
 \begin{pmatrix}
 \frac{1}{m_1} & \frac{1}{m_2} & \cdots & \frac{1}{m_N} \\
 -1 & 1 & 0 & \cdots & 0 \\
 -1 & 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 -1 & 0 & 0 & \cdots & 1 \\
 \end{pmatrix}^{-1}
 = \begin{pmatrix}
 \frac{1}{m_1} & 0 & 0 & \cdots & 0 \\
 -1 & 0 & 1 & \cdots & 0 \\
 -1 & 0 & 0 & \cdots & 1 \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 -1 & 0 & 0 & \cdots & 1 \\
 \end{pmatrix}
\]

Similarly, the last $N$ rows have

\[
 \begin{pmatrix}
 \frac{1}{T'_{d01}} & \frac{1}{T'_{d02}} & \cdots & \frac{1}{T'_{d0N}} \\
 -\frac{1}{r_{d01}} & 0 & 0 & \cdots & 0 \\
 -\frac{1}{r_{d02}} & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 -\frac{1}{r_{d01}} & 0 & 0 & \cdots & 1 \\
 \end{pmatrix}^{-1}
\]

The (3.59) can now expressed as follows in new coordinate system,

\[
 \begin{bmatrix}
 \Delta \delta_{agg} \\
 \Delta \delta_{diff} \\
 \Delta e'_{agg} \\
 \Delta e'_{diff} \\
 \end{bmatrix} =
 \begin{bmatrix}
 \frac{1}{m_1} & \frac{1}{m_2} & \cdots & \frac{1}{m_N} \\
 -1 & 1 & 0 & \cdots & 0 \\
 -1 & 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \ddots & \vdots \\
 -1 & 0 & 0 & \cdots & 1 \\
 \end{bmatrix}
 \begin{bmatrix}
 P_1 & 0 & \cdots & 0 \\
 0 & P_2 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & P_v \\
 \end{bmatrix}
 \begin{bmatrix}
 \Delta \delta_{agg} \\
 \Delta \delta_{diff} \\
 \Delta e'_{agg} \\
 \Delta e'_{diff} \\
 \end{bmatrix}
 + K_{abSum} \Delta y_{sum}
\]

The (3.62a) can now be written as follows in new coordinate system,

\[
 0 = A_{baSum} \begin{bmatrix}
 P_1 & 0 & \cdots & 0 \\
 0 & P_2 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & P_v \\
 \end{bmatrix}
 \begin{bmatrix}
 \Delta \delta_{agg} \\
 \Delta \delta_{diff} \\
 \Delta e'_{agg} \\
 \Delta e'_{diff} \\
 \end{bmatrix}
 + A_{bbSum} \Delta y_{sum}
\]

Under the new coordinate system, the slow and fast dynamic variables can be easily identified. If we define a set of factors

\[
 \epsilon_m = \sqrt{\frac{m_1 m_2 \cdots m_N}{m_{sum}}} \quad \text{and} \quad \epsilon_T = \sqrt{\frac{T'_{d01} T'_{d02} \cdots T'_{d0N}}{T'_{d0sum}}},
\]

then $\epsilon_m \Delta \delta_{agg}$ and $\delta_{diff}$ belong to category $O(\epsilon_m)$, $\epsilon_T \Delta T'_{d0sum}$ and $\Delta T'_{d0diff}$ belong to $O(\epsilon_T)$. Here, we define a ratio $\epsilon_i$ for $i = 1, 2, \ldots, \nu$ characterizing the difference between the slow/fast dynamics.

Finally apply such transformation matrix into all coherent areas, we can get an aggregated rotor angle $\Delta \delta_{agg i}$ and an aggregated transient voltage $e'_{agg i}$ for each area $i$ where $i \in [1, 2, \ldots, \nu]$.

\[
 \epsilon_i = \sqrt{\epsilon_m \epsilon_T}
\]
Finally, it would be necessarily to point out the first \( N \) terms on the left hand side of (3.59) does not contain \( \Delta \dot{\delta} \). Comparing with \( \ddot{\delta} = f_2(\delta, \dot{\delta}, v) \) (or \( f_2(\delta, \omega, v) \)) to be consistent with previous discussion, neglecting \( \dot{\delta} \) terms may sacrifice some degree of accuracy. Specifically, if the generator is in conventional model like the one in Appendix A.1.1 or Appendix A.2.1 and have no additional ancillary controller, \( \dot{\delta} \) influences \( \ddot{\delta} \) exclusively through \( D \dot{\delta} \) where \( D \) the damping factor. Neglecting the damping factor, the first \( N \) rows constitute an exactly second order ordinary differential system. In control theorem, a second differential equations with insufficient damping, no damping in this case, always corresponds to a periodic solution—the oscillation. Such second order models are widely used to for studying the frequency property of the power system, especially poorly damped oscillation modes. This kind of simplification is also suitable for our particular application since the inter-area oscillation modes that should be retained in reduced order system are usually poorly damped. However, the \( D \) might have significant meaning in designing power system controller and economic dispatch [44]. In case such simplification is not desired, we can consider \( \ddot{\delta} \) and \( \delta \) together by replacing (3.59) with the following equations,

\[
\begin{bmatrix}
\Omega \\
M \\
T
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}^{-1}
\begin{bmatrix}
\Delta \delta_{agg} \\
\Delta \delta_{diff} \\
\Delta \delta_{agg} \\
\Delta \delta_{diff} \\
\Delta \dot{e}_{agg} \\
\Delta \dot{e}_{diff}
\end{bmatrix}
= 
\begin{bmatrix}
I \\
0 \\
0 \\
K_{aaSum}^{(11)} \\
K_{aaSum}^{(12)} \\
K_{aaSum}^{(21)} \\
K_{aaSum}^{(22)}
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_{agg} \\
\Delta \delta_{diff} \\
\Delta \delta_{agg} \\
\Delta \delta_{diff} \\
\Delta \dot{e}_{agg} \\
\Delta \dot{e}_{diff}
\end{bmatrix}
+ 
\begin{bmatrix}
K_{abSum} \\
0
\end{bmatrix}
\begin{bmatrix}
y
\end{bmatrix}
\] (3.63)

where \( K_{aaSum}^{(11)}, K_{aaSum}^{(12)}, K_{aaSum}^{(21)}, \) and \( K_{aaSum}^{(22)} \) are from the the appropriate partition

\[
K_{aaSum} = 
\begin{bmatrix}
K_{aaSum}^{(11)} & K_{aaSum}^{(12)} \\
K_{aaSum}^{(21)} & K_{aaSum}^{(22)}
\end{bmatrix}
\]

The \( \Omega \) is defined by

\[
\Omega = 
\begin{bmatrix}
\frac{1}{2\pi f_0} & \cdots \\
\cdots & \frac{1}{2\pi f_0}
\end{bmatrix}
\]

Procedures similar to (3.60)–(3.62) can be applied again to derive a more accurate dynamic system under new coordinate system. In the remaining part of this chapter, we neglect the influence of the \( \dot{\delta} \) (or \( \omega \)) on the system dynamics, but the relevant methods and conclusions should be parallelly applied when neglecting \( \dot{\delta} \) is not desirable.
3. Synchronous generator aggregation

Model Reduction

After the previous stage, we got a well-determined slow/fast dynamic for each coherent area, which is the prerequisite for a final reduced-order model. One of the simplest yet efficient idea [10] is to set non-aggregated equations as 0 (equivalent to setting $\varepsilon = 0$), thus the corresponding differential equations of non-aggregated variables are degenerated into “algebraic equality”. A more accurate method is to neglect $O(\varepsilon^2)$ term, or neglect $\beta_0'$ (c.f. Equation 3.47), by applying transform (3.48). We first look at the first method and then move to second one.

Rearrange numbering of each dynamic variable such that aggregated rotor angle $\Delta \delta_{agg}$ and $\epsilon_d'$ sitting at top while keeping other fast variables to be eliminated at bottom. We denote slow dynamic variables as $x$, fast dynamic variables as $z$,

\begin{align}
\Delta \dot{x} &= A_{11} \Delta x + A_{13} \Delta z + A_{13} \Delta y, \quad (3.64a) \\
\varepsilon \Delta \dot{z} &= A_{21} \Delta x + A_{22} \Delta z + A_{23} \Delta y, \quad (3.64b) \\
0 &= A_{31} \Delta x + A_{32} \Delta z + A_{33} \Delta y. \quad (3.64c)
\end{align}

Setting $\Delta \dot{z} = 0$,

\begin{align}
0 &= A_{21} \Delta x + A_{22} \Delta z + A_{23} \Delta y.
\end{align}

Solve for $\Delta z$,

\begin{align}
\Delta z &= -A_{22}^{-1} A_{21} \Delta x - A_{22}^{-1} A_{23} \Delta y. \quad (3.65)
\end{align}

Take (3.65) into (3.64) to eliminate $\Delta z$,

\begin{align}
\Delta \dot{x} &= A_{11} \Delta x + A_{13} \Delta y, \quad (3.66a) \\
0 &= A_{31} \Delta x + A_{33} \Delta y. \quad (3.66b)
\end{align}

where

\begin{align}
A_{11} &= A_{11} - A_{12} A_{22}^{-1} A_{21} & A_{13} &= A_{12} A_{22}^{-1} A_{23} \\
A_{31} &= A_{31} - A_{33} A_{22}^{-1} A_{21} & A_{33} &= A_{33} - A_{32} A_{22}^{-1} A_{23} \quad (3.67)
\end{align}

This is so-called First-Order Approximation* method in [10]. A even simpler reduction model can be derived by neglecting $z$ completely that arrives at Inertial Aggregation method or Podmore’s Method [13, 20]

\begin{align}
\Delta \dot{x} &= A_{11} \Delta x + A_{13} \Delta y, \quad (3.68a) \\
0 &= A_{31} \Delta x + A_{33} \Delta y. \quad (3.68b)
\end{align}

* Even though the name suggests “First-order” in [10], this method in fact neglects $\varepsilon$ by setting $\varepsilon = 0$. In another words, this method uses solutions on center manifold $M_0$ to approximate the real trajectory.
3.2. Aggregation methods

It is evident in many references [35, 13, 20], the Inertial Aggregation can give a relatively good approximation. A more rigorous method based on perturbation theory and O’Malley-Vasil’eva’s asymptotic expansion can be applied for developing a hopefully more accurate approximation.

If we discard any error of $O(\varepsilon^2)$ or higher, the same procedure in A.1.1 can be applied again (take $x$ and $y$ as slow variables) resulting in a real system approximation expressed by

$$
\begin{bmatrix}
\Delta \dot{x} \\
0
\end{bmatrix} = A_{ss} \begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
$$

(3.69)

where the $A_{ss}$ is defined in such way,

$$
A_{ss} = (I - \varepsilon \begin{bmatrix}
A_{12} \\
A_{32}
\end{bmatrix} A_{22}^{-1} [A_{21} A_{23}]) A_0
$$

in which

$$
A_0 = \begin{bmatrix}
A_{11} & A_{13} \\
A_{31} & A_{33}
\end{bmatrix} - \begin{bmatrix}
A_{12} \\
A_{32}
\end{bmatrix} A_{22}^{-1} [A_{21} A_{23}].
$$

Alternatively write $A_{ss}$ in compact form, it is

$$
A_{ss} = \begin{bmatrix}
I - \varepsilon A_{12} & A_{21}^2 & A_{23} \\
-I\varepsilon & 0 & 0 \\
-I\varepsilon & 0 & 0
\end{bmatrix} \begin{bmatrix}
A_{11} - A_{12} A_{22}^{-1} A_{21} & A_{12} A_{22}^{-1} A_{23} & A_{13} - A_{12} A_{22}^{-1} A_{23} \\
A_{21} & -A_{22} & A_{23} \\
A_{32} & -A_{32} A_{22}^{-1} A_{21} & A_{33} - A_{32} A_{22}^{-1} A_{23}
\end{bmatrix}.
$$

(3.70)

In both equation,

$$
\varepsilon = \sqrt{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_v}.
$$

Compare (3.70) with (3.66), we can find as $\varepsilon \to 0$ the matrix above will degrade into $A_{11}$ to $A_{33}$ in (3.70). For this reason (also see the footnote on page 84), the First-Order Approximation method introduced above will be renamed as Pseudo First-Order Approximation hereinafter, and we name the newly introduced method as First-order Approximation method to give a clear distinction.

3.2.3 Reconstruct non-linear model

All of content introduced in this subsection so for dedicates into reducing linearized system model. For a single predefined operation point, the linearized reduced order model should be able to give a good approximation. However, these coefficient matrices involved in linearized model, like $A_{ss}$ in (3.69), lack a clear defined physical meaning to be fit into any defined power system appliance. Some simulation software even doesn’t provide API for self-defined model. Even fortunately the reduced linear model can be successfully defined in the simulation program, it might not be suitable for application in a different operating point much away from the one used for developing reduced order model. There some some suggestions to handle with this difficulty. The first suggestion is to apply the arguments about slow/fast manifolds directly.
3. Synchronous generator aggregation

Figure 3.1: Augmented generator bus

...to the nonlinear model and result in a reduced order nonlinear. This kind of task is difficult due to the \( \sin \) and \( \cos \) terms. Another suggestion reconstruct the nonlinear model by taking the advantage of system identification. The nonlinear model reconstruction idea is basically suggestion by \cite{10}, which generally includes generators, transmission lines, load, etc. Some necessary steps are suggested by \cite{13}. In what follows, the main attention is drawn into reconstructing aggregated generator.

Unless explicitly specified, the reduced linearized system always has one set of generator variables \([\delta, \omega, e'_q]\) for each coherent area, which serves as representatives of “common mode” of all generator dynamics in that area. We now need to construct a single representative generator from these variables. The first task we have to encounter is to define a terminal bus on which the representative is located.

**Augment generator bus**

\cite{20, 57, 54} and many more references suggest a similar method on constructing the requested terminal bus. Since this bus does not exist in original system, we address it as **Augmented Generator Bus**. The construction method to be introduced below is adapted from the one developed by Podmore\cite{10}.

The overview of such aggregation method is illustrated in Figure 3.1 for a simplest example. There are two generators located in two different terminal buses bus \( A \) and bus \( B \) that may or may not have shunt connection or even an impedance branch connection stretching between them. The figure on the right depicts the reduced generator and the system set up around it. It can be seen Gen \( A \) and Gen \( C \) along with the branch connecting them are removed from their original buses, and a new equivalent generator is added on the augmented Bus \( C \). The presence of idea transformer \( T_1 \) and \( T_2 \) make sure the infinite conductance connections between Bus \( A \) and Bus \( B \) are possible. In a
3.2. Aggregation methods

Simulation program, a bus is used to register voltage phasor at a node expect serving as a “connection port”. For a transformer, it has attributes of voltage and phasor transfer ratios. These attributes and the shunt connections can be determined by following steps.

**Step 1** Decide the voltage magnitude $V_{\text{aug}}$ and phase angle $\theta_{\text{aug}}$ of augmented bus. The voltage magnitude can be chosen as average voltage magnitude value of generators in a single group, magnitude of a selected bus voltage, or a average magnitude weighted against reactive power production at selected working point or nominal value. Because all values are converted into per-unit system, the (two-side) idea generators can be simply neglected. $V_A$, $V_B$ and $V_{\text{aug}}$ should have same value under per-unit system. Therefore, there is no strict requirement, but set a reasonable value due to some consideration of some “ancillary” system, like the overhead limit might be based on this value. In my algorithm, the voltage is set as

$$V_{\text{aug}} = \frac{1}{\sqrt{\sum_i Q_{\text{gen}_i}}} \sqrt{\sum_i V_i}.$$  

Similarly, the phase angle $\theta_{\text{aug}}$ might not have significant influence on final result, nor has any influence on power flow due to infinite conductance and susceptance on the transformer branches. But the power flows are decided from rest of system so it is not infinite. One may choose phase angle of a selected generator bus for the augmented bus, or choose the average value against active power production. In my algorithm, it is decided by,

$$\theta_{\text{aug}} = \frac{1}{\sum_i P_{\text{gen}_i}} \sum_i P_{\text{gen}_i} \theta_i.$$  

**Step 2** Usually there is no direct load on generator buses nor common branch between between them. However, if the system load buses have been aggregated by, for example, Kron reduction, then they may have such common branch like between Bus-$A$ and Bus-$B$, or have directly connected branches on buses like Bus-$A$ an Bus-$B$. The direct shunt connections are kept at original buses, while the common branch should be replaced by two equivalent direct shunts at both ends of branch. The augmented buses have no shunts connection and merely serves as a “port” for the representative generator to sit on. The common branch can be eliminated in the following way. Firstly, assume the branch has admittance $Y_{AB}$ we need to calculate current flow through it,

$$I_{AB} = (V_A - V_B)Y_{AB}.$$  

Then the shunt admittances directly connected at Bus-$A$ can be expressed as,

$$Y_A = \frac{I_{AB}}{V_A} = (1 - \frac{V_B}{V_A})Y_{AB}, \quad Y_B = -\frac{I_{AB}}{V_B} = (1 - \frac{V_A}{V_B})Y_{AB}.$$
3. Synchronous generator aggregation

Step 3 For other components like dynamic loads, they should be kept at original buses. These kept components along with original generator buses, like Bus-A and Bus-B, will be eliminated all together by Kron Reduction and its extension method that includes to some extend dynamic loads (c.f. chapter 4).

Identify parameters of representative generator

After deriving an augmented generator bus (or equivalent generator bus) and its various attributes, we have complete knowledge about \( y \) in (3.33). Now we are ready to identify the parameters of conventional non-linear equivalent generator model from their linearized aggregations that can be derived with the first-order equivalent reduction (3.69), pseudo first-order equivalent reduction (3.66) and the inertial equivalent reduction (3.68). My suggested identification algorithm is based on optimization procedure, of which the numerical calculation is relegated into \texttt{fminsearch} or \texttt{fmincon} functions offered in optimization toolbox of \textit{Matlab} \textsuperscript{®} program. The only concern here is about the fitness function.

To blind the detailed implementation of reduced order linearized model, we rewrite the reduced linearized system as follows,

\[
d\dot{\delta} = \tilde{A}_{aa}d\delta + \tilde{A}_{ab}d\epsilon'_q + \tilde{A}_{ac}dV + \tilde{A}_{ad}d\theta, \tag{3.71a}
\]

\[
d\dot{\epsilon}'_q = \tilde{A}_{ba}d\delta + \tilde{A}_{bb}d\epsilon'_q + \tilde{A}_{bc}dV + \tilde{A}_{bd}d\theta, \tag{3.71b}
\]

\[
0 = \tilde{A}_{ca}d\delta + \tilde{A}_{cb}d\epsilon'_q + \tilde{A}_{cc}dV + \tilde{A}_{cd}d\theta. \tag{3.71c}
\]

All three linearized generator reduction methods previously introduced can be matched into the generalized equation above. On the other hand, we assume that the reduced system comprises of a representative generator for each coherent area except the study area where generators are kept as original, and other generators explicitly marked to be kept in details. Their non-linear system model can be expressed like,

\[
\dot{x} = f_1(\delta, \omega, V, \theta) \tag{3.72a}
\]

\[
\ddot{\delta} = f_2(\delta, \omega, V, \theta) \tag{3.72b}
\]

\[
\dot{\epsilon}'_q = f_3(\delta, \omega, V, \theta) \tag{3.72c}
\]

\[
0 = g(\delta, \omega, V, \theta) \tag{3.72d}
\]

where, in contrast to \( f \) and \( g \) in (3.33) or other functions about generator model, (3.72) are functions on system level. \( x \) is a vector of \( \delta, \dot{\delta} \) and \( \epsilon'_q \) of all generators ordered based on their coherent area. Similar to aggregation on generator level, if the damping factors can be neglected, the first function of \( f \), the one describes the trivial relation between \( \dot{\delta} \) and \( \delta \). The first-order relationship of \( [\dot{x}^T, \ddot{x}^T]^T \) to \( [x^T, \dot{x}^T]^T \) can be expressed in second-order form as
3.2. Aggregation methods

\( \delta \) with regard to \( \delta \). In what follows, we assume the damping factors are negligible in consistent with previous discussion. Here, we adopt the convention that using superscript denotes numbering of different coherent area, and use subscript denote different type of dynamics. For example, considering the dynamics of the first coherent area, they are \( f^1 = [f^1_1, f^1_2, \cdots]^T \), and the complete system dynamics are described by \( f = [f^1, f^2, \cdots, f^\nu]^T \). Similar numbering convention also applies to \( g \). Here, we assume there exists totally \( \nu \) coherent areas and \( N_i \) generators in the \( i \)-area.

We then linearize (3.72) at the same steady-state working point as used for developing linearized model on previous layer. That is

\[
\dot{\delta} = \frac{\partial f_2(\delta, e'_q, V, \theta)}{\partial \delta} d \delta + \frac{\partial f_2(\delta, e'_q, y, \theta)}{\partial \theta} d \theta \tag{3.73a}
\]

\[
\dot{e}'_q = \frac{\partial f_3(\delta, e'_q, V, \theta)}{\partial \delta} d \delta + \frac{\partial f_3(\delta, e'_q, y, \theta)}{\partial \theta} d \theta \tag{3.73b}
\]

\[
0 = \frac{\partial g(\delta, e'_q, V, \theta)}{\partial \delta} d \delta + \frac{\partial g(\delta, e'_q, y, \theta)}{\partial \theta} d \theta \tag{3.73c}
\]

The detail expressions for each partial differential terms can be find in Appendix A. Then the fitness function can be defined as

\[
c = \left\| \frac{\partial f_2(\delta, e'_q, V, \theta)}{\partial \delta} d \sigma \right\|_2 + \left\| \frac{\partial f_3(\delta, e'_q, V, \theta)}{\partial \delta} d \sigma \right\|_2 + \cdots + \left\| \frac{\partial g(\delta, e'_q, V, \theta)}{\partial \sigma} d \sigma \right\|_2 \tag{3.74}
\]

The parameters to be decided by optimization process include \( X_d, X_q, R_g \) and \( X_q \), the dynamic variables, \( \delta, \omega \) and \( e'_q \), voltage magnitudes, \( v \), and phases, \( \theta \) are not included in optimization parameters but needs to be decided beforehand based on our projection matrix \( P_1 \) and \( P_2 \), defined by (3.56) and (3.58) respectively. For the algorithm based on PSAT\(^\text{®} \), from which one can directly get generator terminal current injection on \( d \)- and \( q \)-axes, \( I_d \) and \( I_q \), the algorithm is stretched in the following pseudo-codes where the words in typewriter font almost always indicate the parameters defined in the programming function.

**Step 1** Define variables \( ss_i = \sin(\delta_i - \theta_i) \), \( cc_i = \cos(\delta_i - \theta_i) \), and read data \( I_d = i_d \) and \( I_q = i_q \) from PSAT. Also, if \( X_d'' \) exists (not equal to 0), then assign \( X_d = X_d'' \); else assign \( X_d = X_d' \). If \( X_q'' \) exists (not equal to 0), then assign \( X_q = X_q'' \); else if \( X_q' \) exist, assign \( X_q = X_q' \); otherwise, assign \( X_q = X_q' \); assign \( R = R_g \).

**Step 2** Define

\[
c_1 = \frac{R}{R^2 + X_d X_d}, \quad c_2 = \frac{X_d}{R^2 + X_d X_d}, \quad \text{and} \quad c_3 = \frac{X_q}{R^2 + X_q X_d}
\]

Those parameters always occur together.

**Step 3** Estimate aggregated \( \sin(\delta_{agg} - \theta) \) and \( \cos(\delta_{agg} - \theta) \) according to (3.56) and (3.58),

\[
ss = \sum_i w_{\delta_i} \sin(\delta_i - \theta_i) \quad \text{and} \quad cc = \sum_i w_{\delta_i} \cos(\delta_i - \theta_i)
\]
3. Synchronous generator aggregation

where the \( w_\delta \) is the weighting factor defined by the first row of \( P_1 \) used to calculate aggregated (slow) parameters.

Estimate the aggregated \( d \)-axis and \( q \)-axis current
\[
I_d = \sum_i I_{d_i} \quad \text{and} \quad I_q = \sum_i I_{d_i} I_{q_i}.
\]

Define aggregated generator inertia and time constant
\[
M = \sum_i m_i \quad \text{and} \quad T_{d0} = \sum_i T'_{d0}.
\]

**Step 4** Calculate the partial differential terms occurring in (3.73). The calculation formula can be found in Appendix A.1.1 by using parameters defined above.

**Step 5** Define fitness function (3.74) and initial guesses. If there is no clue to for these better guesses, they can be chosen as harmonic average of original generators
\[
X_{d_{agg}} = \sum_i \frac{1}{X_{d_i}}, \quad X_{1d_{agg}} = \sum_i \frac{1}{X_{1d_i}},
\]
\[
R_{agg} = \sum_i \frac{1}{R_i}, \quad X_{q_{agg}} = \sum_i \frac{1}{X_{q_i}}.
\]

**Step 6** Pass defined fitness function and initial guesses into optimization function, like \texttt{fminsearch} or \texttt{fmincon} in \textit{Matlab}.

The approximation result and its accuracy is compared with other approximation methods in chapter 6.

3.3 The Podmore’s aggregation method and its difference with two-time scale aggregation method

There exists another well-known algorithm that have long been implemented in many existing power system simulation program. It share various similarity with the aforementioned two-time scale methods, like how to augmented generator terminal bus. Though its algorithm is comparatively less complicated—without intricate discussion about slow/fast dynamics or asymptotic expansion and generators are all assumed in second order model, still it deserves our attention to draw a comparison between these two methods. In this section, we will introduce this algorithm and more attention will be focused onto its different properties as compared to two-time-scale algorithm introduced before.

**Podmore’s algorithm** The Podmore’s method is illustrated by the stretch of Figure 3.2a. Considering situations similar to the one expressed in Figure 3.1, where generator Gen-A at Bus-A and Gen-B at Bus-B are from the same coherent area. Common branches stretching different buses and direct shunt connections may exist. Figure 3.2b illustrates the aggregation steps. Under the assumption that all generators of our concerns are in second order model and have no resistive losses, the generator can be expressed as constant
3.3. The Podmore’s aggregation method and its difference with two-time scale aggregation method

Figure 3.2: Podmore’s aggregation procedure

Voltage $E'_1$ and $E'_2$ behind a reactances $X'_d_1$ and $X'_d_2$. As the first step of aggregation we assume $E' = E'_1 = E'_2$ and combine these two “ends” together forming a single internal bus of the aggregated generator. The value of $E'$ including its phase angle will be decided in a moment. For now the $X'_d_1$ and $X'_d_2$ branches are kept as the original model, but we augment them by inserting an idea transformers (and a idea phase shifter) such that we have an new terminal Bus, Bus-C. Since there is no impedance on the transformer/phase-shifter branches, we have the freedom of choosing appropriate voltage magnitude and voltage phase angle of Bus-C, then the transfer ratios and phase shifts of $T_1$ and $T_2$ can be decided accordingly. Usually, similar to the previous case, we can choose the $V_c$ to be the average value of $V_A$ and $V_B$ or the mean value weighted by their active power production. The phase angle $\theta_c$ can be chosen as the averaged voltage phase angles or the weighted average with weights being their reactive power productions. At the final step, like the Figure 3.2c, the $X'_d_1$ and $X'_d_2$ in parallel connection can be replaced by

$$X'_{d_{agg}} = \frac{1}{\frac{1}{X'_d_1} + \frac{1}{X'_d_2}}.$$

Since we have defined common terminal buses voltage magnitude and phase angle and we also have the knowledge of $X'_{d_{agg}}$ and the current through it, the internal voltage $E'_d$ of the aggregated generator can be easily calculated.

**Comparison with two-time scale method** Compared with the two-time scale method discussed previously, the Podomore’s method stands out having the advantage of being a simple aggregation method. Theoretically, it has a clear physical explanation. However, it is based on the assumption that the aggregated generator is in II-order model. Such assumption only gives considera-
3. Synchronous generator aggregation

tion on dynamics of $\delta$ and $\dot{\delta}$, but the dynamics associated with $e'_q$, $e''_q$, etc. are simply neglected. This might result in losing some amount of accuracy.
In this chapter, our attention is drawn onto aggregating load buses, after we identified and aggregated good clusters of coherent generators (Section 2.2) and found the load buses closely related with them (Section 2.3). From the computation point of view, we might already reduced great amount of computation time, the loose power network can be stored sparse matrix, which may also facilitate the simulation. However, there are still much security concerns if we stop here. As recalling that the synchronous generators are represented as an equivalent one sitting at an augmented generator bus and the network structure is largely retained, much crucial information might be inevitably disclosed to other parties when, for example, exchanging network data with neighboring Transmission System Operators (TSO’s). Ideally, we expect to reduced power system into a simple Generator-Load-Ancillary models like

![Diagram](image)

Figure 4.1: Ideal reduced model

where the Bus load and ancillary controller are aggregated model equivalents whose effect on the rest of power system, for the dynamics and power flow calculation, is equivalent to original complete system. The aggregation of the generator is handled by techniques introduced in the last two chapters. In this chapter, some techniques will be introduced for deciding the aggregated network impedance $Z$ and the aggregated load in the form of “ZIP” model.
4. Kron Reduction and extensions

4.1 The Schur complement and Kron reduction

As discussed in the chapter 3, the power system network can be represented by admittance matrix $Y$ in the relation of the well-known Ohm’s law,

$$ I = YV, \quad (4.1) $$

where $V \in \mathbb{C}^n$ is a vector of voltages corresponding to $n$ buses, and $I \in \mathbb{C}^n$ is a vector of current injections. Furthermore, if define an adjacency matrix $A$ for the network graph,

$$ A = [a_{ij}], $$

where $a_{ij} \in \mathbb{C}$ for $i \neq j$ corresponding to the the branch from $(i,j)$ is defined as the branch impedance and phase shift, if phase shifter exists; and $a_{ii} \in \mathbb{C}$ for $i \in \{1,2,\ldots\}$ is shunt directly connected to bus $i$. We then have the Laplacian matrix,

$$ L = D - A. $$

The admittance matrix is

$$ Y = L + \text{diag} \left\{ \{A_{ii}, \text{ for } i = 1,2,\ldots\} \right\}. $$

Particularly, Laplacian $L$ is non-singular with an eigenvector, eigenvalue pair $(\lambda_0, v_0)$ being

$$ \lambda_0 = 0 \quad v_0 = 1. $$

Though admittance matrix $Y$ might also be singular, its generalized inverse, $Y^\dagger$, has a very important significance in the definition of resistance distance or effective resistance, \[22, 15\]

$$ r_{ij} = (e_i - e_j)^T Y^\dagger (e_i - e_j) = L_{ii}^\dagger + L_{jj}^\dagger - L_{ij}^\dagger - L_{ji}^\dagger. $$

In what follows, the traditional Kron reduction will be developed in the form of admittance matrix, and the equivalent network graph reduction is also illustrated in figures. Essentialiy, the Kron reduction in network corresponds to Schur complement in matrix.

**Schur complement** One way the Schur complement frequently occurs is through the Gaussian elimination (row reduction). Take the equality of Ohm’s law (4.1) for an example. Assume $\mathfrak{S}$ with the cardinality $|\mathfrak{S}| = n$ is the index set. The buses that we want to keep in the reduced equation are indexed by set $a \subseteq \mathfrak{S}$, and denote $a^c = \mathfrak{S} \setminus a$. Permute the bus numbering, the (4.1) can be expanded like

$$ \begin{bmatrix} I[a] \\ I[a^c] \end{bmatrix} = \begin{bmatrix} Y[a, a] & Y[a, a^c] \\ Y[a^c, a] & Y[a^c, a^c] \end{bmatrix} \begin{bmatrix} V[a] \\ V[a^c] \end{bmatrix}. \quad (4.2) $$
From the last equation, we can pre-multiply \([ J - X J ] \), where the \( J \) is identity matrix to distinguish with \( I \) that is usually current vector,

\[
\begin{bmatrix}
J & 0 \\
-X & J
\end{bmatrix}
\begin{bmatrix}
I[\alpha^c] \\
I[\alpha]
\end{bmatrix}
= \begin{bmatrix}
J & 0 \\
-X & J
\end{bmatrix}
\begin{bmatrix}
Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] \\
Y[\alpha, \alpha^c] & Y[\alpha, \alpha]
\end{bmatrix}
\begin{bmatrix}
V[\alpha^c] \\
V[\alpha]
\end{bmatrix}
= \begin{bmatrix}
Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] \\
Y[\alpha, \alpha^c] - XY[\alpha^c, \alpha^c] & Y[\alpha, \alpha] - XY[\alpha^c, \alpha]
\end{bmatrix}
\begin{bmatrix}
V[\alpha^c] \\
V[\alpha]
\end{bmatrix}
\]

If the \( X \) is chosen in such way that \( Y[\alpha, \alpha^c] - XY[\alpha^c, \alpha^c] = 0 \). If \( Y[\alpha^c, \alpha^c] \) is invertible, it is simply

\[
X = Y[\alpha, \alpha^c]Y[\alpha^c, \alpha^c]^{-1} \tag{4.3}
\]

. Then the equation above becomes,

\[
\begin{bmatrix}
I[\alpha^c] \\
I[\alpha] - XI[\alpha^c]
\end{bmatrix}
= \begin{bmatrix}
Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] \\
0 & Y[\alpha, \alpha] - XY[\alpha^c, \alpha]
\end{bmatrix}
\begin{bmatrix}
V[\alpha^c] \\
V[\alpha]
\end{bmatrix}
\]

It can been see the second equation is decoupled from \( V[\alpha^c] \). If, furthermore, we assume no net current injection exists in vertices indexed by \( \{\alpha\} \), that is no constant current injections and shunt connections, the \( I[\alpha^c] = 0 \). In this case, we can safely eliminate the the first row and get the reduced Ohm’s law. Accordingly, in the network connections, the buses indexed by \( \alpha^c \) can be eliminated by reconstructing the network from the new admittance matrix,

\[
Y_{\text{new}} = Y[\alpha, \alpha] - XY[\alpha^c, \alpha].
\]

Usually, the \( Y[\alpha, \alpha] - XY[\alpha^c, \alpha] \) is abbreviated as \( Y/Y[\alpha^c, \alpha^c] \).

**Augmented admittance matrix** The preceding discussion is solely based on the admittance matrix \( Y \), which permits us to develop Schur complement and decouple the \( V[\alpha] \) with \( I[\alpha] \) under specific conditions (the current injection \( I[\alpha^c] \) should be zero). If we have no interest on \( I[\alpha] \) and \( V[\alpha^c] \), we can also get a reduced network from the newly constructed admittance matrix \( Y_{\text{new}} \). However, the Schur complement and the corresponding reduction on network is not immediately clear. To this end, [15] suggests the Augmented Laplacian matrix that explicitly encode the ground, seen as a reference bus, into the matrix. Given an admittance matrix \( Y \), the augmented admittance matrix is defined in the following way,

\[
\hat{Y} = \begin{bmatrix}
Y \\
-1^T \text{diag (} \{A_i\} \} & -\text{diag (} \{A_i\} \} \sum_i A_{ii}
\end{bmatrix}.
\tag{4.4}
\]

The augmented row and column corresponds to the ground node, and the augmented \( \hat{Y} \) now holds for the following relation,

\[
\hat{Y} \mathbb{1} = 0.
\]
4. Kron Reduction and Extensions

Figure 4.2: Graph representation of Kundur’s 4-machine system with explicit branches to the ground

Figure 4.3: Reduced network of Kundur’s 4-machine system, assuming load buses Bus 7 and Bus 9 have net current extraction, generator buses Bus 1–4 have net current injection, current injections at other buses are balanced, i.e. no net current injection or extraction

Hence, \( \hat{Y} \) is Laplacian matrix, which can clearly induce a graph with explicit branches to the ground like Figure 4.2 for Kundur’s system.

Similarly, the Schur complement can be applied on the augmented matrix. In this case, because of the augmented ground “bus”, all current injections and extractions are balanced, we can write

\[
0 = \hat{Y} V = \begin{bmatrix}
Y - \text{diag} \left( \{ A_{ii} \} \right) V \\
-1^T \text{diag} \left( \{ A_{ii} \} \right) \sum_i A_{ii}
\end{bmatrix}
\begin{bmatrix}
V \alpha^c \\
0
\end{bmatrix}.
\]

where the voltage of ground, the reference voltage, is set to be 0. With some necessary matrix permutation, the equation above can be written in sub-matrix form

\[
0 = \begin{bmatrix}
Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] - \text{diag}\left(\{ A_{ij}, \forall i \in \alpha^c \}\right) I \\
-1^T \text{diag}\left(\{ A_{ii}, \forall i \in \alpha^c \}\right) & -1^T \text{diag}\left(\{ A_{ij}, \forall i \in \alpha^c \}\right) \sum_i A_{ii}
\end{bmatrix}
\begin{bmatrix}
V[\alpha^c] \\
V[^a[\alpha^c]]
\end{bmatrix}.
\]

Here, the ground “bus” is also included in bus set to be kept in reduced system, the submatrix at the lower right corner is equivalent to \( Y[\alpha, \alpha] \), a submatrix of \( Y \) and the submatrix at lower left corner has the equivalent role as \( Y[\alpha, \alpha^c] \).
Thus, we have the following abbreviations,

\[ \hat{Y}[\alpha, \alpha] = Y[\alpha, \alpha] \]
\[ \hat{Y}[\alpha^c, \alpha] = [Y[\alpha^c, \alpha] - \text{diag}([A_{ii}, \forall i \in \alpha^c])\mathbb{I}] \]
\[ \hat{Y}[\alpha, \alpha^c] = \begin{bmatrix} \hat{Y}[\alpha, \alpha] & -\text{diag}([A_{ii}, \forall i \in \alpha]) \sum_i A_{ii} \\ -\mathbb{I}^T \text{diag}([A_{ii}, \forall i \in \alpha^c]) \end{bmatrix} \]

The voltage for the ground is set to be 0 as in previous case. Apply again the pre-multiplier \([-\hat{X}^T J\]).

\[ 0 = \begin{bmatrix} J & 0 \\ -\hat{X}^T & J \end{bmatrix} \begin{bmatrix} \hat{Y}[\alpha^c, \alpha] & \hat{Y}[\alpha^c, \alpha] \\ \hat{Y}[\alpha, \alpha^c] & \hat{Y}[\alpha, \alpha] \end{bmatrix} \begin{bmatrix} Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] \\ Y[\alpha, \alpha^c] & Y[\alpha, \alpha] \end{bmatrix} \begin{bmatrix} \hat{Y}[\alpha^c, \alpha^c] - \hat{X} \hat{Y}[\alpha^c, \alpha^c] \\ \hat{Y}[\alpha, \alpha^c] - \hat{X} \hat{Y}[\alpha, \alpha^c] \end{bmatrix} \]

As in the case of \(X\) in the reducing matrix \(Y\), the \(\hat{X}\) can be defined as

\[ \hat{X} = \hat{Y}[\alpha, \alpha^c]\hat{Y}[\alpha^c, \alpha^c]^{-1}. \]

Thus, the equation above becomes,

\[ 0 = \begin{bmatrix} \hat{Y}[\alpha^c, \alpha^c] & \hat{Y}[\alpha^c, \alpha] \\ 0 & \hat{Y}[\alpha, \alpha^c] - \hat{X} \hat{Y}[\alpha^c, \alpha] \end{bmatrix} \begin{bmatrix} V[\alpha^c] \\ [V(\alpha)]_0 \end{bmatrix} \]

if \(\hat{Y}[\alpha^c, \alpha^c]\) is invertible. Comparing with the definition of \(X\) in (4.3), notice that \(Y[\alpha^c, \alpha^c] = \hat{Y}[\alpha^c, \alpha^c]\),

\[ \hat{X} = \begin{bmatrix} Y[\alpha, \alpha^c] \\ -\mathbb{I}^T \text{diag}([A_{ii}, \forall i \in \alpha^c]) \end{bmatrix} Y[\alpha^c, \alpha^c]^{-1} \]
\[ = \begin{bmatrix} Y[\alpha, \alpha^c]Y[\alpha^c, \alpha^c]^{-1} \\ -\mathbb{I}^T \text{diag}([A_{ii}, \forall i \in \alpha^c])Y[\alpha^c, \alpha^c]^{-1} \end{bmatrix} \]
\[ = \begin{bmatrix} X \\ -\mathbb{I}^T \text{diag}([A_{ii}, \forall i \in \alpha^c])Y[\alpha^c, \alpha^c]^{-1} \end{bmatrix}. \]
Then the red admittance matrix $\hat{Y}_{\text{red}}$ is defined as

$$
\hat{Y}_{\text{red}} = \hat{Y}[\alpha, \alpha] - \hat{X} \hat{Y}[\alpha^c, \alpha] - \mathbb{I} \text{diag}([A_{ii}, \forall i \in \alpha])^{-1} \sum_i A_{ii} \mathbb{I} - \mathbb{I} \text{diag}([A_{ij}, \forall i \in \alpha^c])^{-1}
$$

where $\hat{Y}_{\text{red}(1,1)}$, $\hat{Y}_{\text{red}(1,2)}$, $\hat{Y}_{\text{red}(2,1)}$, and $\hat{Y}_{\text{red}(2,2)}$ are defined in the following ways

$$
\begin{align*}
\hat{Y}_{\text{red}(1,2)} &= Y[\alpha, \alpha^c] - X Y[\alpha^c, \alpha]^{-1} \text{diag}([A_{ij}, \forall i \in \alpha^c])^{-1} \mathbb{I} - \text{diag}([A_{ij}, \forall i \in \alpha])^{-1} \\
\hat{Y}_{\text{red}(2,1)} &= \mathbb{I} \text{diag}([A_{ij}, \forall i \in \alpha^c]) Y[\alpha^c, \alpha]^{-1} Y[\alpha^c, \alpha]^{-1} \mathbb{I} - \mathbb{I} \text{diag}([A_{ij}, \forall i \in \alpha])^{-1} \\
\hat{Y}_{\text{red}(2,2)} &= \sum_i A_{ii} - \mathbb{I} \text{diag}([A_{ij}, \forall i \in \alpha^c]) Y[\alpha^c, \alpha]^{-1} \mathbb{I}
\end{align*}
$$

It can be simply verified that

$$
\hat{Y}_{\text{red}} \mathbb{I} = 0.
$$

Therefore, $Y$ and $Y_{\text{red}}$ are all Laplacian matrices that can induce well-defined graphs, like the graph for the Kundur’s 4-machine system Figure 4.2 and the graph corresponding to the reduced system Figure 4.3 [12].

**Connections with $\Delta - \mathbb{Y}$ transformation and the quotient property**

Given a closer look into the Kron reduction, one might find the process is parallel to the well-known $\Delta - \mathbb{Y}$ transformation in circuit reduction. To compare with them, we need to study the Quotient property.

**Theorem 4.1 (Quotient Property of Schur Complement)**

Given two matrices $M$ and $A$, defined in the following way,

$$
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}
$$

assuming submatrices $A$ and $E$ are nonsingular. Then the quotient property states that

$$
M/A = (M/E)/(A/E)
$$

The proof can be found in various references like [72, 73]. To build up the connection, we first look at the $\Delta - \mathbb{Y}$ transformation illustrated in Figure 4.4.
4.1. The Schur complement and Kron reduction

Assuming no net current injection or extraction at Bus 4, we can eliminate it by solving for the following equations,

\[
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
0
\end{bmatrix} =
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots \\
\sum_i b_{1i} & -b_{12} & -b_{13} & -b_{14} \\
-\sum_i b_{2i} & \sum_i b_{2i} & -b_{23} & -b_{24} \\
-\sum_i b_{3i} & \sum_i b_{3i} & -b_{34} & \vdots \\
-\sum_i b_{4i} & \sum_i b_{4i} & -b_{43} & -b_{44}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\]

where \(b_{ij}\) is the admittance of branch between \(i\) and \(j\), self-admittance to the ground is denoted by \(b_{ii}\). Solve for \(V_4\) in terms of \(I_1, I_2\) and \(I_3\), and substitute it in the original equation. Thus, we get the (generalized) \(\Delta - Y\) transformation,

\[
I_a = (Y_{aa} - y_{a4}y_4^{-1}y_4^T) V_a
\]

It can be easily identified that \(\Delta - Y\) transformation is nothing but a Schur complement (Kron reduction speaking in network system) of a single one-dimensional entry (or bus). The \((i, j)\) entry of \(Y_{\text{red}(1)}\) is

\[
[Y_{\text{red}(1)}]_{ij} = [Y_{aa}]_{ij} - b_{ia}y_4^{-1}b_{aj}.
\]

If \(b_{ia}\) and \(b_{aj}\) are nonzero, it is the usual \(\Delta - Y\) transformation.

If, taking one step further, we start to eliminate Bus 3 with the assumption that \(I_3 = 0\) and \(\sum_i b_{3i} - b_{34}(\sum_i b_{4i}b_{4i}) \neq 0\). There are two parallel ways, one is to calculate the Schur complement of buses \(a^c = \{\text{Bus 3, Bus 4}\}\). We abbreviate \(A/A[a, a]\) by \(A[a]\), then the reduced network admittance matrix after eliminating Bus 4 and Bus 3 at one time all together,

\[
Y_{\text{red}(2)} = Y/a = Y/\begin{bmatrix}
y_{33} & -b_{34} \\
-b_{43} & y_{44}
\end{bmatrix}.
\]
4. Kron Reduction and Extensions

or we eliminate Bus 3 continually after eliminating the first Bus 4 assumption

\[ Y'_{\text{red(2)}} = Y_{\text{red(1)}} / \{\text{Bus 3}\} = Y_{\text{red(1)}} / (y_{33} - b_{34}y_{44}^{-1}b_{43}) \]

where \( y_{33} = \sum_i b_{3i} \). We can verify \( \begin{bmatrix} y_{33} & -b_{34} \\ -b_{43} & y_{44} \end{bmatrix} / y_{44} = (y_{33} - b_{34}y_{44}^{-1}b_{43}) \). From the quotient property (4.6), we have

\[ Y'_{\text{red(2)}} = Y_{\text{red(1)}} / (y_{33} - b_{34}y_{44}^{-1}b_{43}) \]

\[ = (Y / y_{44}) / \begin{bmatrix} y_{33} & -b_{34} \\ -b_{43} & y_{44} \end{bmatrix} / y_{44} = Y / \begin{bmatrix} y_{33} & -b_{34} \\ -b_{43} & y_{44} \end{bmatrix} \]

We conclude that the Schur complement or Kron reduction is equivalent to \( \Delta - Y \) transformation by removing current balanced buses one by one.

4.2 The extension of Kron reduction

The traditional Kron reduction have a great application in various aspects including power system reduction and circuit designs. However, it has its limitations that only current balanced buses can be eliminated. One way to remedy it is to convert the loads into constant impedances and apply the Kron reduction on the augmented admittance matrix, which explicitly include the ground as reference bus. This remedy is acceptable as long as the time constants associated with load dynamics are beyond the range of frequency of our interests. The inclusion of renewable power sources are usually modeled as constant power injections in power system stability analysis. Modeling these power sources into constant (negative) impedances may no longer be appropriate. In this section, we will try extending the traditional Kron reduction such that the loads at buses to be eliminated can be equivalently transferred into remaining buses.

In our first step, the “algebraic” equation of power balance \( 0 = g(\delta, V) \) can be expressed in matrix form,

\[ 0 = g(\delta, V) = V \circ (\bar{Y}V) - S(\delta) \]

\[ = \begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \circ \begin{bmatrix} J & 0 \\ -\bar{X} & J \end{bmatrix} \begin{bmatrix} I[\alpha^c] \\ I[\alpha] \end{bmatrix} \] (4.7)

where the “\( \circ \)” denotes the Hadamard product, the element-wise product, of two matrices. Because of the complexity of handling (4.7) directly. We can instead look back at the Ohm’s law (4.1) in the matrix form. Assume the current \( I[\alpha^c] \) in not zero, we again pre-multiply matrix \( \begin{bmatrix} J & 0 \\ -\bar{X} & J \end{bmatrix} \) from the left onto the equation of Ohm’s law, we get the new current injection on the left
4.2. The extension of Kron reduction

hand side of the second row of the result,

\[ I = \begin{bmatrix} I[a] \\ I[a] - X[I^c] \end{bmatrix}. \]

substitute \( \overline{YY} = \overline{I} \) for \( YY \) on (4.7), we get

\[
V \circ \overline{I} - S = \begin{bmatrix} V[a^c] \\ V[a] \end{bmatrix} \circ \left( \begin{bmatrix} J & 0 \\ -X & J \end{bmatrix} \begin{bmatrix} \overline{I[a^c]} \\ \overline{I[a] - X[I^c]} \end{bmatrix} \right) - \begin{bmatrix} S[a^c] \\ S[a] \end{bmatrix}
\]

\[
= \begin{bmatrix} V[a] \circ \begin{bmatrix} 0 \\ -X[I^c] \end{bmatrix} \end{bmatrix}
\]

Obviously, \( V \circ I \) and \( S \) are not balanced because of the additional term \(-X[I^c]\) introduced by the Kron Reduction in the current vector \( I \). Thus, we can define a compensation term \( S_{cI} \),

\[
S_{cI}(I[a^c]) = -V[a] \circ (X[I^c]). \tag{4.8}
\]

Thus, we get the power balance equation for reduced network that includes the aggregation of constant current loads by adding the compensation term \( S_{cI} \),

\[
0 = V[a] \circ (\overline{Y_{new}}V[a]) - S[a] - S_{cI}(I[a^c]). \tag{4.9}
\]

This equation can be safely applied onto situations where the eliminated buses have known constant current injection or extraction loads.

Then, we look at aggregation of buses containing constant power loads. Surprisingly, aggregation of constant power loads is even harder than aggregating constant current loads or constant impedance loads thanks to its nonlinear dependence on voltages. We can decompose power balance equations (4.7) as follows

\[
V \circ \overline{I} - S = \begin{bmatrix} V[a^c] \\ V[a] \end{bmatrix} \circ \begin{bmatrix} \overline{I[a^c]} \\ \overline{I[a]} \end{bmatrix} - \begin{bmatrix} S[a^c] \\ S[a] \end{bmatrix}
\]

\[
= \begin{bmatrix} V[a] \\ V[a] \end{bmatrix} \circ \begin{bmatrix} \overline{I[a^c]} \\ \overline{I[a]} \end{bmatrix} - \left( \begin{bmatrix} V[a^c] \\ V[a] \end{bmatrix} \circ \begin{bmatrix} V[a^c]^{-1} \circ S[a] \\ V[a]^{-1} \circ S[a^c] \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} V[a^c] \\ V[a] \end{bmatrix} \circ \begin{bmatrix} \overline{I[a^c]} \\ \overline{I[a]} \end{bmatrix} - \begin{bmatrix} V[a^c]^{-1} \circ S[a] \\ V[a]^{-1} \circ S[a^c] \end{bmatrix} = 0.
\]
4. Kron Reduction and extensions

When applying the transformation \[ \left[ \begin{array}{c} J - X \end{array} \right] \] onto the second argument of the Hadamard product, it is equivalent to applying Kron Reduction.

\[
\begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \ast \left( \begin{bmatrix} J & 0 \\ -X & J \end{bmatrix} - \begin{bmatrix} I[\alpha^c] & -S[\alpha^c] \\ -V[\alpha^{-1} \ast S[\alpha]] & V[\alpha] \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \ast \left( \begin{bmatrix} J & 0 \\ -X & J \end{bmatrix} - \begin{bmatrix} I[\alpha^c] & -S[\alpha^c] \\ 0 & 1 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \ast \left( \begin{bmatrix} V[\alpha^c] \\ Y[\alpha^c, \alpha, \alpha^c] \end{bmatrix} - \begin{bmatrix} V[\alpha^{-1} \ast S[\alpha]] & Y[\alpha^c, \alpha, \alpha] \\ 0 & 1 \end{bmatrix} \right) = 0.
\]

Comparing (4.2) and (4.2), we find the first row of equations are the same, while the only difference dwells on the second row

\[
0 = V[\alpha] \ast \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -V[\alpha^c] \\ 0 & 1 \end{bmatrix} \right) = 0.
\]

where the existence of \( V[\alpha^c] \) in the last term is pretty annoying since the ultimate purpose of applying Kron Reduction is to eliminate dependence on \( V[\alpha^c] \), that is, the voltages of eliminated buses. This problem roots deep in the intrinsic property of constant power model which can be expressed as follows by using Ohm’s law in matrix form

\[
S = V \ast (YY)
\]

\[
= \text{diag}(V)YY = \text{diag}(V)YY \text{diag}(V)1
\]

\[
= \text{diag} \left\{ \begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \right\} \begin{bmatrix} Y[\alpha^c, \alpha^c] & Y[\alpha^c, \alpha] \\ Y[\alpha^c, \alpha] & Y[\alpha, \alpha] \end{bmatrix} \text{diag} \left\{ \begin{bmatrix} V[\alpha^c] \\ V[\alpha] \end{bmatrix} \right\} \begin{bmatrix} 1_{|\alpha^c|} \\ 1_{|\alpha|} \end{bmatrix}
\]

where (4.11) employs diagonal matrix \( \text{diag}(V) \) to remove the painful Hadamard production in the first row. It can be seen that \( \text{diag}(V) \) or \( \text{diag}(V) \) occurs twice before and after \( Y \), which is not symmetric, so the Kron Reduction cannot totally remove \( V[\alpha^c] \) in the final expression by simply solving quadratic equations. Or, as an alternative, we come back to the second row of (4.2). If every parameters are in p.u. we can express \( V[\alpha^c]^{-1} \) by its Neumann series,

\[
V[\alpha^c]^{-1} = (1 - (1 - V[\alpha^c]))^{-1} = 1 + (1 - V[\alpha^c]) + (1 - V[\alpha^c])^2 + \sigma \left( (1 - V[\alpha^c])^2 \right)
\]

\[
\approx 1 + (1 - V_{\text{approx}}) + (1 - V_{\text{approx}})^2
\]

where \( (1 - V_{\text{approx}})^2 = [1 - V_{\text{approx}}] \ast [1 - V_{\text{approx}}] \). \( V_{\text{approx}} \) is approximated voltages of \( V[\alpha^c] \) for which a convenient choice might be

\[
V_{\text{approx}} = KV[\alpha] \approx V[\alpha^c]
\]

where \( K \) is the “ratio” matrix between \( V_{\text{approx}} \) and \( V[\alpha] \). \( K \) might be solved from the linearization of differential algebraic equations \( V, \dot{g} \), the power flow
sensitivity matrix. Thus, the overall equations including constant load aggregation become

\[0_{|\alpha|} = V[\alpha] \cdot \left( Y_{\text{red}} V[\alpha] - V[\alpha]^{-1} \circ S[\alpha^c] + \bar{X}(V[\alpha^c]^{-1} \circ S[\alpha^c]) \right)\]

\[= V[\alpha] \cdot \left( Y_{\text{red}} V[\alpha] \right) - S[\alpha] + V[\alpha] \circ \bar{X}(V[\alpha^c]^{-1} \circ S[\alpha^c])\]

\[\approx V[\alpha] \cdot \left( Y_{\text{red}} V[\alpha] \right) - S[\alpha] + V[\alpha] \circ \bar{X} \left( 1 + (1 - V_{\text{approx}}) + (1 - V_{\text{approx}})^2 \right) \circ S[\alpha^c]\]

If furthermore taking into account the compensation for aggregation of constant current load, then the algebraic equations become

\[0_{|\alpha|} \approx V[\alpha] \cdot \left( Y_{\text{red}} V[\alpha] \right) - S[\alpha] - S_{cI}(I[\alpha^c]) - S_{c0}(V[\alpha], S[\alpha^c])\]  \hspace{1cm} (4.16)

where

\[S_{c0} = S_{c0}(V_{\text{approx}}, S[\alpha^c]) = V[\alpha] \circ \bar{X} \left( 1 + (1 - V_{\text{approx}}) + (1 - V_{\text{approx}})^2 \right) \circ S[\alpha]\]

and,

\[S_{cI}(I[\alpha^c]) = -V[\alpha] \circ \left( \bar{X} I[\alpha] \right)\]

are the compensation terms for constant power loads and constant current load respectively.

In the end, there are several things that are worth mentioning. For the first, if our selection of \(\{\alpha\}\) only includes “balanced buses” rather than generator buses or load buses, then \(S_{cI} = 0\) and \(S_{c0} = 0\), which was proved equivalent to traditional \(Y - \Delta\) conversion but processed in batches as discussed in the first section of this chapter. For the second, if the graph induced by the power system is connected, then the submatrix \(Y[\alpha^c, \alpha^c]\) is invertible. The choice of \(X\) is unique that is \(X = Y[\alpha, \alpha^c] Y[\alpha, \alpha]^{-1}\). For the last, if any branch has pure phase shifter component, the graph induced by the power system is no longer undirected and \(Y\) is no longer symmetric. However, there is no obstacle to the application of such Kron Reduction technique because the derivation above doesn’t depend on the symmetricity of \(Y\), but this setup is not considered in traditional \(\Delta - Y\) transformation or Kron reduction.
Chapter 5

System identification method

The methods and discussions introduced so far are all based on the study on dynamic system. In recent year there are increasing trend of studying on system identification method in hope to give a better result despite more complexity is added into the modern power system. There are many justifications for using system identification method rather than studying the dynamics. For the first, renewable generation and active loads prevail in the modern power grid and in the future. They are sometimes hard to be modeled in traditional ZIP load, or are even hard controllable, like the power turbines in the early stage are not integrated with powerful power electronics controllers or wind tracing controllers. Secondly, in the case of interconnected system where one TSO has no knowledge about neighboring systems and the responsible TSOs have not provide any information, we have no choose but try to model the neighboring system dynamics out of measurements (usually active and reactive power) through interconnecting transmission lines to other systems. Out of these concerns, the system identification might be a good idea. Many researches exist in the fields like [9, 41]. Owing to the development of Wide-Area Measurement System (WAMS), higher accuracy and faster practical solution is feasible. In the following, the basic technique of model reduction in using system identification is summarized.

5.1 Model identification

In the parlance of standard system identification problem, based on how much knowledge we have regarding the system, we might classify our identification model as black box, gray box and white box. For the black box model, we have no theoretical knowledge about the underlying structure. The exact underlying system structure must be inferred from input and output recording in an experiment. The underlying structures might be assumed having Hammerstein structure [74] or Hammerstein-Winner models [70, 67]. Recently,
the neutral network structure widely used in artificial intelligent might also be seen as an example of black box modeling [33]. On the other hand, the gray box has some knowledge of underlying theoretic system structure, but not enough to decide the complete input/output system. It is said the theoretical basis must be completed by additional structures from measurement data sets. In the case of our application in power system identification, the individual types of load and their modeling are well developed, but how they constitute the total loads dynamic, i.e. the portions each component takes in the mixture of loads is unknown and is pending for exploration from data set, the neutral network structure can also be classified into this category. In the end, the white box model is derived from the first principle, like the models deduced from physical theories, energy conservation etc., but their parameters need to be decided from given data set and the number of parameters is definite. The gray box modeling will be the focus of following discussion.

Gray box model formulation
The basic structure of gray box can be formulated in the following equations,

\[ m(f, p, q) = 0 \]  \hspace{1cm} (5.1)

with,

\[ q = Ac, \]

where \( f \) is a vector of inputs, \( p \) is vector of output. The \( m \) is the known part of the system and vector \( q \) is the unknown part of the system. Sometimes, it is assumed that \( q \) linearly depend on operating condition \( c \). To solve for the unknown part of the system, several methods are available. Among them, the most efficient one might be linear or nonlinear least square techniques, by minimizing the model residual,

\[ \min_A \sum_t |\dot{p}(t) - \dot{p}(t|A)|^2. \]

Such minimization process can be fulfilled and refined by (linear) least square or refined by nonlinear least square techniques. Recent development on the global optimization technique like generic algorithm also provides some solutions.

5.2 Application in power system aggregation

Following steps indicated in chapter 3, we have identified a coherent area where synchronous generators and associated load share a same pattern of behavior in the dynamic profile. For this reason, we are trying to set up a single Generator-Load-Ancillary model like Figure 4.1. Their nonlinear gray box structure is developed as follows,
5.2. Application in power system aggregation

**Synchronous generator** The generator is assumed having III-order modeling,

\[
\dot{\delta} = 2\pi f_0 (\omega - 1) \tag{5.2a}
\]

\[
e'_q = \frac{1}{T'_d} \left( -e'_q x'_d + \frac{x'_d - x'_d}{x'_d} v_g \cos(\delta - \theta) + v_f \right) \tag{5.2b}
\]

\[
\dot{\omega} = \frac{1}{M} \left( p_m - \frac{v_g}{x'_d} e'_q \sin(\delta - \theta) - D(\omega - 1) \right) \tag{5.2c}
\]

and the active/reactive power extraction from study area at the boundary between two different areas is set up as

\[
p_g = -\frac{v_g}{x'_d} e'_q \sin(\delta - \theta) + p_{ZIP0} \left( k_{pz} \left( \frac{v_g}{v_0} \right)^2 + k_{pu} \left( \frac{v_g}{v_0} \right) + k_{p0} \right) \tag{5.3a}
\]

\[
q_g = -\frac{v_g}{x'_d} e'_q \cos(\delta - \theta) + \frac{v_g^2}{x'_d} + q_{ZIP0} \left( k_{qz} \left( \frac{v_g}{v_0} \right)^2 + k_{qi} \left( \frac{v_g}{v_0} \right) + k_{q0} \right) \tag{5.3b}
\]

where the system parameters, \( \theta \) is

\[
\theta = [x_d, x'_d, T'_d, M, D, p_{load}, q_{load}, k_{p}, k_{q}, k_{pu}, k_{qi}, k_{p0}, k_{q0}] .
\]

The output active and reactive power vectors calculated from the preceding model, denoted by \( \hat{p}_g(t|\hat{\theta}) \) and \( \hat{q}_g(t|\hat{\theta}) \), are compared against measured active and reactive power passing through boundary lines. The \( \hat{\theta} \) is the model estimator. A good parameter estimation can minimize the the difference between \( [\hat{p}_g(t), \hat{q}_g(t)] \) and \( [p_g(t), q_g(t)] \). Conforming to (5.1), \( [p_g(t), q_g(t)] \) is the outputs \( p \).

**Cost function** To characterize the quality of the estimated model, the *cost function* can be defined by using *Normalized Root-Mean Square Error*. Let \( y \) be the real measured data of active/reactive power transfer through boundary lines.

\[
NRMS(\hat{\theta}) = \sqrt{\frac{\text{MSE}(\hat{\theta})}{y_{\text{max}} - y_{\text{min}}}} \tag{5.4}
\]

where the \( \text{MSE}(\hat{\theta}) \) can be expressed in the following way

\[
\text{MSE} = \mathbb{E} \frac{1}{n} \sum_{t} \left( \hat{y}(t|\hat{\theta}) - y(t) \right)^2
\]

\[= \cdots \]

\[= \text{Var}(\hat{\theta}) + \text{Bia}(\hat{\theta}, \theta)^2 . \tag{5.5}
\]

It is to say that such fitness function encodes the variance of the output forecasts and the bias. For more discussions, readers are referred to [25].

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5. **System identification method**

**Decide the estimators**  To decide the estimators, a common efficient method resort to *nonlinear least square estimation* [37] or converting the nonlinear system into a linear one, i.e. the so-called *model inversion* [68]. Some other method is based on global optimization techniques like genetic algorithm [41]. All of methods are already integrated into **Matlab**. In the result and validation session in Chapter 6, the simulation of the identified model, that is, the aggregated model, is compared with the real one. The estimator optimization procedure took about 6 hours in single core of **Intel Xeon E5-2697v2** (2.75 GHz) provided by Euler cluster at ETH Zürich, while the result is not insignificant better than the method introduced in Chapter 3 based on slow/faster manifolds decomposition and perturbation theory.
Chapter 6

Results and validation

To clarify and validate various methods included in preceding discussion, this chapter includes some test cases and comparisons, followed by some comments. Due to the limited time of the thesis project, not all methods are fully tested in large systems and some advantages of certain methods are not immediately evident. Nevertheless, these test cases prove to some extend the reliability of the introduced methods. All the simulation tests are performed on PSAT (Power system analysis toolbox) on Matlab® platform.

6.1 Coherent area identification

This section dedicates to verifying the coherent area identification methods introduced in Chapter 2 and Chapter 3. Some of the test cases are included before in the relevant sections.

Because different working operations demand different set-ups, the resulting coherent area divisions are only suitable for a limited range around the initial condition. Two different cases are consider, the first one is Kundur’s 4 machine system and the other one is 14-IEEE system based on portion of American Power System in Midwestern US.

6.1.1 Kundur’s 4 machine system

The Kundur’s system is shown in Figure 6.1, where different coherent areas labeled with different colors. An appropriate coherent area identification method is able to correctly find out such classification. The basic data about this system is included in Table 6.1. The eigenvalue analysis shows the interarea oscillation of frequency $0.5455 \text{Hz}$ is associated with eigenvalue $-0.1295 + j3.4272$. Neglecting damping factors, the eigen-analysis of the sec-

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Results and validation

Table 6.1 The generator specification of Kundur’s 4 machine system

<table>
<thead>
<tr>
<th>Gen</th>
<th>Rated power</th>
<th>[M, D]</th>
<th>$p_m$</th>
<th>$v_f$</th>
<th>[xd, x’d]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen 1</td>
<td>900 MVA</td>
<td>[12.35, 0.00]</td>
<td>7.0144</td>
<td>1.9659</td>
<td>[8.00, 0.03]</td>
</tr>
<tr>
<td>Gen 2</td>
<td>900 MVA</td>
<td>[13.00, 0.00]</td>
<td>7.0148</td>
<td>2.0148</td>
<td>[8.00, 0.03]</td>
</tr>
<tr>
<td>Gen 3</td>
<td>900 MVA</td>
<td>[13.00, 0.00]</td>
<td>7.0137</td>
<td>1.9406</td>
<td>[8.00, 0.03]</td>
</tr>
<tr>
<td>Gen 4</td>
<td>900 MVA</td>
<td>[12.35, 0.00]</td>
<td>7.2034</td>
<td>1.9528</td>
<td>[8.00, 0.03]</td>
</tr>
</tbody>
</table>

The tight and loose coherent area identification both classify Gen 1 and Gen 4 as a coherent group and other generators as the other coherent group.

6.1.2 IEEE 14 bus system

Another example is applied on IEEE 14 bus system based on a portion of American Electrical Power System (Midwestern US) [49]. The network structure is sketched in Figure 6.2. The generators specifications and operating point are listed in Table 6.2. This system has no obvious interarea structure, but studying this case aims to test the sensitivity of loose/tight coherent area identification. In practice, we may encounter some situations when we have to dissect an area that is already closely connected into two.

The eigenanalysis shows the “interarea” oscillation mode (the least damped oscillation mode) is associated with eigenvector

$$v = [0.1803 -0.6901 -0.2334 -0.5116 -0.4184].$$

Figure 6.1: Kundur’s original 4 machine system
6.1. Coherent area identification

![Diagram of IEEE 14 bus system]

Table 6.2 The generator specification of IEEE 14 bus system

<table>
<thead>
<tr>
<th>Gen</th>
<th>Rated power</th>
<th>$[M, D]$</th>
<th>$T_d'$</th>
<th>$P_m$</th>
<th>$v_f$</th>
<th>$[x_d', x_d]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen 1</td>
<td>615 MVA</td>
<td>[10.296, 2.00]</td>
<td>7.40</td>
<td>3.5203</td>
<td>1.1227</td>
<td>[0.8979, 0.2396]</td>
</tr>
<tr>
<td>Gen 2</td>
<td>60 MVA</td>
<td>[13.080, 2.00]</td>
<td>6.10</td>
<td>0.0018</td>
<td>2.0450</td>
<td>[1.0500, 0.1850]</td>
</tr>
<tr>
<td>Gen 3</td>
<td>60 MVA</td>
<td>[13.080, 2.00]</td>
<td>6.10</td>
<td>0.4050</td>
<td>2.7181</td>
<td>[1.0500, 0.1850]</td>
</tr>
<tr>
<td>Gen 4</td>
<td>25 MVA</td>
<td>[10.120, 2.00]</td>
<td>4.75</td>
<td>0.0005</td>
<td>2.6222</td>
<td>[1.2500, 0.2320]</td>
</tr>
<tr>
<td>Gen 5</td>
<td>25 MVA</td>
<td>[10.120, 2.00]</td>
<td>4.75</td>
<td>0.0009</td>
<td>3.1463</td>
<td>[1.2500, 0.2320]</td>
</tr>
</tbody>
</table>

and the coefficient matrix with tolerance being $\gamma$,

$$C = \begin{bmatrix}
0.0500 & -0.7594 & -0.3007 & -0.6211 & -0.5282 \\
-0.7594 & 0.0500 & -0.0796 & 0.0397 & 0.0205 \\
-0.3007 & -0.0796 & 0.0500 & -0.0182 & 0.0135 \\
-0.6211 & 0.0397 & -0.0182 & 0.0500 & 0.0450 \\
-0.5282 & 0.0205 & 0.0135 & 0.0450 & 0.0500
\end{bmatrix} \quad (6.3)$$

Apparently, the loose coherent area identification result in classifying Gen 1 as a coherent area of its own and the others as another coherent area as can be seen from the eigenvector (6.2). But the tight coherent area identification or a closer look into the coefficient matrix gives a different generator grouping,

<table>
<thead>
<tr>
<th>Results from tight coherent area identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area 1</td>
</tr>
<tr>
<td>Area 2</td>
</tr>
<tr>
<td>Area 3</td>
</tr>
</tbody>
</table>

Due to relatively simple structure of the test case, we can visually identify
theree submatrices from (6.3) such that,

\[ C_1 = C[1, 1] = \begin{bmatrix} 0.0500 \end{bmatrix} \]
\[ C_2[3, 3] = \begin{bmatrix} 0.0500 \\ 0.0397 \\ 0.0205 \end{bmatrix} \]
\[ C_3 = C [(2, 4, 5), (2, 4, 5)] = \begin{bmatrix} 0.0500 & 0.0397 & 0.0205 \\ 0.0397 & 0.0500 & 0.0450 \\ 0.0205 & 0.0135 & 0.0500 \end{bmatrix} \]

have only positive entries. The result perfectly matches the test system setup—the Gen 3 at Bus 2 is synchronous generator, whereas the others Gen 3–5 are synchronous condensors which is also modeled as special kind of synchronous generator in PSAT.

### 6.2 Load bus aggregation

One of the test cases has already been performed on Kundur’s 4 bus system in Chapter 2.3 during the introduction of identification algorithm. In what follows, the algorithm is applied again in the IEEE 14 bus system Figure 6.2 based on the generator grouping identified by tight coherent area identification algorithm. In the following case, we apply the load bus aggregation algorithm to the IEEE 14 bus system based on shortest path betweenness in Subsection 2.3.2. The betweenness result is shown in the following table, where the edge weight is normalized value \([A']_{ij}\) for \((i, j) \in \mathcal{E}\) as in (2.22).

<table>
<thead>
<tr>
<th>Edge</th>
<th>edge betweenness</th>
<th>Edge</th>
<th>edge betweenness</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2)</td>
<td>26.77</td>
<td>(6, 11)</td>
<td>83.43</td>
</tr>
<tr>
<td>(1, 5)</td>
<td>13.42</td>
<td>(6, 12)</td>
<td>87.59</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>91.31</td>
<td>(6, 13)</td>
<td>56.28</td>
</tr>
<tr>
<td>(2, 4)</td>
<td>134.66</td>
<td>(7, 8 )</td>
<td>204.30</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>111.31</td>
<td>(7, 9 )</td>
<td>44.45</td>
</tr>
<tr>
<td>(3, 4)</td>
<td>129.22</td>
<td>(9, 10)</td>
<td>35.52</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>32.73</td>
<td>(9, 14)</td>
<td>137.85</td>
</tr>
<tr>
<td>(4, 7)</td>
<td>213.91</td>
<td>(10, 11)</td>
<td>42.58</td>
</tr>
<tr>
<td>(4, 9)</td>
<td>310.16</td>
<td>(13, 12)</td>
<td>15.76</td>
</tr>
<tr>
<td>(5, 6)</td>
<td>290.07</td>
<td>(13, 14)</td>
<td>99.35</td>
</tr>
</tbody>
</table>

The branches having highest are Bus 4–9, Bus 4–7 and Bus 4–9, which cut off the system in two components. The first one contains generator Gen 1 and Gen 3 and other one has no active power injections.

However, compared with loose/tight coherent area identification result, the load bus aggregation did not assigns load buses into the coherent group \{Area 1, Area 2\} and Area 3 respectively. This can be explained by observing that Gen 2 and Gen 3–5 are synchronous condensors. Therefore, they do not participate in the low frequency oscillation that is primarily dominated by active
Figure 6.3: Reduced Kundur’s system where the Area 2 (marked in blue) is represented by their equivalent model comprising of a synchronous generator, Gen 3 and ZIP load, Load 9

Table 6.3 Results of generator aggregation

<table>
<thead>
<tr>
<th>Method</th>
<th>Rated power</th>
<th>([M, D])</th>
<th>(T_d')</th>
<th>(P_m)</th>
<th>(v_f)</th>
<th>([x_d, x_d'])</th>
</tr>
</thead>
<tbody>
<tr>
<td>two time-scale</td>
<td>100 MVA</td>
<td>([222.30, 0.00])</td>
<td>16.00</td>
<td>14.25</td>
<td>4.2357</td>
<td>([0.3889, 0.0312])</td>
</tr>
<tr>
<td>Podomore’s method</td>
<td>100 MVA</td>
<td>([222.30, 0.00])</td>
<td>8.00</td>
<td>14.22</td>
<td>1.9600</td>
<td>([0.1000, 0.0167])</td>
</tr>
<tr>
<td>Identification method</td>
<td>100 MVA</td>
<td>([163.37, -4.80])</td>
<td>9.95</td>
<td>7.19</td>
<td>2.1712</td>
<td>([0.2857, 0.0391])</td>
</tr>
</tbody>
</table>

power balance. Nevertheless, the load bus aggregation algorithm successfully identified the weak links in the graph, which might be helpful to updating transmission lines.

6.3 Generator aggregation

Based on the coherent area identified above on Kundur’s 4 machine system, the following contain demonstrate the synchronous generator aggregation by three methods, namely two-time-scale, Podmore’s, and identification methods. Compared with the original system Figure 6.1, we expect the final aggregation will be like Figure 6.3.

In the original system, the generators are all in VI-order model. The two time-scale aggregation procedures involve, for the first, representing the VI-order model in reduced III-order, and, for the second, replacing several generator with an equivalent one. The Podomore’s method share many similarities with the Two-Times scale method, but much of dynamics reduction is based on averaging techniques. The last method is based on system identification techniques whose global optimization procedure is handled by genetic algorithm with 150 population size and 10 elites. Compared with generators in original Kundur’s system whose specifications are list in Table 6.1, the aggregated generator data is shown in Table 6.3. It can be seen that the two time-scale and Podomore’s method have different way of handling the dynamic of \(e_q\), thus resulting in much different value of \(T_d'\), \(v_f\) and \([x_d, x_d']\) between these two cases. Furthermore, the identification method results in \(P_m\) value being approximately half of that in the other two cases. The experiments showed that optimal values for the aggregated generator, to which the identification
6. Results and validation

Figure 6.4: Plot of parts simulation results: from the top to bottom and from left to right, the subplots correspond to 1) Active power though boundary branch(es), i.e. Bus 6–7 in Figure 6.3 or Bus 8–9 in Figure 6.1, 2) Reactive power through boundary branch(es) 3) Voltage profile at boundary bus, i.e. Bus 6 in Figure 6.3, 4) Voltage profile at Bus 1, 5) Active power production of Gen 1, and 6) power angle $\delta$ of Gen 1. The three cases from Podmore’s method, two time-scale method and System identification based method are marked with blue, red and green color respectively, “measurement data” refers to the “real” simulation result from unreduced system

procedure lead, to a large extend depend on the initial guesses. A poor specification for the initial populations may lay in another region of attraction. One may set population guesses based on results from Two-time scale method or Podmore’s method, then resort to system identification method for improvement.

The simulation plot for some of variables are included in the Figure 6.4. To incite system oscillation while keep the system at least transient stable, a necessary requirement for studying interarea oscillation and small signal stability analysis, a three-phase-to-ground fault is assumed at the Bus 8 of Figure 6.1. Fault happens at $t = 1\,\text{s}$ and is cleared at $t = 1.05\,\text{s}$. The generator controllers like AVR are disabled for this moment because only the
6.3. Generator aggregation

identification based method and two time-scale method (the first layer) explicitly considered the dynamic simplification in presence of generator controller.
Chapter 7

Conclusion

The thesis presents various methods for generator dynamics aggregation and its relevant task of coherent area identification. Some simple test cases with various methods are also included and compared.

The loose/tight coherent aggregation is applied for grouping generators with large coherency. Tests have been applied on the Kundur’s 4 machine system and, for a special case, on IEEE 14 bus system. In both cases, algorithm correctly identified generator belong to different predefined groups. Especially, in the case IEEE 14 bus system, it accurately classify a big generator representing external power source and an internal generator representing local power production into two different groups, whereas other synchronous machines that are synchronous condenser in reality are set aside as separate group. The tolerance involved in the algorithm gives a leeway to “granularity” of the resulting generator groups.

Two methods are proposed to complement the coherent generator grouping from the previous result by extending the range of generator grouping into load buses. The first one based on newly suggested generator influence factor on load buses, including potentially the buses of renewable power sources. The other one is inspired by previous study in complex networks, the concept edge betweenness assisted by power flow sensitivity analysis is applied to separate power system network graph into different partitions. Comparing the partition with the generator grouping, one can easily assign load buses into different coherent group. Both methods are tested in Kundur’s 4 machine system, and both got correct results. The later method is also tested in a case of IEEE 14 bus system, a system with no particular interarea structure. It can successfully identify the weakest branches that in turn roughly demarcate the load buses according to coherent areas.

The Kron reduction and its application is also considered. In the discussion of traditional Kron reduction, the relation between Kron reduction widely used
in circuit study and Schur complement in algebra has been the focus. Such analysis leads to an extension of Kron reduction for use of ZIP load model reduction.

Finally three methods are considered in developing synchronous generator aggregation. For each coherent area, unless explicit specification, all three methods aggregate all generators existing in the coherent area into one equivalent. The Podmore’s method proposes augmenting system buses for sitting the aggregated generators and removing generators from their original buses. Usually, Kron reduction ensues for removing unnecessary buses. For dynamic reduction, the Podmore’s method is based on averaging. The two-time scale method takes one step future by removing part of dynamics related with the fast manifold. The two time-scale method is also closely related to perturbation theory where the two time-scale property is evident. Particularly, a new equation (3.70) is proposed, which includes the conclusion of previous two time-scale aggregation method as a special case. Its two layer structure handles dynamic reduction within generator model and reduction among system level separately. The identification based method makes use of nonlinear gray box identification. For the method involved in this study, it is based on global optimization, where prolonged calculation time limits its application. However, other method like nonlinear least square might be used for improvement. Methods are tested in Kundur’s 4 machine system, showing good results for all three cases.

The suggested future studies should analyze the reduction of dynamics involved in power system controllers and renewable power sources based on the same principle of slow/fast manifold argument extensively exploited in deriving the two time-scale aggregation. More refinement on the Kron reduction extension in use of load aggregation is necessary.
Appendix A

Generator Models

A.1 VI-order generator model

A.1.1 Generator model

\[ \dot{\delta} = 2\pi f_0(\omega - 1) \]
\[ \dot{\omega} = \frac{p_a m}{m} - \frac{P_{gen}}{m(\omega - 1)} - \frac{r_2}{r_2 + x_q^a x_d} \left( \left( r_a \left( e_q^a - v_g \sin(\delta - \theta) \right) + x_q^a \left( e_q^a - v_g \cos(\delta - \theta) \right) \right)^2 + \left( r_d \left( e_d^a - v_g \cos(\delta - \theta) \right) - x_d^a \left( e_d^a - v_g \sin(\delta - \theta) \right) \right)^2 \right) \]
\[ e_q' = \frac{1}{T_{d0}} \left( e_q' - f_1(e_q') - \left( x_d - x_q' - \frac{T_n}{T_{d0}} \frac{x_q^a}{x_d} (x_d - x_q') \right) \left( \frac{e_q^a - v_g \sin(\delta - \theta)}{r_2 + x_q^a x_d} \frac{x_q^a}{x_d} - \frac{x_q^a - v_g \cos(\delta - \theta)}{r_2 + x_q^a x_d} \right) \right) \]
\[ e_d' = \frac{1}{T_{d0}} \left( e_d' - f_1(e_d') - \left( x_d - x_q' + \frac{T_n}{T_{d0}} \frac{x_q^a}{x_d} (x_d - x_q') \right) \left( \frac{e_d^a - v_g \cos(\delta - \theta)}{r_2 + x_q^a x_d} \frac{x_q^a}{x_d} - \frac{x_q^a - v_g \sin(\delta - \theta)}{r_2 + x_q^a x_d} \right) \right) \]
\[ e_q'' = \frac{1}{T_{q0}} \left( -e_q'' + e_q' - \left( x_d - x_q'' + \frac{T_n}{T_{q0}} \frac{x_q^a}{x_q} (x_d - x_q'') \right) \left( \frac{e_q^a - v_g \sin(\delta - \theta)}{r_2 + x_q^a x_d} \frac{x_q^a}{x_q} - \frac{x_q^a - v_g \cos(\delta - \theta)}{r_2 + x_q^a x_d} \right) \right) \]
\[ e_d'' = \frac{1}{T_{q0}} \left( -e_d'' + e_d' + \left( x_q' - x_q'' + \frac{T_n}{T_{q0}} \frac{x_q^a}{x_q} (x_q' - x_q'') \right) \left( \frac{e_d^a - v_g \cos(\delta - \theta)}{r_2 + x_q^a x_d} \frac{x_q^a}{x_q} - \frac{x_q^a - v_g \sin(\delta - \theta)}{r_2 + x_q^a x_d} \right) \right) \]
A. Generator Models

A.1.2 Jacobian matrix

\[
\begin{align*}
\frac{\partial f_1}{\partial \omega} &= 2\pi f_0 \\
\frac{\partial f_2}{\partial \omega} &= 2\pi \left( \frac{r_x \cos \delta - \theta - x^* \sin \delta - \theta}{r^2 + x^* y^*} \right) \\
\frac{\partial f_3}{\partial \omega} &= \frac{1}{m} \left( \frac{r_x}{r^2 + x^* y^*} \right) \\
\frac{\partial f_4}{\partial \omega} &= -2\pi \left( \frac{r_x - y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_5}{\partial \omega} &= -\frac{1}{T_{d0}} \left( -x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_6}{\partial \omega} &= \frac{1}{T_{d0}} \left( x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_7}{\partial \omega} &= -1 \left( -x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_8}{\partial \omega} &= -\frac{1}{T_{d0}} \left( x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_9}{\partial \omega} &= 1 \left( x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_{10}}{\partial \omega} &= -1 \left( -x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right) \\
\frac{\partial f_{11}}{\partial \omega} &= 1 \left( x^* - \frac{T_d}{T_{d0}} y^* (x_d - x^*) \right) \left( \frac{x^*/r^2 + x^* y^*}{r^2 + x^* y^*} \right)
\end{align*}
\]

A.2 III-order generator model

A.2.1 Generator model

\[
\dot{\delta} = 2\pi f_0 (m - 1)
\]

\[
\dot{\omega} = \frac{p_m}{m} - \frac{p_{gen}}{m} - \frac{d}{m} (\omega - 1) - \frac{r_a}{m} \left( \frac{r_a v_y \sin (\delta - \theta) + x_q (v'_q - v_y \cos (\delta - \theta))}{r^2_a + x_q x_d^2} \right)^2 + \frac{r_a (v'_q - v_y \cos (\delta - \theta)) + x_q (v'_q - v_y \sin (\delta - \theta))}{r^2_a + x_q x_d^2}
\]

\[
e'_q = \frac{1}{T_{d0}} \left( v'_q - f_3 (e'_q) - (x_d - x'_d) \left( \frac{r_a v_y \sin (\delta - \theta)}{r^2_a + x_q x_d^2} + x_q (v'_q - v_y \cos (\delta - \theta)) \right) \right)
\]
A.2.2 Jacobian matrix

\[ \frac{\partial f_1}{\partial \omega} = 2\pi f_0 \]

\[ \frac{\partial f_2}{\partial \delta} = \frac{2r_a x'_d}{m} \left(i_d \left( \frac{r_a}{r_a^2 + x_q x'_d} \cos(\delta - \theta) - \frac{x_q}{r_a^2 + x_q x'_d} \sin(\delta - \theta) \right) + \frac{x'_d}{r_a^2 + x_q x'_d} \cos(\delta - \theta) + \frac{r_a}{r_a^2 + x_q x'_d} \sin(\delta - \theta) \right) \]

\[ \frac{\partial f_2}{\partial \omega} = \frac{1}{m} \]

\[ \frac{\partial f_2}{\partial e'_q} = -\frac{2r_a}{m} \left( i_d \frac{x_q}{r_a^2 + x_q x'_d} + \frac{r_a}{r_a^2 + x_q x'_d} \right) \]

\[ \frac{\partial f_3}{\partial \delta} = \frac{1}{T_d} \left( x_d - x'_d \right) \left( \frac{r_a}{r_a^2 + x_q x'_d} \cos(\delta - \theta) - \frac{x_q}{r_a^2 + x_q x'_d} \sin(\delta - \theta) \right) \]

\[ \frac{\partial f_3}{\partial \omega} = \frac{1}{T_d} \left( x_d - x'_d \right) \frac{x_q}{r_a^2 + x_q x'_d} \]
Appendix B

Some proofs

B.1 Existence of (real number) common divisor and commensurable

Theorem B.1 Given two nonzero real numbers, \( x, y \in \mathbb{R} \), there exists a real number \( g \) such that \( \frac{x}{g} \) and \( \frac{y}{g} \) are integer if and only if \( x \) and \( y \) are commensurable.

Proof For the first part ("\( \Rightarrow \)"), we need to show that if \( \{x, y\} \) having common divisor, that is the \( g \), then \( x \) and \( y \) are commensurable. From the definition of (real number) common divisor, the footnote (†) on page 59, we get

\[
\frac{x}{g}, \frac{y}{g} \in \mathbb{Z}
\]

thus the quotient,

\[
\frac{x}{g} = \frac{y}{g} \in \mathbb{Q}
\]

that is to say \( x \) and \( y \) are commensurable.

For the second part ("\( \Leftarrow \)"), we need to show that if \( x \) and \( y \) is commensurable we can find a common divisor. Due to the hypothesis \( \frac{x}{y} \in \mathbb{Q} \) we can have

\[
\frac{x}{y} = \frac{p}{q} \quad \text{for} \quad p, q \in \mathbb{Q} \quad \text{and} \quad p, q \neq 0
\]

due to the our hypothesis. Then we choose \( g = \frac{x}{y} \). Verifying

\[
\frac{x}{x/p} = p \in \mathbb{Z} \quad \frac{y}{y/p} = y/p = q \in \mathbb{Z}
\]

we are assured that \( g = \frac{x}{y} \) is indeed a common divisor of \( x \) and \( y \). \( \square \)
Bibliography


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