Natural Deduction for Non-Classical Logics*

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Abstract

We present a framework for machine implementation of families of non-classical logics with Kripke-style semantics. We decompose a logic into two interacting parts, each a natural deduction system: a base logic of labelled formulae, and a theory of labels characterizing the properties of the Kripke models. By appropriate combinations we capture both partial and complete fragments of large families of non-classical logics such as modal, relevance, and intuitionistic logics. Our approach is modular and supports uniform proofs of soundness, completeness and proof normalization. We have implemented our work in the Isabelle Logical Framework.

1 Introduction

Natural deduction (ND) is the outcome of a combination of philosophical and practical concerns [11, 22, 42, 51]. Its philosophical roots are in the attempts of intuitionistic philosophers to provide a theory of meaning for the logical connectives in terms of deduction. This analysis is closely related to the claim that ND directly reflects the reasoning of a working mathematician, unlike, e.g, Hilbert systems. Independently of philosophical concerns, ND can be seen

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simply as a practically usable notation for formal derivation, which provides the deduction theorem directly, rather than as a metatheorem.

Our work is on the practical, not the philosophical, side: we are interested in exploiting *logical frameworks* [26, 34, 39, 41], i.e. formal notations providing support for the uniform implementation of different logics, which are particularly suitable for ND. Specifically, we address the problem of how to present families of non-classical logics so that they can be implemented in a logical framework in a usable way.

The problem is not trivial: these logics are usually presented as Hilbert systems and, even if a presentation is a ND system, it is a modified form of ND that cannot be encoded directly in a logical framework. Further, there may be little or no regularity in the proofs of standard metatheorems for related proof systems (e.g. soundness and completeness with respect to the intended semantics), so that techniques have to be developed from scratch for each member. As a result, finding 'good' presentations is still specialist work.[1]

The particular non-classical logics we consider are those with non-classical connectives (we also use *modality* and *non-local connective* as synonyms for *non-classical connective*), which can, e.g., assert the necessity or contingency of propositions, or take account, in some way, of context. Many of these logics can be interpreted using a Kripke-style semantics consisting of a set of 'worlds' between which a relation has been defined, where the meaning of a non-classical connective at some world is defined in terms of conditions at others. Thus, e.g., the $\Box$ and $\Diamond$ of modal logic can be interpreted in terms of a binary relation [29], relevant implication can be interpreted using a ternary relation [12], and non-classical negation can be interpreted again using a binary relation [10, 13]. In each case a class of logics is defined by variations of the behavior of the relation alone. We show how to exploit this view of non-classical logics as a basis for ND presentations, where we can (i) exploit modularity in the semantics so that related logics result from modifications just to the behavior of the relations, and (ii) prove metatheoretic results in a modular fashion; i.e. the soundness and completeness of encodings, and proof normalization results, are parameterized, along with the presentations themselves, over the properties of the relations.

ND, even though recognized as usually one of the more practical notations for a proof system, is often considered badly suited for non-classical logics, because it builds in too many assumptions. The problem is that proof under assumption needs a deduction theorem:

> if assuming A true we can show B true, then $A \to B$ is true.

But for implications weaker or substantially different from intuitionistic $\to$, this fails (at least for the conventional reading of 'if-then' that we get in ND). Attempts to build ND presentations of non-classical logics other than intuitionistic logic have introduced various technical devices to get around the problem;

[1] With sufficient effort, a logical framework can implement *any* (recursively enumerable) proof system, but the resulting encoding does not necessarily 'fit' well (see [21], where a concept of a *natural representation* in a framework is formalized and investigated).

e.g. Dunn [12], for relevance logics, considers 'relevant' ND, where rules have side conditions on discharged assumptions, and Prawitz [42], for (some) modal logics, proposes rules for $\Box$ that impose side conditions on the set of hypotheses.[2] The continuing primacy of Hilbert presentations in non-classical logics, despite the difficulty in actually using them, is evidence that these inventions have not been completely successful.

Nevertheless, in this paper we present non-classical logics as ND proof systems. However our systems, unlike those of Dunn or Prawitz, fit well in a standard framework: all our rules are *ordinary* (insensitive to thinning or contraction of assumptions), *pure* (have no non-local side conditions), and *single-conclusioned*.[3] Our presentations are partitioned into interacting parts: a *base logic* of propositional[4] logic and a *relational theory* characterizing the properties of the relation(s); the base logic stays fixed for a given family, and we generate the particular logic we want by 'plugging in' the appropriate relational theory. The task we address here is how to reduce non-classical logics to these interacting well-behaved ND systems for which we can prove general metatheorems. To carry out this program, we combine the language of ND with that of a *Labelled Deductive System* (LDS), as proposed by Gabbay [18] and others (e.g. [17, 50]). We show that for many non-classical logics with Kripke-style semantics it is possible, using an LDS, to provide a presentation partitioned in such a way as to support the modularity we want.

To illustrate, take the example of modal logic, where the usual deduction theorem fails. The standard Kripke completeness theorem tells us that A is provable ($\vdash A$) iff A is true at every world in every suitable frame (W, R), where W is the set of possible worlds, and R is the (accessibility) relation between worlds, or, in symbols, $\vdash A$ iff $\forall w \in \mathsf{W}(w \vDash A)$. The deduction theorem, as formulated above, then corresponds to

$$(\forall w \in \mathsf{W}(w \vDash A) \Rightarrow \forall w \in \mathsf{W}(w \vDash B)) \Rightarrow \forall w \in \mathsf{W}(w \vDash A \to B)\,,$$

where $\Rightarrow$ is implication in the meta-language and $\to$ is implication in the object language. But this is false, the semantics of $\to$ in a Kripke frame is just the weaker:

$$\forall w \in \mathsf{W}((w \vDash A \Rightarrow w \vDash B) \Rightarrow w \vDash A \to B)\,. \tag{1}$$

Thus a naive attempt to embed modal logic in a ND system will fail. Suppose, however, that we extend ND to be over pairs drawn from the language of modal logic *and labels*; i.e. instead of $\vdash A$, we consider $\vdash w{:}A$, where w is a 'world', and $\vdash A$ iff $\forall w \in \mathsf{W}(\vdash w{:}A)$. This provides a language in which we can formulate

[2] Different approaches to proof under assumption in modal logics, based on modified sequent systems, have been proposed, e.g. [3, 7, 9, 17, 32, 33, 57]. We do not consider such systems here, but see [56] for detailed comparison and discussion.

[3] We use the vocabulary of [2], which should be consulted for a technical discussion of consequence relations, and which notes (§5.5) that 'every ordinary, pure single-conclusioned ND system can, e.g., quite easily be implemented in the Edinburgh LF.'

[4] We do not, here, consider quantified non-classical logics, which can be presented by means of quantifier rules similar to those of free logic.

a deduction theorem corresponding to (1), and provides a basis for a standard ND system of the sort we need. Moreover, we can use the same notation to express the general behavior of modal operators like $\Box$ in a way that is independent of the (relational) details of the Kripke model providing the semantics, i.e. $\vdash w{:}\Box A$ iff $\vdash w'{:}A$ for all $w' \in \mathsf{W}$ accessible from w. Then by formalizing the details of particular accessibility relations we can produce particular modal logics. This treatment has obvious similarities to traditional semantic embeddings (i.e. translations into predicate logic [27, 35, 36]), but it offers substantial advantages: our formalization does not require all of first-order logic and it yields structured ND systems where the separation between the base logic and the relational theory gives us better proof normalization results (see Theorem 29 and its commentary).

In [5] we investigated labelled ND presentations of modal logics based on classical propositional logic. This paper, which is an expanded account of [4], explores the generalizations needed to build proof systems for both positive and full fragments of large families of non-classical logics, including relevance and intuitionistic logic, and to treat non-classical negation as a modality (the metatheory of positive logics is different from that for full logics, see, e.g., Dunn's semantic analysis of positive modal logics in [15]). We provide a framework for a uniform treatment of a wide range of non-classical connectives ($\Box$, $\Diamond$, relevant and intuitionistic implication, non-classical negation, etc.). Our framework is based on an abstract classification of modalities as 'universal' or 'existential', and associated general metatheorems. We have implemented our approach in the Isabelle Logical Framework [40], which supports management of separate theories and their structured combination, and the result is a parameterized proof development system where (although this is not a formally quantifiable property) proof construction is natural and intuitive.

Organization The remainder of our paper is organized as follows. In Section 2 we formalize our presentations of labelled non-classical logics as base logics extended with relational theories; we provide an example, a presentation of the relevance logic **R**, which also shows the advantages of our approach over Hilbert axiomatizations. In Section 3 we modularly prove soundness and completeness of our presentations with respect to Kripke semantics (Theorem 12), and discuss the possible incompleteness of unrestricted positive fragments (Theorem 21). In Section 4 we consider the proof-theoretic properties, including proof normalization, of our encodings (Theorem 29) and contrast, in these terms, our systems with related formalisms such as semantic embedding. Finally, in Section 5 we compare our work with other approaches based on LDS presentations or algebra, and discuss future work. An Appendix contains a sketch of our implementation in Isabelle, its application, and its correctness proof.

2 Labelled Non-Classical Logics

In this section we formalize our presentations. We introduce the fundamentals of how a labelled ND presentation relates to a Kripke semantics (Section 2.1). Then we define the base logic (Section 2.2) and the associated class of relational theories over which it is parameterized (Section 2.3). Finally, we give examples of labelled ND presentations for non-classical logics (Section 2.4).

2.1 Labels and Kripke Models

Let W be a set of *labels* ranging over worlds in a Kripke model, and R an $n+1$-ary relation over W. If $a, a_1, \ldots, a_n$ are labels and A is a formula, then we call $R\,a\,a_1 \ldots a_n$ a *relational formula* (*rwff*), and $a{:}A$ a *labelled formula* (*lwff*). Formulae are built from connectives, which are partitioned into two families: *local* and *non-local*. If a formula A is built from a local connective $\mathcal{C}$ of arity n, $A = \mathcal{C}(A_1, \ldots, A_n)$, then the truth of the lwff $a{:}A$ depends only on the (local) truth of $a{:}A_1, \ldots, a{:}A_n$. Typical local connectives are conjunction ($\wedge$), disjunction ($\vee$), material implication ($\supset$), and 'local' negation ($\sim$). Where $\models^{\mathsf{M}}$ is the truth relation for lwffs in the model M, we have:[5]

$$\models^{\mathsf{M}} a{:}A \wedge B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ and } \models^{\mathsf{M}} a{:}B \tag{2}$$

$$\models^{\mathsf{M}} a{:}A \vee B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ or } \models^{\mathsf{M}} a{:}B \tag{3}$$

$$\models^{\mathsf{M}} a{:}A \supset B \text{ iff } \models^{\mathsf{M}} a{:}A \text{ implies } \models^{\mathsf{M}} a{:}B \tag{4}$$

$$\models^{\mathsf{M}} a{:} \sim A \text{ iff } \not\models^{\mathsf{M}} a{:}A \tag{5}$$

For notational simplicity, we omit parentheses where possible and write binary connectives in infix notation (as above).

A non-local connective $\mathcal{M}$ of arity n is associated with an $n+1$-ary relation R on worlds, and the truth of $a{:}\mathcal{M}A_1 \ldots A_n$ is evaluated non-locally at the worlds R-accessible from a; i.e. in terms of the truth of $a_1{:}A_1, \ldots, a_n{:}A_n$ where $R\,a\,a_1 \ldots a_n$.

Examples of non-local connectives and associated relations are the unary modal operator $\Box$ and the binary accessibility relation on possible worlds, or relevant implication $\rightarrow$ and the ternary compossibility relation. We extend $\models^{\mathsf{M}}$ to express truths for rwffs in a Kripke model M with an $n+1$-ary relation R as

$$\models^{\mathsf{M}} R\,a\,a_1 \ldots a_n \text{ iff } (a, a_1, \ldots, a_n) \in \mathsf{R}\,, \tag{6}$$

and we call $\mathcal{M}$ a *universal non-local connective* when the metalevel quantification in the evaluation clause of $\mathcal{M}$ is universal (and the body is an implication):

$$\begin{aligned} \models^{\mathsf{M}} a{:}\mathcal{M}(A_1, \ldots, A_n) \text{ iff for all } a_1, \ldots, a_n ((&\models^{\mathsf{M}} R\,a\,a_1 \ldots a_n \\ &\text{and } \models^{\mathsf{M}} a_1{:}A_1 \text{ and} \ldots \text{and } \models^{\mathsf{M}} a_{n-1}{:}A_{n-1}) \text{ imply } \models^{\mathsf{M}} a_n{:}A_n)\,. \end{aligned} \tag{7}$$

[5] $\sim$ can also be defined in terms of $\supset$ and *falsum* ($\bot$). Then we can compare, like for modal logics in [5], the logics obtained when (i) $\models^{\mathsf{M}} a{:} \sim A$ iff $\models^{\mathsf{M}} a{:}A$ implies $\models^{\mathsf{M}} b{:}\bot$, and (ii) $\bot$ is 'global', i.e. $\models^{\mathsf{M}} a{:}\bot$ implies $\models^{\mathsf{M}} b{:}A$, with the (paraconsistent) logics where (i′) $\models^{\mathsf{M}} a{:} \sim A$ iff $\models^{\mathsf{M}} a{:}A$ implies $\models^{\mathsf{M}} a{:}\bot$, and (ii′) $\bot$ is 'local', i.e. $\models^{\mathsf{M}} a{:}\bot$ implies $\models^{\mathsf{M}} a{:}A$.

Similarly, $\mathcal{M}$ is an *existential non-local connective* when the metalevel quantification is existential (and the body is a conjunction):

$$\vDash^{\mathsf{M}} a{:}\mathcal{M}(A_1,\ldots,A_n) \text{ iff there exist } a_1,\ldots,a_n(\vDash^{\mathsf{M}} R\,a\,a_1\ldots a_n \text{ and } \vDash^{\mathsf{M}} a_1{:}A_1 \text{ and}\ldots\text{and } \vDash^{\mathsf{M}} a_{n-1}{:}A_{n-1} \text{ and } \vDash^{\mathsf{M}} a_n{:}A_n)\,. \quad (8)$$

In these terms $\Box$ and (relevant) $\rightarrow$ are universal non-local connectives, $\Diamond$ is existential, and their evaluation clauses are special cases of (7) and (8), e.g.

$$\vDash^{\mathsf{M}} a{:}A_1 \rightarrow A_2 \text{ iff for all } a_1, a_2((\vDash^{\mathsf{M}} R\,a\,a_1\,a_2 \text{ and } \vDash^{\mathsf{M}} a_1{:}A_1) \text{ imply } \vDash^{\mathsf{M}} a_2{:}A_2)\,. \quad (9)$$

A uniform treatment of negation plays a central role in our framework. However, in the Kripke semantics for relevance (and other) logics, both a formula and its negation may be 'true' at a world, which cannot be the case with $\sim$. Thus a new connective is introduced, a *non-local negation* $\neg$, formalized by a unary function $*$ on worlds [12]:

$$\vDash^{\mathsf{M}} a{:}\neg A \text{ iff } \nvDash^{\mathsf{M}} a^*{:}A\,. \quad (10)$$

(Note that we overload the symbol $*$ by using it in both the syntax and the semantics.) Informally, a^* is the world that does not deny what a asserts, i.e. a and a^* are compatible worlds. We generalize this by introducing the constant $\perp\!\!\!\perp$ that expresses *incoherence* of compatible worlds and replace (10) with

$$\vDash^{\mathsf{M}} a{:}\neg A \text{ iff for all } b(\vDash^{\mathsf{M}} a^*{:}A \text{ implies } \vDash^{\mathsf{M}} b{:}\perp\!\!\!\perp)\,, \quad (11)$$

where $\nvDash^{\mathsf{M}} b{:}\perp\!\!\!\perp$ for every world b.

Some remarks. First, when relevant implication is present, we can define $a{:}\neg A$ as $a{:}A \rightarrow \perp\!\!\!\perp$, and postulate $R\,a\,a^*\,b$ for every b, so that (11) is just a special case of (9).[6] Second, when $a = a^*$, e.g. for modal or classical logic, $\perp\!\!\!\perp$ reduces to $\perp$, $\neg$ to $\sim$, and (11) to (5). Finally, there is an alternative approach to non-local negation, e.g. for relevance, linear and ortho-logic [10, 13, 24, 25, 45], which uses an *incompatibility relation* N between worlds:

$$\vDash^{\mathsf{M}} a{:}\neg A \text{ iff for all } b(\vDash^{\mathsf{M}} b{:}A \text{ implies } b\ N\ a)\,. \quad (12)$$

Then a^* is the 'strongest' world b for which $b\ N\ a$ does *not* hold. This can be shown to be equivalent to our approach (for a comparison of (12) with (10) see [13]).

We define the language of a non-classical logic L as follows:

Definition 1 *Let I, J be two finite sets of indices. The language of a non-classical logic L is a tuple $(W, *, S, O, F)$. W is a set of labels closed under $*$ of type $W \rightarrow W$. S is a denumerably infinite set of propositional variables. O*

[6] That a and a^* are 'compossible' according to every b, as stated by $R\,a\,a^*\,b$, is justified by the meaning of $*$.

is the set whose members are (i) the constant $\perp\!\!\!\perp$ (and/or $\perp$); (ii) local and/or non-local negation (or neither for positive logics); (iii) a set of local connectives $\{\mathcal{C}_j \mid j \in J\}$; (iv) a set of non-local connectives $\{\mathcal{M}_i \mid i \in I\}$ with an associated set $\overrightarrow{R} = \{R_i \mid i \in I\}$ of relations of the appropriate arities. F is the set of rwffs and lwffs: if $a, a_1, \ldots, a_n$ are labels, R_i has arity $n+1$, and A is a formula built up from members of S and O, then $R_i\, a\, a_1 \ldots a_n$ is an rwff and $a{:}A$ is an lwff.

Note that by associating different relations to universal and existential non-local connectives, we make no a priori assumptions about their interrelationships (when the relations are not independent, incompleteness may arise, cf. Theorem 21 in Section 3.3).

Definition 2 *A* frame *(for the logic L) is a tuple* $(\mathsf{W}, \mathsf{G}, \overrightarrow{\mathsf{R}}, *)$*, where* W *is a non-empty set of* worlds*,* $\mathsf{G} \in \mathsf{W}$ *is the* actual world*,* $\overrightarrow{\mathsf{R}} = \{\mathsf{R}_i \mid i \in I\}$ *is the set of relations over* W *corresponding to* $\overrightarrow{R}$*, and* $*$ *is a function of type* $\mathsf{W} \to \mathsf{W}$*. A* model M *(for L) consists of a frame and a function* V *mapping elements of* W *and propositional variables to truth values (*0 *or* 1*), where*

$$\models^{\mathsf{M}} a{:}p \text{ iff } \mathsf{V}(a, p) = 1 \tag{13}$$

and $\models^{\mathsf{M}}$ *is extended to lwffs with local and non-local connectives and to rwffs as above. When* $\models^{\mathsf{M}} \varphi$*, for φ an lwff or an rwff, we say that φ is* true *in* M*.*

A non-classical logic L is characterized by its language and by its models, i.e. the conditions independently imposed on $*$, each R_i, etc. Moreover, some logics, e.g. intuitionistic and relevance logics, require truth to be monotonic. To express this, we define a partial order $\sqsubseteq$ on worlds, where for intuitionistic logic $\sqsubseteq$ coincides with the accessibility relation, while for relevance logics it can be defined in terms of R, i.e. $a \sqsubseteq b$ iff $R\, G\, a\, b$; for modal logic $\sqsubseteq$ reduces to equality. Then we require that V satisfy the *atomic monotony condition*, i.e. for any a_i, a_j and for any propositional variable p

$$\text{if } \models^{\mathsf{M}} a_i{:}p \text{ and } \models^{\mathsf{M}} a_i \sqsubseteq a_j, \text{ then } \models^{\mathsf{M}} a_j{:}p\,. \tag{14}$$

One might be tempted to generalize this immediately to arbitrary formulae; this is in fact the case for 'usual' non-classical logics, such as intuitionistic and relevance logics, where we can prove by induction on the structure of A that

$$\text{if } \models^{\mathsf{M}} a_i{:}A \text{ and } \models^{\mathsf{M}} a_i \sqsubseteq a_j, \text{ then } \models^{\mathsf{M}} a_j{:}A\,. \tag{15}$$

But there are logics for which (15) does not hold for every formula; e.g. [16, 30] combine intuitionistic implication $\to$ with classical implication $\supset$, and show that (15) holds for $A \to B$ (in fact it holds, as one would expect, for every intuitionistic formula) but it fails for $A \supset B$. This problem is solved there by restricting (15) to *persistent* formulae. Formally, a formula A of the 'intuitionistic/classical' logic in [16] is persistent if (i) it is atomic, or (ii) if it is of the form $B \to C$ or $\neg B$, where $\neg$ is intuitionistic (and thus non-local) negation, or (iii) it is of the form $B \wedge C$ or $B \vee C$, and B and C are both persistent. Similar

definitions of persistency can be given for other non-classical logics, depending on the particular language we are considering, and we can then restrict (15) to the following general property (provable from (14) by induction on the structure of A).

Property 3 *For any a_i, a_j and for any persistent formula A, if $\models^{\mathsf{M}} a_i{:}A$ and $\models^{\mathsf{M}} a_i \sqsubseteq a_j$, then $\models^{\mathsf{M}} a_j{:}A$.*

Monotony is defined also for rwffs: for an $n+1$-ary relation R_i we require that

$$\text{for all } j < n, \text{ if } \models^{\mathsf{M}} R_i\, a_0 \ldots a_{j-1}\, a_j\, a_{j+1} \ldots a_n \text{ and } \models^{\mathsf{M}} a \sqsubseteq a_j\,, \quad \text{then } \models^{\mathsf{M}} R_i\, a_0 \ldots a_{j-1}\, a\, a_{j+1} \ldots a_n \tag{16}$$

and

$$\text{if } \models^{\mathsf{M}} R_i\, a_0 \ldots a_{n-1}\, a_n \text{ and } \models^{\mathsf{M}} a_n \sqsubseteq a, \text{ then } \models^{\mathsf{M}} R_i\, a_0 \ldots a_{n-1}\, a\,. \tag{17}$$

In the following we assume formulae of the form $a \sqsubseteq b$ to be special cases of relational formulae, but we note that we could introduce them explicitly as a third kind of formulae, independent of lwffs and rwffs (proof-theory and semantics are then extended accordingly). This assumption allows us to treat the properties of the partial order, reflexivity and transitivity, as instances of (16) and (17).

As a notational simplification, we introduce the following definition.

Definition 4 *We will often restrict our attention to non-classical logics with a restricted language containing the local connectives $\wedge, \vee, \supset$, one universal non-local connective $\mathcal{M}^u$ of arity u associated with a $u+1$–ary relation R^u, one existential non-local connective $\mathcal{M}^e$ of arity e associated with an $e+1$–ary relation R^e, non-local negation $\neg$, and the constant $\perp\!\!\!\perp$.*[7] *From Definition 2, a model for such a logic is the tuple $\mathsf{M} = (\mathsf{W}, \mathsf{G}, \mathsf{R}^u, \mathsf{R}^e, *, \mathsf{V})$, and truth for an rwff or lwff φ in M, $\models^{\mathsf{M}} \varphi$, is the smallest relation $\models^{\mathsf{M}}$ satisfying* (2), (3), (4), (6) *and* (7) *for R^u and $\mathcal{M}^u$,* (6) *and* (8) *for R^e and $\mathcal{M}^e$,* (11), (13), (14), (16) *and* (17) *for R^u and R^e.*

Finally note that we do not, here, consider logics like the relevance logic **E** for which models with more than one actual world are needed. These logics can be formalized by employing a set P of actual worlds and modifying the postulates of the relational theory with a precondition testing membership in P; for instance, for identity the postulate $R\,G\,a\,a$ is replaced with '$x \in P$ implies $R\,x\,a\,a$'.

2.2 The Base Logic $\mathcal{B}$

We now introduce the base logic $\mathcal{B}$ that provides the rules we need to reason about lwffs. Our formalization is motivated by pragmatic concerns: the base logic should

[7] Since the language might not contain (the non-local) $\rightarrow$, we take $\neg$ as a primitive operator as opposed to defined by means of $\perp\!\!\!\perp$ and $\rightarrow$.

(i) make no assumptions about the relational theories extending it,

(ii) be adequate for the logics we are interested in, and

(iii) have good proof-theoretic properties.

In [5] we provide a base logic for the modal logics of the Geach hierarchy that satisfies all these criteria. Unfortunately, in the more general case considered here, things are not so clear cut: the base logic depends on the particular family of non-classical logics we formalize, and thus to achieve (ii) and (iii) we have to replace (i) with

(i') make as few assumptions as possible about the relational theories extending it and ideally none at all.

(See Section 2.3, where we discuss 'complementary rules', and Section 4, where we discuss extensions with first-order relational theories.)

We start by considering the simplest connective: classical (local) implication. We adapt the traditional ND rules [42] by adding labels to get the rules $\supset I$ and $\supset E$:

$$\frac{\begin{array}{c}[a{:}A_1]\\ \vdots\\ a{:}A_2\end{array}}{a{:}A_1 \supset A_2}\ \supset I \qquad \frac{a{:}A_1 \supset A_2 \quad a{:}A_1}{a{:}A_2}\ \supset E \tag{18}$$

Rules for $\wedge$, $\vee$ and other local connectives are adapted similarly.

Then we give the rules for $\mathcal{M}^u$ and $\mathcal{M}^e$:

$$\frac{\begin{array}{c}[a_1{:}A_1]\cdots[a_{u-1}{:}A_{u-1}]\,[R^u\, a\, a_1 \ldots a_u]\\ \vdots\\ a_u{:}A_u\end{array}}{a{:}\mathcal{M}^u A_1 \ldots A_u}\ \mathcal{M}^u I$$

$$\frac{a{:}\mathcal{M}^u A_1 \ldots A_u \quad a_1{:}A_1 \cdots a_{u-1}{:}A_{u-1} \quad R^u\, a\, a_1 \ldots a_u}{a_u{:}A_u}\ \mathcal{M}^u E \tag{19}$$

$$\frac{a_1{:}A_1 \cdots a_e{:}A_e \quad R^e\, a\, a_1 \ldots a_e}{a{:}\mathcal{M}^e A_1 \ldots A_e}\ \mathcal{M}^e I$$

$$\frac{a{:}\mathcal{M}^e A_1 \ldots A_e \qquad \begin{array}{c}[a_1{:}A_1]\cdots[a_e{:}A_e]\,[R^e\, a\, a_1 \ldots a_e]\\ \vdots\\ b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e E$$

where, in $\mathcal{M}^u I$ and $\mathcal{M}^e E$, each a_k, a_l $(1 \leq k \leq u,\ 1 \leq l \leq e)$ is fresh; e.g. in $\mathcal{M}^e E$, $a_1, \ldots, a_e$ are all different from a, b and each other, and do not occur in

any assumption on which the upper occurrence of $b{:}B$ depends other than those listed.

Comparing these rules with (4), (7) and (8), we see that they reflect the semantic definitions. When we treat negation, however, the correspondence between the rules and the semantics is more subtle, and we must choose which kind of negation we want to encode. We begin by providing the rules $\neg I$ and $\neg E$, cf. (11):

$$\frac{\begin{array}{c}[a^*{:}A]\\ \vdots\\ b{:}\perp\!\!\!\perp\end{array}}{a{:}\neg A}\ \neg I \qquad \frac{a{:}\neg A \quad a^*{:}A}{b{:}\perp\!\!\!\perp}\ \neg E \tag{20}$$

These rules capture only a *minimal* non-local negation. If we want a base logic capable of formalizing *intuitionistic* or *classical* non-local negation we need the additional rules $\perp\!\!\!\perp Ei$ and $\perp\!\!\!\perp Ec$, respectively:

$$\frac{b{:}\perp\!\!\!\perp}{a{:}A}\ \perp\!\!\!\perp Ei \qquad \frac{\begin{array}{c}[a{:}\neg A]\\ \vdots\\ b{:}\perp\!\!\!\perp\end{array}}{a^*{:}A}\ \perp\!\!\!\perp Ec \tag{21}$$

Finally, we require the rule *monl*, expressing monotony at the level of lwffs:

$$\frac{a_i{:}A \quad a_i \sqsubseteq a_j}{a_j{:}A}\ monl \tag{22}$$

where A is a persistent formula.[8] Since *monl* reflects Property 3, the details of its definition, including the proviso on its application, depend on the logic we are considering.

2.3 Relational Theories

We formalize a logic L by extending (the appropriate) $\mathcal{B}$ with a *relational theory* axiomatizing the properties of $*$ and of the relations R_i in a Kripke model. Correspondence theory [53, 54] and known correspondence results [47] allow us to determine which possible axioms correspond to which properties of $*$ and R_i. Some of these properties can only be expressed using higher-order logic (e.g. the McKinsey axiom $\Box\Diamond A \supset \Diamond\Box A$, and the Löb axiom $\Box(\Box A \to A) \to \Box A$ of provability logic), but for other properties first-order logic, or even fragments of it, is enough. We restrict our attention to properties axiomatizable using *(Horn) relational rules*, i.e. those of the form

$$\frac{R_i\, t_0^1 \ldots t_m^1 \quad \cdots \quad R_i\, t_0^n \ldots t_m^n}{R_i\, t_0 \ldots t_m}$$

[8] If this restriction is not imposed, then the result is not sound for some logics, e.g. an (attempted) encoding of intuitionistic implication collapses to classical implication, similar to what is shown to happen for Hilbert systems in [16, 30], as remarked above.

where the t_k^j are terms built from labels and function symbols. (Some properties of R_i, e.g. *ass1* and *ass2* below, can be expressed as Horn relational rules only after the introduction of Skolem function constants; see [5], where we use the theorem on functional extensions [49, p. 55] to show that the introduction of Skolem constants is a conservative extension.) A *(Horn) relational theory* $\mathcal{T}$ is a theory generated by a set of such rules.

Even with such a restriction, we are able to capture many families of non-classical logics used in practice. For example, relational theories corresponding to logics in the modal Geach hierarchy (e.g. K, D, T, B, $S4$, $S4.2$, $KD45$, and $S5$), and various relevance logics (e.g. **B**, **N**, **T**, and **R**); cf. Section 2.4.[9]

Thus, for example, the modal axiom $\Box A \supset A$ corresponds to the reflexivity of the accessibility relation,

$$\frac{}{x\ R\ x}\ \mathit{refl}$$

while the axioms of relevance logic $A \to A$ and $(A \to B) \to ((B \to C) \to (A \to C))$ correspond to identity (*iden*) and associativity (*ass1* and *ass2*) for the compossibility relation:

$$\frac{}{R\,G\,a\,a}\ \mathit{iden} \qquad \frac{R\,a\,b\,e \quad R\,e\,c\,d}{R\,b\,h(a,b,c,d,e)\,d}\ \mathit{ass1} \qquad \frac{R\,a\,b\,e \quad R\,e\,c\,d}{R\,a\,c\,h(a,b,c,d,e)}\ \mathit{ass2} \tag{23}$$

where h is a 5–ary Skolem function constant.

For negation we do not extend the properties of R_i directly, but instead refine the behavior of the $*$ function. To encode intuitionistic ($**i$), classical ($**i$, $**c$), or ortho (*ortho1*, *ortho2*) negation, we add the following rules, which impose different behaviors on $*$:

$$\frac{}{a \sqsubseteq a^{**}}\ **i \qquad \frac{}{a^{**} \sqsubseteq a}\ **c \qquad \frac{}{a \sqsubseteq a^{*}}\ \mathit{ortho1} \qquad \frac{}{a^{*} \sqsubseteq a}\ \mathit{ortho2}$$

Finally, corresponding to (16) and (17), we have $n+1$ rules for the monotonic properties of rwffs:

$$\frac{R_i\,a_0 \ldots a_{j-1}\,a_j\,a_{j+1} \ldots a_n \quad a \sqsubseteq a_j}{R_i\,a_0 \ldots a_{j-1}\,a\,a_{j+1} \ldots a_n}\ \mathit{monR}_i(j) \qquad \frac{R_i\,a_0 \ldots a_{n-1}\,a_n \quad a_n \sqsubseteq a}{R_i\,a_0 \ldots a_{n-1}\,a}\ \mathit{monR}_i(n)$$

where $j < n$ in the schematic rule $\mathit{monR}_i(j)$.

Negation and monotony again raise the question of what exactly a base logic should be. The rules we have just given can be seen as rwff complements of the lwff rules given earlier. For instance, for an intuitionistic negation, i.e. where the base logic contains $\perp\!\!\!\perp Ei$, we need $**i$, while for a classical negation, i.e. with $\perp\!\!\!\perp Ec$, we need $**i$ and $**c$; similarly, the monR_i rules complement *monl*. Moreover, only by requiring these complementary rules can we establish

[9] Also, Horn relational rule sets can be directly implemented in the Horn fragment of the metalogics we use for our implementation (it is not necessary first to embed first-order logic or formalize additional judgements, cf. the appendix and [20, 26]). This restriction also yields good proof-theoretic properties, cf. Theorem 29.

L	$\mathcal{B}$	$\mathcal{T}$ (includes at least)
$\mathcal{M}L$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$	$monR_i$ rules (for both R^u and R^e)
$\mathcal{J}L$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$ $\perp\!\!\!\perp Ei$	$monR_i$ rules (for both R^u and R^e) $**i$
$\mathcal{C}L$	rules for $\wedge, \vee, \supset, \mathcal{M}^u, \mathcal{M}^e, \neg$ $monl$ $\perp\!\!\!\perp Ec$	$monR_i$ rules (for both R^u and R^e) $**i$, $**c$

Figure 1: The systems $\mathcal{M}L$, $\mathcal{J}L$ and $\mathcal{C}L$

desired proof-theoretic results (cf. the proof of Theorem 29). Thus it is convenient, on pragmatic grounds, to assume that a base logic $\mathcal{B}$ is extended with a theory that includes these minimal relational rules (a characterization of the logics in which this complementarity is not satisfied, e.g. $\perp\!\!\!\perp Ei$ without $**i$, or $\perp\!\!\!\perp Ec$ with only $**c$, is out of the scope of this paper).

A logic $L = \mathcal{B} + \mathcal{T}$ is the extension of an appropriate base logic $\mathcal{B}$ with a Horn relational theory $\mathcal{T}$. Consider the restricted language of Definition 4 (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\!\perp$). Following Prawitz [42, 43], in Figure 1 we distinguish three families of ND systems according to their treatment of negation: minimal, intuitionistic or classical (we make the distinction by considering the $\perp\!\!\!\perp$ rules, as opposed to Prawitz's $\perp$ rules).

The *minimal system* $\mathcal{M}L$ is determined by a base logic containing $monl$ (with the appropriate restrictions) and introduction and elimination rules for local, e.g. (18), and non-local connectives, cf. (19) and (20), and by a relational theory containing, at least, the $monR_i$ rules, to complement $monl$.[10] The *intuitionistic system* $\mathcal{J}L$ is then obtained by extending $\mathcal{M}L$ with $\perp\!\!\!\perp Ei$ and the complementary rule $**i$, and the *classical system* $\mathcal{C}L$ is obtained by extending $\mathcal{M}L$ with $\perp\!\!\!\perp Ec$ and the complementary rules $**i$ and $**c$. Alternatively, we can obtain $\mathcal{C}L$ by extending $\mathcal{J}L$ with $\perp\!\!\!\perp Ec$ and $**c$ (we can easily show that $\perp\!\!\!\perp Ei$ can be derived from these rules). Furthermore, we can extend $\mathcal{C}L$ with the rules *ortho1* and *ortho2*, to formalize ortho negation, where $a = a^* = a^{**}$. For each of these systems, we can then further extend the relational theory $\mathcal{T}$ with rules expressing properties of the relations to obtain particular logics.

Definition 5 *Let $L = \mathcal{B} + \mathcal{T}$ be an arbitrary non-classical logic. If Γ is a set of lwffs, Δ a set of rwffs, and φ an lwff or an rwff, a* derivation of φ from Γ and Δ in L *is a tree formed using the rules in L, ending with φ and depending only on $\Gamma \cup \Delta$. We write $\Gamma, \Delta \vdash_L \varphi$ when φ can be so derived. A derivation of φ in L depending on the empty set, $\vdash_L \varphi$, is a* proof of φ in L, *and we say that φ is a* theorem *of L.*

[10] Note that, unlike Prawitz's, our minimal system does not satisfy the *inversion principle*, since it contains $monl$ which is neither an introduction nor an elimination rule.

Fact 6 $\Gamma, \Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ *iff* $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$.

Notation 7 *We systematically use* Π*, with or without indices, to range over derivations, and we write* $\begin{matrix}\Pi\\ \varphi\end{matrix}$ *to specify that the formula* φ *is the conclusion of the derivation* Π*. Similarly, we write* $\begin{matrix}\varphi\\ \Pi\end{matrix}$ *or* $\left[\begin{matrix}[\varphi]\\ \Pi\end{matrix}\right]$ *to distinguish a possibly empty set of occurrences of the open [discharged] assumption* φ *in* Π*. Furthermore, we write* $\Pi[b/a]$ *for the systematic substitution of* b *for* a *in* Π*, with a suitable renaming of the variables to avoid clashes, and we use superscripts to associate discharged assumptions with rule applications. Finally, we assume the reader is familiar with the terminology of ND [42, 43, 55], e.g. major and minor premises in an inference.*

2.4 Examples of Non-Classical Logics

Our framework can be specialized to implement fragments and full presentations of large classes of modal and relevance logics. The important (though relatively simple) case of modal logics is discussed at length in [5]. There we show how the base logic K, which consists of the rules

$$\frac{\begin{matrix}[x{:}A]\\ \vdots\\ x{:}B\end{matrix}}{x{:}A \supset B}\ \supset I \qquad \frac{x{:}A \supset B \quad x{:}A}{x{:}B}\ \supset E$$

$$\frac{\begin{matrix}[x\ R\ y]\\ \vdots\\ y{:}A\end{matrix}}{x{:}\Box A}\ \Box I \qquad \frac{x{:}\Box A \quad x\ R\ y}{y{:}A}\ \Box E$$

$$\frac{y{:}A \quad x\ R\ y}{x{:}\Diamond A}\ \Diamond I \qquad \frac{x{:}\Diamond A \qquad \begin{matrix}[y{:}A]\ [x\ R\ y]\\ \vdots\\ z{:}B\end{matrix}}{z{:}B}\ \Diamond E$$

$$\frac{\begin{matrix}[x{:}A \supset \bot]\\ \vdots\\ y{:}\bot\end{matrix}}{x{:}A}\ \bot E$$

where y is fresh in $\Box I$ and $\Diamond E$ ($\Box$ and $\Diamond$ are instances of $\mathcal{M}^u$ and $\mathcal{M}^e$), can be extended with relational rules to yield, e.g., logics in the Geach hierarchy. For instance, $S4$ is obtained by extending K with relational rules expressing reflexivity and transitivity of the accessibility relation:

$$\frac{}{x\ R\ x}\ \textit{refl} \qquad \frac{x\ R\ y \quad y\ R\ z}{x\ R\ z}\ \textit{trans}$$

Here we consider, as an example from relevance logics, the logic **R**: we compare our system with the Hilbert system $\mathbf{R_H}$ of Routley and Meyer [46], and show the advantages of our approach in the modular way we present the system so that it can be extended to obtain (positive and full) intuitionistic and classical logic.

Definition 8 *We define* **R** *as follows. Since Routley and Meyer consider a classical non-local negation (i.e.* $\neg\neg A \to A$ *is an axiom of* $\mathbf{R_H}$*), we use the classical version of* $\mathcal{B}$ *with* $\perp\!\!\!\perp Ec$*.* $\mathcal{B}$ *also includes monl and the rules for* $\wedge$*,* $\vee$*,* $\neg$*, and* $\to$*;* $A \to B$ *is defined as the binary universal modality* $\mathcal{M}^u AB$ *associated with the ternary relation* R*, for which we provide a relational theory generated by:* $**i$ *and* $**c$*, monR*(1) *and*[11] *monR*(3)*, iden, ass1, ass2 (cf.* (23)*), and the additional rules*

$$\frac{}{R\,a\,a\,a}\ idem \qquad \frac{R\,a\,b\,c}{R\,a\,c^*\,b^*}\ anti \qquad \frac{R\,a\,b\,c}{R\,b\,a\,c}\ comm$$

$$\frac{R\,a\,b\,c}{R\,a\,b\,g(a,b,c)}\ cont1 \qquad \frac{R\,a\,b\,c}{R\,g(a,b,c)\,b\,c}\ cont2$$

where g *is a ternary Skolem function constant.* $a \sqsubseteq b$ *is defined to be* $R\,G\,a\,b$*.*

We obtain the positive fragment $\mathbf{R}^+$ *simply by deleting all the rules involving non-local negation (*$\neg I$*,* $\neg E$*,* $\perp\!\!\!\perp Ec$*,* $**i$*,* $**c$*, anti).*

We postpone proofs that **R** and $\mathbf{R}^+$ are what we claim they are, i.e. equivalent to $\mathbf{R_H}$ and ${\mathbf{R_H}}^+$ (which we get from $\mathbf{R_H}$ by deleting the axioms for negation), until Section 3, where we show the soundness and completeness of our presentations with respect to Kripke-style semantics. Here we are interested rather in comparing the modularity of the two presentations.

Routley and Meyer show that there is a problem with their presentations: ${\mathbf{R_H}}^+$ is a subsystem of positive intuitionistic logic, but $\mathbf{R_H}$ is a subsystem *only* of classical logic. That is, full intuitionistic logic **J** cannot be *modularly* obtained by simply adding new axioms to $\mathbf{R_H}$. (**J** can be obtained from $\mathbf{R_H}$, but only in a non-modular fashion, if relevant negation is rejected in favor of an intuitionistic one [46, p. 227].)

Now consider our systems. Positive intuitionistic logic $\mathbf{J}^+$ is obtained from $\mathbf{R}^+$ by adding the rule

$$\frac{}{R\,G\,G\,a}\ int$$

corresponding to the (intuitionistically valid 'thinning') axiom $A \to (B \to B)$, so that the ternary R reduces to a binary partial order (in fact to the usual accessibility relation for Kripke models of intuitionistic logic), and $\to$ reduces to intuitionistic implication. However, extending **R** with the rule *int* yields classical logic: we are able to derive $R\,G\,a\,G$, so that, essentially, all the worlds collapse; i.e. $a = a^* = a^{**}$, $\to$ reduces to $\supset$, and $\neg$ to $\sim$. This should not come as a

[11] *monR*(2) is derivable from *monR*(1), since $R\,a\,b\,c \vdash_{\mathbf{R}} R\,b\,a\,c$.

surprise: in Definition 8 we explicitly defined $\mathbf{R}$ to contain, like $\mathbf{R_H}$, a classical treatment of negation. That is, with reference to Figure 1, we defined $\mathbf{R} = \mathcal{C}\mathbf{R}$. But Figure 1 also tells us that this problem can be naturally overcome in our setting: to restore the *modularity* of the extensions and obtain full intuitionistic logic $\mathbf{J}$, we just need to consider the system $\mathcal{J}\mathbf{R}$ (*intuitionistic* $\mathbf{R}$), which we obtain from $\mathbf{R}$ by substituting $\perp\!\!\!\perp Ec$ with $\perp\!\!\!\perp Ei$, and deleting $**c$. Indeed, $\mathcal{J}\mathbf{R}$ is an intermediate system between $\mathbf{R}^+$ and $\mathbf{R}$, i.e. $\mathbf{R}^+ \subset \mathcal{J}\mathbf{R} \subset \mathbf{R}$, and we can extend it with *int* to obtain full intuitionistic logic.

Proposition 9 *Adding the rule int to* $\mathcal{J}\mathbf{R}$ *results in* $\mathbf{J}$.

We show this by (i) proving that R reduces to a partial order, and (ii) that relevant $\rightarrow$, $\neg$, $\perp\!\!\!\perp$ and the corresponding relevance rules reduce to intuitionistic $\rightarrow$, $\sim$, $\perp$ and the corresponding intuitionistic rules.

To conclude this section we give a couple of derivations in $\mathbf{R}$. We show, first, *contraposition*, i.e. that $G{:}A \rightarrow B$ implies $G{:}\neg B \rightarrow \neg A$, when R is *antitonic*

$$\dfrac{\dfrac{[a{:}\neg B]^2 \quad \dfrac{G{:}A \rightarrow B \quad [b^*{:}A]^1 \quad \dfrac{[R\,G\,a\,b]^2}{R\,G\,b^*\,a^*}\,anti}{a^*{:}B}\,\rightarrow E}{\dfrac{c{:}\perp\!\!\!\perp}{b{:}\neg A}\,\neg I^1}\,\neg E}{G{:}\neg B \rightarrow \neg A}\,\rightarrow I^2$$

and then that $G{:}\neg\neg A \rightarrow A$ is provable (note the use of $\perp\!\!\!\perp Ec$ and $**c$):

$$\dfrac{\dfrac{\dfrac{[b{:}\neg\neg A]^2 \quad [R\,G\,b\,c]^2}{c{:}\neg\neg A}\,monl \quad [c^*{:}\neg A]^1}{\dfrac{d{:}\perp\!\!\!\perp}{c^{**}{:}A}\,\perp\!\!\!\perp Ec^1 \qquad \dfrac{}{R\,G\,c^{**}\,c}\,{**c}}\,\neg E}{\dfrac{c{:}A}{G{:}\neg\neg A \rightarrow A}\,\rightarrow I^2}\,monl$$

This proof, formalized in Isabelle, is given in Appendix A.

3 Soundness and completeness of the presentations

In this section we show that every non-classical logic $L = \mathcal{B} + \mathcal{T}$ is sound and complete with respect to the corresponding Kripke-style semantics. We consider here only the case where $\mathcal{T}$ is a Horn relational theory; extensions of $\mathcal{B}$ with first-order relational theories are discussed in Section 4.

For notational simplicity, we consider again the restricted language of Definition 4 (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\!\perp$); the results generalize easily to unrestricted languages.

Definition 10 *Given a set of lwffs* Γ *and a set of rwffs* Δ*, we call the ordered pair* (Γ, Δ) *a* proof context *(*pc*). When* $\Gamma_1 \subseteq \Gamma_2$ *and* $\Delta_1 \subseteq \Delta_2$*, we write* $(\Gamma_1, \Delta_1) \subseteq (\Gamma_2, \Delta_2)$*. When* $a{:}A \in \Gamma$*, we write* $a{:}A \in (\Gamma, \Delta)$ *irrespective of* Δ*, and when* $R_i\, a\, a_1 \ldots a_n \in \Delta$*, we write* $R_i\, a\, a_1 \ldots a_n \in (\Gamma, \Delta)$ *irrespective of* Γ*. Moreover, if there exists an* A *such that* $a{:}A \in \Gamma$*, or if* a *is an argument of an rwff in* Δ*, we write* $a \in (\Gamma, \Delta)$ *and say that the label* a occurs in (Γ, Δ)*. Finally, we extend the definition of* $\models^{\mathsf{M}}$ *as follows:* $\models^{\mathsf{M}} \Gamma$ *means that* $\models^{\mathsf{M}} a{:}A$ *for all* $a{:}A \in \Gamma$*;* $\models^{\mathsf{M}} \Delta$ *means that* $\models^{\mathsf{M}} R_i\, a\, a_1 \ldots a_n$ *for all* $R_i\, a\, a_1 \ldots a_n \in \Delta$*;* $\models^{\mathsf{M}} (\Gamma, \Delta)$ *means that* $\models^{\mathsf{M}} \Gamma$ *and* $\models^{\mathsf{M}} \Delta$*;* $\Delta \models R_i\, a\, a_1 \ldots a_n$ *means that* $\models^{\mathsf{M}} \Delta$ *implies* $\models^{\mathsf{M}} R_i\, a\, a_1 \ldots a_n$ *for any model* M*; and* $\Gamma, \Delta \models a{:}A$ *means that* $\models^{\mathsf{M}} (\Gamma, \Delta)$ *implies* $\models^{\mathsf{M}} a{:}A$ *for any model* M*.*

The explicit embedding of properties of the models and the possibility of explicitly reasoning about them, via rwffs and relational rules, require us to consider soundness and completeness also for rwffs, where we show that $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ iff $\Delta \models R_i\, a\, a_1 \ldots a_n$.

Definition 11 *Let* Γ *be a set of lwffs and* Δ *a set of rwffs. The logic* L *is* sound *iff (i)* $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ *implies* $\Delta \models R_i\, a\, a_1 \ldots a_n$*, and (ii)* $\Gamma, \Delta \vdash_L a{:}A$ *implies* $\Gamma, \Delta \models a{:}A$*.* L *is* complete *iff the converses hold.*

We have

Theorem 12 *L is sound and complete.*

Proof By Lemma 13 and Lemma 20 below. ■

3.1 Soundness

Lemma 13 *L is sound.*

Proof Throughout the proof let $\mathsf{M} = (\mathsf{W}, \mathsf{G}, \mathsf{R}^u, \mathsf{R}^e, *, \mathsf{V})$ be an arbitrary model for the logic L. We prove (i) by induction on the structure of the derivation of the rwff $R_i\, a\, a_1 \ldots a_n$ from Δ. The base case, $R_i\, a\, a_1 \ldots a_n \in \Delta$, is trivial, and there is one step for each application of a Horn relational rule. We treat only one example, which involves Skolem functions; soundness of the other rules follows similarly.[12] Consider applications of the rules *ass1* and *ass2* for a ternary relation R^u:

$$\frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ R^u\, a\, b\, e & R^u\, e\, c\, d\end{array}}{R^u\, b\, h(a,b,c,d,e)\, d}\ \mathit{ass1} \qquad\qquad \frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ R^u\, a\, b\, e & R^u\, e\, c\, d\end{array}}{R^u\, a\, c\, h(a,b,c,d,e)}\ \mathit{ass2}$$

[12] Note that our models do not contain functions corresponding to possible Skolem functions in the signature. When such constants are present the appropriate Skolem expansion of the model is required [55, p. 137]; e.g. for (relevant) associativity the signature of the relational theory is conservatively extended with a 5-ary Skolem function constant h, and h is also added to the model.

where Π_1 is the derivation $\Delta_1 \vdash_L R^u\, a\, b\, e$, and Π_2 is the derivation $\Delta_2 \vdash_L R^u\, e\, c\, d$, with $\Delta = \Delta_1 \cup \Delta_2$. Assume that R^u is associative and that $\models^{\mathsf{M}} \Delta$. Then from the induction hypotheses we obtain $\models^{\mathsf{M}} R^u\, a\, b\, e$ and $\models^{\mathsf{M}} R^u\, e\, c\, d$, and we conclude $\models^{\mathsf{M}} R^u\, b\, h(a,b,c,d,e)\, d$ and $\models^{\mathsf{M}} R^u\, a\, c\, h(a,b,c,d,e)$.

We prove (ii) by induction on the structure of the derivation of $a{:}A$ from Γ and Δ. The base case, $a{:}A \in \Gamma$, is trivial, and there is one step for each inference rule. We treat only applications of $\mathcal{M}^u I$, $\mathcal{M}^e E$, and $\perp\!\!\!\perp Ec$; soundness of the other rules follows similarly (soundness of *monl* with respect to Property 3 is immediate by the restriction on its application).

For $\mathcal{M}^u I$, consider

$$\dfrac{\begin{array}{c}[a_1{:}A_1]\ \cdots\ [a_{u-1}{:}A_{u-1}]\ [R^u\, a\, a_1 \ldots a_u] \\ \Pi_1 \\ a_u{:}A_u\end{array}}{a{:}\mathcal{M}^u A_1 \ldots A_u}\ \mathcal{M}^u I$$

where Π_1 is the derivation $\Gamma_1, \Delta_1 \vdash_L a_u{:}A_u$, with $\Gamma_1 = \Gamma \cup \{a_1{:}A_1, \ldots, a_{u-1}{:}A_{u-1}\}$ and $\Delta_1 = \Delta \cup \{R^u\, a\, a_1 \ldots a_u\}$. The induction hypothesis is $\Gamma_1, \Delta_1 \vdash_L a_u{:}A_u$ implies $\Gamma_1, \Delta_1 \models a_u{:}A_u$. Assume $\models^{\mathsf{M}} (\Gamma, \Delta)$. Considering the restriction on the application of $\mathcal{M}^u I$, we can extend Γ and Δ to $\Gamma' = \Gamma \cup \{a'_1{:}A_1, \ldots, a'_{u-1}{:}A_{u-1}\}$ and $\Delta' = \Delta \cup \{R^u\, a\, a'_1 \ldots a'_u\}$ for arbitrary $a'_1, \ldots, a'_u \notin (\Gamma, \Delta)$, and assume $\models^{\mathsf{M}} \Gamma'$ and $\models^{\mathsf{M}} \Delta'$. Since $\models^{\mathsf{M}} \Gamma'$ implies $\models^{\mathsf{M}} \Gamma_1$ and $\models^{\mathsf{M}} \Delta'$ implies $\models^{\mathsf{M}} \Delta_1$, from the induction hypothesis we obtain $\models^{\mathsf{M}} a'_u{:}A_u$ for arbitrary $a'_1, \ldots, a'_u \notin (\Gamma, \Delta)$ such that $\models^{\mathsf{M}} R^u\, a\, a'_1 \ldots a'_u$ and $\models^{\mathsf{M}} a'_1{:}A_1, \ldots, \models^{\mathsf{M}} a'_{u-1}{:}A_{u-1}$. We conclude $\models^{\mathsf{M}} a{:}\mathcal{M}^u A_1 \ldots A_u$ from the definition of $\models^{\mathsf{M}}$.

For $\mathcal{M}^e E$, let Π be the derivation

$$\dfrac{\begin{array}{cc}\Pi_1 & [a_1{:}A_1] \cdots [a_e{:}A_e]\ [R^e\, a\, a_1 \ldots a_e] \\ a{:}\mathcal{M}^e A_1 \ldots A_e & \Pi_2 \\ & b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e E$$

That is, Π is $\Gamma, \Delta \vdash_L b{:}B$, where, by the restriction on $\mathcal{M}^e E$, $a_1, \ldots, a_e$ do not occur in (Γ, Δ) and are different from a, b. Moreover, Π_1 is the derivation $\Gamma, \Delta \vdash_L a{:}\mathcal{M}^e A_1 \ldots A_e$, and Π_2 is the derivation $\Gamma \cup \{a_1{:}A_1, \ldots, a_e{:}A_e\}, \Delta \cup \{R^e\, a\, a_1 \ldots a_e\} \vdash_L b{:}B$. By the induction hypothesis for Π_1, we have that $\Gamma, \Delta \models^{\mathsf{M}} a{:}\mathcal{M}^e A_1 \ldots A_e$, and thus, from the definition of $\models^{\mathsf{M}}$, there exist $b_1, \ldots, b_e$ such that $\models^{\mathsf{M}} b_1{:}A_1, \ldots, \models^{\mathsf{M}} b_e{:}A_e$ and $\models^{\mathsf{M}} R^e\, a\, b_1 \ldots b_e$. We can then extend Γ and Δ to $\Gamma' = \Gamma \cup \{a'_1{:}A_1, \ldots, a'_e{:}A_e\}$ and $\Delta' = \Delta \cup \{R^e\, a\, a'_1 \ldots a'_e\}$ for arbitrary $a'_1, \ldots, a'_e \notin (\Gamma, \Delta)$, and from the induction hypothesis for Π_2 we conclude $\Gamma, \Delta \models^{\mathsf{M}} b{:}B$.

For $\perp\!\!\!\perp Ec$, consider

$$\dfrac{\begin{array}{c}[a{:}\neg A] \\ \Pi_1 \\ b{:}\perp\!\!\!\perp\end{array}}{a^*{:}A}\ \perp\!\!\!\perp Ec$$

where Π_1 is the derivation $\Gamma_1, \Delta \vdash_L b{:}\perp\!\!\!\perp$, with $\Gamma_1 = \Gamma \cup \{a{:}\neg A\}$. The induction hypothesis is $\Gamma_1, \Delta \vdash_L b{:}\perp\!\!\!\perp$ implies $\Gamma_1, \Delta \models^{\mathsf{M}} b{:}\perp\!\!\!\perp$. We assume $\models^{\mathsf{M}} (\Gamma, \Delta)$ and

prove $\models^{\mathsf{M}} a^*{:}A$. Since $\not\models^{\mathsf{M}} b{:}\perp\!\!\!\perp$, from the induction hypothesis we obtain $\not\models^{\mathsf{M}} \Gamma_1$, and therefore $\not\models^{\mathsf{M}} \{a{:}\neg A\}$. We conclude $\models^{\mathsf{M}} a^*{:}A$ from the definition of $\models^{\mathsf{M}}$. ■

3.2 Completeness

Completeness follows by a Henkin-style proof, where a canonical model $\mathsf{M}^C = (\mathsf{W}^C, \mathsf{G}^C, \mathsf{R}^{uC}, \mathsf{R}^{eC}, *^C, \mathsf{V}^C)$ is built to show the contrapositives of the conditions in Definition 11, i.e.

$$\Delta \nvdash_L R_i\, a\, a_1 \ldots a_n \text{ implies } \Delta \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n\,,$$
$$\Gamma, \Delta \nvdash_L a{:}A \text{ implies } \Gamma, \Delta \not\models^{\mathsf{M}^C} a{:}A\,.$$

In standard proofs for 'unlabelled' non-classical logics (with Kripke semantics), a countermodel for underivable formulae is built by defining a notion of maximality for sets of formulae, and then using an extension result (such as the Lindenbaum Lemma, the Zorn Lemma or the Belnap Extension Lemma, e.g. [12]) to show that every set of formulae is contained in some maximal set; the canonical model is then obtained by repeated applications of the extension lemma. There are several possible definitions of maximality that can be considered, depending on the logic. For instance, maximality can be defined in terms of consistency (as is usually done, e.g., for modal logics), in terms of notions weaker than consistency for paraconsistent logics such as relevance logics, or we can simply build the canonical model by extending disjoint theory–countertheory pairs [1, 12, 15].

The latter approach is more general than the other ones as it does not rely on negation and thus applies also to positive fragments. We have taken a similar approach, but instead of introducing countertheories, we start by defining what it means for a proof context (Γ, Δ) to be maximal with respect to an underivable lwff $a{:}A$. Then, given the presence of labelled formulae and explicit assumptions on the relations between the labels, i.e. the rwffs in Δ, we modify the Lindenbaum Lemma (cf. Lemma 15 below) to extend (Γ, Δ) to a single proof context $(\Gamma^\bullet, \Delta^\bullet)$ maximal w.r.t. $a{:}A$, where maximality is 'globally' checked also against the additional assumptions in Δ. The elements of W^C are then defined by partitioning $\Gamma^\bullet$ with respect to the labels, and the relations are defined by exploiting the information in Δ. Therefore only one application of the extension lemma is needed, in that we simultaneously build all elements of W^C. Moreover, and most importantly, our proof is completely independent of L: exactly the same procedure applies to any fragment of any logic.

Definition 14 *Given a logic $L = \mathcal{B} + \mathcal{T}$, let Δ_L be the* deductive closure *of Δ under $\mathcal{T}$, i.e. $\Delta_L = \{R_i\, a\, a_1 \ldots a_n \mid \Delta \vdash_L R_i\, a\, a_1 \ldots a_n\}$. A pc (Γ, Δ) is* maximal w.r.t. $a{:}A$ *iff (i) $\Delta = \Delta_L$, and (ii) $b{:}B \notin (\Gamma, \Delta)$ iff $\Gamma \cup \{b{:}B\}, \Delta \vdash_L a{:}A$.*

Note that, when (Γ, Δ) is maximal w.r.t. $a{:}A$, both $a{:}A \notin (\Gamma, \Delta)$ and $\Gamma, \Delta \nvdash_L a{:}A$. Moreover, $b{:}\perp\!\!\!\perp \notin (\Gamma, \Delta)$ and $\Gamma, \Delta \nvdash_L b{:}\perp\!\!\!\perp$ for any b, for otherwise $\Gamma, \Delta \vdash_L a{:}A$. Also note that $\Gamma, \Delta \vdash_L b{:}B$ iff $\Gamma, \Delta_L \vdash_L b{:}B$, and that $\Delta \vdash_L R_i\, a\, a_1 \ldots a_n$ iff $\Delta_L \vdash_L R_i\, a\, a_1 \ldots a_n$.

In the Lindenbaum lemma for first-order logic, a maximally consistent set of formulae is inductively built by adding for every formula $\exists x.P$ a *witness* to its truth, namely a formula $P[c/x]$ for some new constant c. A similar procedure applies here in the case of existential non-local connectives: if the addition of $w{:}\mathcal{M}^e A_1 \ldots A_e$ does not yield a derivation of $a{:}A$, then we also add $t_1{:}A_1, \ldots, t_e{:}A_e$ and $R^e\, w\, t_1 \ldots t_e$, for some new $t_1, \ldots, t_e$, which act as *witness worlds* to the truth of $w{:}\mathcal{M}^e A_1 \ldots A_e$. This ensures that the pc $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$, as shown in Lemma 15 below. As a comparison, in the standard completeness proof for unlabelled modal logics one shows instead that if $\models^{\mathsf{M}^C} w{:}\Diamond A$, then, by the extension lemma, there is a world w' accessible from w that serves as a witness to the truth of $\Diamond A$, i.e. $\models^{\mathsf{M}^C} w'{:}A$.

Lemma 15 *If* $\Gamma, \Delta \nvdash_L a{:}A$, *then* (Γ, Δ) *can be extended to a pc* $(\Gamma^\bullet, \Delta^\bullet)$ *that is maximal w.r.t.* $a{:}A$.

Proof We first extend the language of the logic L with infinitely many new constants for witness worlds. Systematically let t range over the new constants for witness worlds, and w range over labels (including a) and over the new constants; t and w may be subscripted. Let $l_1, l_2, \ldots$ be an enumeration of all lwffs in the extended language. Starting from $(\Gamma_0, \Delta_0) = (\Gamma, \Delta)$, we inductively build a sequence of pcs by defining $(\Gamma_{i+1}, \Delta_{i+1})$ as follows:

- if $\Gamma_i \cup \{l_{i+1}\}, \Delta_i \vdash_L a{:}A$, then $(\Gamma_{i+1}, \Delta_{i+1}) = (\Gamma_i, \Delta_i)$
- if $\Gamma_i \cup \{l_{i+1}\}, \Delta_i \nvdash_L a{:}A$, then
  - if l_{i+1} is $w{:}\mathcal{M}^e A_1 \ldots A_e$, then we add witnesses to the truth of $w{:}\mathcal{M}^e A_1 \ldots A_e$, i.e. for $t_1, \ldots, t_e \notin (\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e\}, \Delta_i)$,

$$\begin{array}{lcl} \Gamma_{i+1} & = & \Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e, t_1{:}A_1, \ldots, t_e{:}A_e\} \\ \Delta_{i+1} & = & \Delta_i \cup \{R^e\, w\, t_1 \ldots t_e\} \end{array}$$

  - if l_{i+1} is not $w{:}\mathcal{M}^e A_1 \ldots A_e$, then $(\Gamma_{i+1}, \Delta_{i+1}) = (\Gamma_i \cup \{l_{i+1}\}, \Delta_i)$

Every (Γ_i, Δ_i) is such that $\Gamma_i, \Delta_i \nvdash_L a{:}A$. To prove this we show that if $\Gamma_i, \Delta_i \nvdash_L a{:}A$ then $\Gamma_{i+1}, \Delta_{i+1} \nvdash_L a{:}A$. The only non-trivial case is the addition of witnesses to the truth of $w{:}\mathcal{M}^e A_1 \ldots A_e$. Suppose that

$$\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e, t_1{:}A_1, \ldots, t_e{:}A_e\}, \Delta_i \cup \{R^e\, w\, t_1 \ldots t_e\} \vdash_L a{:}A$$

where $t_1, \ldots, t_e \notin (\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e\}, \Delta_i)$. Then we can apply $\mathcal{M}^e E$, and thus $\Gamma_i \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e\}, \Delta_i \vdash_L a{:}A$. Contradiction.

Now define

$$\Gamma^\bullet = \bigcup_{i \geq 0} \Gamma_i \quad \text{and} \quad \Delta^\bullet = \bigcup_{i \geq 0} (\Delta_i)_L\,.$$

Then, $(\Gamma, \Delta) \in (\Gamma^\bullet, \Delta^\bullet)$ and $a{:}A \notin (\Gamma^\bullet, \Delta^\bullet)$. Moreover, $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$. Condition (i) in Definition 14 is satisfied by definition of $\Delta^\bullet$, and we

show that condition (ii) holds as well. $\Gamma^\bullet \cup \{b{:}B\}, \Delta^\bullet \nvdash_L a{:}A$ implies $b{:}B \in \Gamma^\bullet$ by construction. For the converse, assume that $b{:}B \in \Gamma^\bullet$. If $\Gamma^\bullet \cup \{b{:}B\}, \Delta^\bullet \vdash_L a{:}A$, then, since $\Gamma^\bullet, \Delta^\bullet \vdash_L b{:}B$, by transitivity of derivations we have that $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. ■

When $\Delta \nvdash_L R_i\, a\, a_1 \ldots a_n$, then we simply extend Δ to $\Delta^\bullet = \Delta_L$, where $R_i\, a\, a_1 \ldots a_n \notin \Delta^\bullet$, since by definition of the deductive closure $\Delta^\bullet \vdash_L R_i\, w\, w_1 \ldots w_n$ iff $R_i\, w\, w_1 \ldots w_n \in \Delta^\bullet$.

Lemma 16 *Let* $(\Gamma^\bullet, \Delta^\bullet)$ *be maximal w.r.t.* $a{:}A$. *Then*

(i) $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}B$ *iff* $w{:}B \in (\Gamma^\bullet, \Delta^\bullet)$.

(ii) $w{:}\mathcal{M}^u A_1 \ldots A_u \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $R^u\, w\, w_1 \ldots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and ... and* $w_{u-1}{:}A_{u-1} \in (\Gamma^\bullet, \Delta^\bullet)$ *imply* $w_u{:}A_u \in (\Gamma^\bullet, \Delta^\bullet)$, *for all* $w_1, \ldots, w_u$.

(iii) $w{:}\mathcal{M}^e A_1 \ldots A_e \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $R^e\, w\, w_1 \ldots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and ... and* $w_e{:}A_e \in (\Gamma^\bullet, \Delta^\bullet)$, *for some* $w_1, \ldots, w_e$.

(iv) $w{:}\neg A \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w^*{:}A \notin (\Gamma^\bullet, \Delta^\bullet)$.

(v) $w{:}A_1 \wedge A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *and* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$.

(vi) $w{:}A_1 \vee A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *or* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$.

(vii) $w{:}A_1 \supset A_2 \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ *implies* $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$.

Proof We only show (i), (iii), and (vi); the other cases follow similarly.

(i) Suppose that $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}B$. If $w{:}B \notin (\Gamma^\bullet, \Delta^\bullet)$, then, since $(\Gamma^\bullet, \Delta^\bullet)$ is maximal w.r.t. $a{:}A$, $\Gamma^\bullet \cup \{w{:}B\}, \Delta^\bullet \vdash_L a{:}A$, and thus, by transitivity, $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. The converse holds by definition.

(iii) Suppose that $R^e\, w\, w_1 \ldots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{e-1}{:}A_{e-1} \in (\Gamma^\bullet, \Delta^\bullet)$ imply $w_e{:}A_e \notin (\Gamma^\bullet, \Delta^\bullet)$, for all $w_1, \ldots, w_e$. Then, by (i),

$$\Gamma^\bullet \cup \{w_1{:}A_1, \ldots, w_{e-1}{:}A_{e-1}\} \cup \{w_e{:}A_e\}, \Delta^\bullet \cup \{R^e\, w\, w_1 \ldots w_e\} \vdash_L a{:}A$$

for all $w_1, \ldots, w_e$. Now, if $w{:}\mathcal{M}^e A_1 \ldots A_e \in (\Gamma^\bullet, \Delta^\bullet)$, then, by (i), $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}\mathcal{M}^e A_1 \ldots A_e$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$, by $\mathcal{M}^e E$. Contradiction. For the converse suppose that $w{:}\mathcal{M}^e A_1 \ldots A_e \notin (\Gamma^\bullet, \Delta^\bullet)$. Then $\Gamma^\bullet \cup \{w{:}\mathcal{M}^e A_1 \ldots A_e\}, \Delta^\bullet \vdash_L a{:}A$. Now, if for some $w_1, \ldots, w_e$, $R^e\, w\, w_1 \ldots w_e \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_e{:}A_e \in (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}\mathcal{M}^e A_1 \ldots A_e$ by (i) and $\mathcal{M}^e I$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$ by transitivity. Contradiction.

(vi) Suppose that $w{:}A_1 \vee A_2 \in (\Gamma^\bullet, \Delta^\bullet)$. Now, if $w{:}A_1 \notin (\Gamma^\bullet, \Delta^\bullet)$ and $w{:}A_2 \notin (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet \cup \{w{:}A_1\}, \Delta^\bullet \vdash_L a{:}A$ and $\Gamma^\bullet \cup \{w{:}A_2\}, \Delta^\bullet \vdash_L a{:}A$, and thus $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$ by (i) and $\vee E$. Contradiction. For the converse suppose that $w{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$. Now, if $w{:}A_1 \vee A_2 \notin (\Gamma^\bullet, \Delta^\bullet)$, then $\Gamma^\bullet \cup \{w{:}A_1 \vee A_2\}, \Delta^\bullet \vdash_L a{:}A$. But by (i) and $\vee I$ the assumption yields $\Gamma^\bullet, \Delta^\bullet \vdash_L w{:}A_1 \vee A_2$, and thus, by transitivity, $\Gamma^\bullet, \Delta^\bullet \vdash_L a{:}A$. Contradiction. We conclude analogously when we assume that $w{:}A_2 \in (\Gamma^\bullet, \Delta^\bullet)$. ■

Definition 17 *Given a pc* $(\Gamma^\bullet, \Delta^\bullet)$ *maximal w.r.t.* $a{:}A$*, we define the* canonical model M^C *(w.r.t.* $(\Gamma^\bullet, \Delta^\bullet)$ *and* $a{:}A$*) for the logic* L *as follows:*

- $\mathsf{W}^C = \{w \mid w \in (\Gamma^\bullet, \Delta^\bullet)\}$*, where* $\mathsf{G}^C = G$ *and* ${w^*}^C = w^*$*;*
- $(w, w_1, \ldots, w_u) \in \mathsf{R}^{uC}$ *iff* $R^u\, w\, w_1 \ldots w_u \in \Delta^\bullet$*, and* $(w, w_1, \ldots, w_e) \in \mathsf{R}^{eC}$ *iff* $R^e\, w\, w_1 \ldots w_e \in \Delta^\bullet$*;*
- $\mathsf{V}^C(w, p) = 1$ *iff* $w{:}p \in \Gamma^\bullet$*.*

The standard definition of R^{uC}, i.e. $(w, w_1, \ldots, w_u) \in \mathsf{R}^{uC}$ iff

$$\{A_u \mid \mathcal{M}^u A_1 \ldots A_u \in w,\ A_1 \in w_1, \ldots, A_{u-1} \in w_{u-1}\} \subseteq w_u\,, \qquad (24)$$

is not applicable in our setting, since (24) does *not* imply $\vdash_L R^u\, w\, w_1 \ldots w_u$. We would therefore be unable to prove completeness for rwffs, since there would be cases where $\nvdash_L R^u\, w\, w_1 \ldots w_u$, but $(w, w_1, \ldots, w_u) \in \mathsf{R}^{uC}$, and thus $\models^{\mathsf{M}^C} R^u\, w\, w_1 \ldots w_u$. Hence, we instead define $(w, w_1, \ldots, w_u) \in \mathsf{R}^{uC}$ iff $R^u\, w\, w_1 \ldots w_u \in \Delta^\bullet$; note that therefore $R^u\, w\, w_1 \ldots w_u \in \Delta^\bullet$ implies (24). Similarly for R^{eC}.[13] Moreover, we immediately have that:

Fact 18 $R_i\, w\, w_1 \ldots w_n \in \Delta^\bullet$ *iff* $\Delta^\bullet \models^{\mathsf{M}^C} R_i\, w\, w_1 \ldots w_n$*.*

The deductive closure of $\Delta^\bullet$ ensures not only completeness for rwffs (cf. Lemma 20 below), but also that the conditions on R^{uC} and R^{eC} are satisfied, so that M^C is really a model for L. As an example, we show that if L contains *ass1* and *ass2* for a ternary relation R^u, then R^{uC} is associative. Consider an arbitrary pc (Γ, Δ), from which we build M^C. Assume $(a, b, e) \in \mathsf{R}^{uC}$ and $(e, c, d) \in \mathsf{R}^{uC}$. Then $R^u\, a\, b\, e \in \Delta^\bullet$ and $R^u\, e\, c\, d \in \Delta^\bullet$. But $\Delta^\bullet$ is deductively closed, and thus $R^u\, b\ h(a, b, c, d, e)\ d \in \Delta^\bullet$ and $R^u\, a\, c\ h(a, b, c, d, e) \in \Delta^\bullet$, by *ass1* and *ass2*. Hence, there exists an x such that $(b, x, d) \in \mathsf{R}^{uC}$ and $(a, c, x) \in \mathsf{R}^{uC}$, and R^{uC} is indeed associative.

Let the *degree* of an lwff be the number of connectives (both local and non-local) that occur in it. By Lemma 16 and Fact 18, it is easy to show that:

Lemma 19 $w{:}B \in (\Gamma^\bullet, \Delta^\bullet)$ *iff* $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w{:}B$*.*

The proof is by induction on the degree of $w{:}B$; as an example, consider the case when $w{:}B$ is $w{:}\mathcal{M}^u A_1 \ldots A_u$. Assume $w{:}\mathcal{M}^u A_1 \ldots A_u \in (\Gamma^\bullet, \Delta^\bullet)$. Then, by Lemma 16, $R^u\, w\, w_1 \ldots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{u-1}{:}A_{u-1} \in (\Gamma^\bullet, \Delta^\bullet)$ imply $w_u{:}A_u \in (\Gamma^\bullet, \Delta^\bullet)$, for all $w_1, \ldots, w_u$. Fact 18 and the induction hypotheses yield $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_u{:}A_u$ for all $w_1, \ldots, w_u$ such that $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} R^u\, w\, w_1 \ldots w_u$ and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_1{:}A_1$ and ... and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_{u-1}{:}A_{u-1}$, i.e. $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w{:}\mathcal{M}^u A_1 \ldots A_u$ by the definition of truth.

For the converse, assume $w{:}\mathcal{M}^u A_1 \ldots A_u \notin (\Gamma^\bullet, \Delta^\bullet)$. Then, by Lemma 16, $R^u\, w\, w_1 \ldots w_u \in (\Gamma^\bullet, \Delta^\bullet)$ and $w_1{:}A_1 \in (\Gamma^\bullet, \Delta^\bullet)$ and ... and $w_{u-1}{:}A_{u-1} \in$

[13] As a further comparison with the standard definition of the canonical model, note also that the label w can be identified with the set of formulae $\{B \mid w{:}B \in \Gamma^\bullet\}$.

$(\Gamma^\bullet, \Delta^\bullet)$ and $w_u{:}A_u \notin (\Gamma^\bullet, \Delta^\bullet)$, for some $w_1, \ldots, w_u$. Fact 18 and the induction hypotheses yield $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} R^u\, w\, w_1 \ldots w_u$ and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_1{:}A_1$ and ... and $\Gamma^\bullet, \Delta^\bullet \models^{\mathsf{M}^C} w_{u-1}{:}A_{u-1}$ and $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} w_u{:}A_u$, i.e. $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} w{:}\mathcal{M}^u A_1 \ldots A_u$ by the definition of truth.

We can now finally show that:

Lemma 20 *L is complete.*

Proof (i) If $\Delta \nvdash_L R_i\, a\, a_1 \ldots a_n$, then $R_i\, a\, a_1 \ldots a_n \notin \Delta^\bullet$, and thus $\Delta^\bullet \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n$, by Fact 18. Hence, $\Delta \not\models^{\mathsf{M}^C} R_i\, a\, a_1 \ldots a_n$. (ii) If $\Gamma, \Delta \nvdash_L a{:}A$, then we extend (Γ, Δ) to a pc $(\Gamma^\bullet, \Delta^\bullet)$ maximal w.r.t. $a{:}A$. Then, by Lemma 19, $\Gamma^\bullet, \Delta^\bullet \not\models^{\mathsf{M}^C} a{:}A$, and thus $\Gamma, \Delta \not\models^{\mathsf{M}^C} a{:}A$. ■

3.3 Positive Fragments and Interrelated Relations

In Section 2 we argued that an unrestricted *monl* rule produces an unsound system in which intuitionistic and classical implication are equivalent, and that soundness is restored when applications of *monl* are restricted to persistent formulae. We show now that the soundness and completeness of our presentations (Theorem 12) depends on another restriction we imposed in Section 2, that there are no a priori assumptions on the interrelationships of the different relations associated with universal and existential modalities. If this restriction is withdrawn and the relations are interrelated, e.g. $\mathsf{R}^u \subseteq \mathsf{R}^e$, then incompleteness may arise.

To illustrate this, we consider the positive fragments of (classical) modal logics. Without negation we cannot define $\Diamond$ in terms of $\Box$ and derive the rules for $\Diamond$. Indeed, there is no a priori reason why $\Box$ and $\Diamond$ must be related at all. Therefore, we characterize the positive fragments containing both $\Box$ and $\Diamond$ by the interrelationships between $R^\Box$ and $R^\Diamond$, which are specified by a (possibly empty) collection of the following Horn relational rules:

$$\frac{x\ R^\Diamond\ y}{x\ R^\Box\ y}\ (\Diamond\Box) \qquad\qquad \frac{x\ R^\Box\ y}{x\ R^\Diamond\ y}\ (\Box\Diamond)$$

Using these rules, we can prove theorems that relate $\Box$ and $\Diamond$. For instance, using $(\Diamond\Box)$ we can prove

$$x{:}(\Diamond A \wedge \Box B) \supset \Diamond(A \wedge B)\,, \tag{25}$$

and using $(\Box\Diamond)$ we can prove

$$x{:}(\Diamond A \supset \Box B) \supset \Box(A \supset B)\,. \tag{26}$$

That these theorems are provable is not surprising: correspondence theory [53, 54] provides a means of showing that (25) corresponds to the semantic condition $\mathsf{R}^\Diamond \subseteq \mathsf{R}^\Box$ and that (26) corresponds to $\mathsf{R}^\Box \subseteq \mathsf{R}^\Diamond$.

Now consider

$$x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)\,, \tag{27}$$

which corresponds to $\mathsf{R}^{\Box} \subseteq \mathsf{R}^{\Diamond}$, and therefore is true in the models satisfying this property. By analysis of normal form proofs (see Section 4), we can show that (27) is not provable using $(\Box\Diamond)$.[14] Hence, positive modal logics where $R^{\Box}$ and $R^{\Diamond}$ are not independent but are related by $(\Box\Diamond)$ are incomplete with respect to Kripke models $(\mathsf{W}, \mathsf{R}^{\Box}, \mathsf{R}^{\Diamond}, \mathsf{V})$ where $\mathsf{R}^{\Box} \subseteq \mathsf{R}^{\Diamond}$. This illustrates that:

Theorem 21 *If the R_is associated with the modalities are not independent, then there are positive fragments of non-classical logics that are incomplete with respect to the corresponding Kripke-style semantics.*

A similar problem holds for Hilbert presentations, as pointed out by Dunn in [15]; he ensures the completeness of the 'absolutely' positive fragment of modal logic (i.e. without negation and implication) by extending his Hilbert-style deductive system with postulates equivalent to (25) and (27). Similarly, we could restore completeness in our setting by giving up our claim to a fixed base logic extended with relational theories and adding a rule directly encoding (27), e.g.

$$\frac{}{x{:}\Box(A \vee B) \supset (\Diamond A \vee \Box B)}$$

However, such a rule is not in the spirit of ND since it does not contribute to the theory of meaning of the connectives. Moreover, it complicates proof normalization arguments.

4 Normalization

In this section, following, where possible, Prawitz [42, 43], we show that the derivation of an lwff can be reduced to a normal form where there are no unnecessary detours and satisfying a subformula property.

There are two possible forms of detours in a derivation and we eliminate them by the reduction operations defined below. For brevity, we consider again the restricted language of Definition 4 (with $\wedge$, $\vee$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^e$, $\neg$, $\perp\!\!\!\perp$), and we only show the part of the derivation where the reduction actually takes place: the missing parts remain unchanged. Moreover, we extend Notation 7 by writing $\Pi \rightsquigarrow \Pi'$ when the derivation Π reduces to the derivation Π' by one or more reductions.

The first, and simplest, form of detour is the application of an elimination rule immediately below the application of the corresponding introduction rule. That is, if an lwff is introduced and then immediately eliminated, then we can avoid introducing it in the first place. Formally, we have:

[14] This is because the proof of (27) requires properties of classical negation. Thus, instead of 'strengthening' the proof system, we could try to restore completeness by adopting a semantics with a 'weaker' negation.

Definition 22 *A* maximal lwff *in a derivation is an lwff that is both the conclusion of an introduction rule and the major premise of an elimination rule.*

Maximal lwffs are removed from a derivation by (finitely many applications of) *proper reductions*. There is one proper reduction for each connective. The proper reductions for universal and existential non-local connectives are as follows.

Proper reduction for $\mathcal{M}^u$:

$$\dfrac{\begin{array}{c}[a_1{:}A_1]\cdots[a_{u-1}{:}A_{u-1}]\ [R^u\,a\,a_1\ldots a_u]\\ \Pi\\ a_u{:}A_u\end{array}}{\begin{array}{l}a{:}\mathcal{M}^u A_1\ldots A_u\end{array}}\ \mathcal{M}^u I \qquad \dfrac{\Pi_1 \quad \Pi_{u-1} \quad \Pi_r}{b_u{:}A_u}\ \mathcal{M}^u E$$

$$\rightsquigarrow \quad \begin{array}{c}\Pi_1 \quad \Pi_{u-1} \quad \Pi_r\\ b_1{:}A_1\cdots b_{u-1}{:}A_{u-1}\ R^u\,a\,b_1\ldots b_u\\ \Pi[b_1/a_1,\ldots,b_u/a_u]\\ b_u{:}A_u\end{array}$$

Proper reduction for $\mathcal{M}^e$:

$$\dfrac{\begin{array}{c}\Pi_1 \quad \Pi_e \quad \Pi_r\\ a_1{:}A_1\cdots a_e{:}A_e \quad R^e\,a\,a_1\ldots a_e\end{array}}{a{:}\mathcal{M}^e A_1\ldots A_e}\ \mathcal{M}^e I \qquad \dfrac{\begin{array}{c}[b_1{:}A_1]\cdots[b_e{:}A_e]\ [R^e\,a\,b_1\ldots b_e]\\ \Pi\\ c{:}C\end{array}}{c{:}C}\ \mathcal{M}^e E$$

$$\rightsquigarrow \quad \begin{array}{c}\Pi_1 \quad \Pi_e \quad \Pi_r\\ a_1{:}A_1\cdots a_e{:}A_e\ R^e\,a\,a_1\ldots a_e\\ \Pi[a_1/b_1,\ldots,a_e/b_e]\\ c{:}C\end{array}$$

where the substitutions in the above reductions are allowed by the provisos on $\mathcal{M}^u I$ and $\mathcal{M}^e E$.

The proper reductions for negation and for local connectives can be easily adapted from the standard 'unlabelled' reductions, e.g. for negation:

$$\dfrac{\dfrac{\begin{array}{c}[a^*{:}A]\\ \Pi_1\\ b{:}\perp\!\!\!\perp\end{array}}{a{:}\neg A}\ \neg I \quad \begin{array}{c}\Pi_2\\ a^*{:}A\end{array}}{b{:}\perp\!\!\!\perp}\ \neg E \qquad \rightsquigarrow \qquad \begin{array}{c}\Pi_2\\ a^*{:}A\\ \Pi_1\\ b{:}\perp\!\!\!\perp\end{array}$$

Let us call *indirect rules* the rules $\mathcal{M}^e E$, $\vee E$, $\perp\!\!\!\perp Ei$ and *monl*. The second form of detour arises when the conclusion of an indirect rule is the major premise of an elimination rule. Consider the different cases. At applications of $\mathcal{M}^e E$, occurrences of the same lwff appear immediately below each other, and this can constitute a detour in which lwffs that potentially interact in a proper reduction

are too far apart. The same problem holds for applications of $\vee E$, and a similar one for applications of *monl*. Finally, when the conclusion of $\perp\!\!\!\perp Ei$ is the major premise of an elimination, then we can easily show that the elimination is an unnecessary inference.

To remove this second form of detour we permute the order of application of indirect and elimination rules. Formally we define:

Definition 23 *A* permutable lwff *in a derivation is an lwff that is both the conclusion of an indirect rule and the major premise of an elimination.*

Permutable lwffs are removed from a derivation by (finitely many applications of) *permutative reductions*. The difference with respect to Prawitz is twofold. First, we explicitly define $\perp\!\!\!\perp Ei$ to be an indirect rule, since, unlike his $\perp$ elimination rule for intuitionistic logic, we cannot restrict $\perp\!\!\!\perp Ei$ to applications where the conclusion is an atomic lwff. For instance, to replace

$$\dfrac{\begin{array}{c}\Pi\\ b{:}\perp\!\!\!\perp\end{array}}{a{:}\mathcal{M}^e A_1 \ldots A_e}\ \perp\!\!\!\perp Ei$$

with

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ b{:}\perp\!\!\!\perp\end{array}}{a_1{:}A_1}\ \perp\!\!\!\perp Ei \quad \cdots \quad \dfrac{\begin{array}{c}\Pi\\ b{:}\perp\!\!\!\perp\end{array}}{a_e{:}A_e}\ \perp\!\!\!\perp Ei \quad \dfrac{\begin{array}{c}\Pi\\ b{:}\perp\!\!\!\perp\end{array}}{R^e\ a\ a_1 \ldots a_e}\ (\dagger)}{a{:}\mathcal{M}^e A_1 \ldots A_e}\ \mathcal{M}^e I$$

we would need a rule $(\dagger)$ that would violate the separation between base logic and relational theory. Second, although it is not an elimination rule, we define *monl* to be an indirect rule since, like $\mathcal{M}^e E$, $\vee E$, and $\perp\!\!\!\perp Ei$, it can interrupt a potential reduction.

As notation, we write

$$\dfrac{a{:}A \quad \Sigma}{b{:}B}\ (r)$$

for an application of an elimination (or indirect) rule (r) with major premise $a{:}A$ and conclusion $b{:}B$, where Σ represents the finite sequence of derivations of the minor premises of the rule. The permutative reductions for $\mathcal{M}^e E$, $\vee E$, and $\perp\!\!\!\perp Ei$ are as follows.

Permutative reductions for $\mathcal{M}^e E$:

$$\dfrac{\dfrac{\begin{array}{c}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{array} \qquad \dfrac{\begin{array}{c}[a{:}A_1]\ \cdots\ [a{:}A_e]\ [R^e\ a\ a_1 \ldots a_e]\\ \Pi_1\\ b{:}B\end{array}}{b{:}B}\ \mathcal{M}^e E \qquad \Sigma}{c{:}C}\ (r)$$

$$\rightsquigarrow \quad \dfrac{\begin{array}{c}\Pi\\ a{:}\mathcal{M}^e A_1 \ldots A_e\end{array} \qquad \dfrac{\begin{array}{c}[a{:}A_1]\ \cdots\ [a{:}A_e]\ [R^e\ a\ a_1 \ldots a_e]\\ \Pi_1\\ b{:}B\end{array} \qquad \Sigma}{c{:}C}\ (r)}{c{:}C}\ \mathcal{M}^e E$$

Permutative reductions for $\vee E$:

$$\dfrac{\dfrac{\Pi \quad \overset{[a:A]}{\Pi_1} \quad \overset{[a:B]}{\Pi_2}}{\dfrac{a{:}A \vee B \quad c{:}C \quad c{:}C}{c{:}C}\ \vee E \qquad \Sigma}{d{:}D}\ (r)}{} \quad \rightsquigarrow \quad \dfrac{\Pi \quad \overset{[a:A]}{\Pi_1} \quad \dfrac{c{:}C \quad \Sigma}{d{:}D}\ (r) \qquad \overset{[a:B]}{\Pi_2} \quad \dfrac{c{:}C \quad \Sigma}{d{:}D}\ (r)}{a{:}A \vee B \qquad d{:}D \qquad\qquad d{:}D}\ \vee E$$

Permutative reductions for $\perp\!\!\!\perp Ei$:

$$\dfrac{\dfrac{\Pi}{\dfrac{b{:}\perp\!\!\!\perp}{a{:}A}\ \perp\!\!\!\perp Ei} \qquad \Sigma}{c{:}C}\ (r) \quad \rightsquigarrow \quad \dfrac{\Pi}{\dfrac{b{:}\perp\!\!\!\perp}{c{:}C}\ \perp\!\!\!\perp Ei}$$

The permutative reductions for *monl* are more complex and we consider them in detail. First, note that since *monl* can be only applied to persistent formulae, we need not consider permuting it with $\supset E$ (cf. the discussions in Section 2.1 and Section 2.3). Now let $A \wedge B$ and $A \vee B$ be persistent formulae. In the permutative reductions of *monl* with applications of $\wedge E$ or $\vee E$ the application of *monl* is 'pushed' to lwffs of smaller degree, e.g.

$$\dfrac{\dfrac{\Pi_1 \quad \Pi_2}{a{:}A \vee B \quad a \sqsubseteq b}\ monl \qquad \overset{[b:A]}{\Pi_3} \quad \overset{[b:B]}{\Pi_4}}{b{:}A \vee B \qquad\qquad c{:}C \qquad\qquad c{:}C}\ \vee E$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Pi_1}{a{:}A \vee B} \qquad \dfrac{\dfrac{\overset{[a:A]}{\Pi_2}}{\dfrac{a \sqsubseteq b}{b{:}A}\ monl} \qquad \overset{[a:B]}{\Pi_3}}{c{:}C} \qquad \dfrac{\dfrac{\overset{[a:B]}{\Pi_2}}{\dfrac{a \sqsubseteq b}{b{:}B}\ monl} \qquad \Pi_4}{c{:}C}}{c{:}C}\ \vee E$$

The permutative reductions of *monl* with $\perp\!\!\!\perp Ei$ or $\perp\!\!\!\perp Ec$ simply result in the deletion of the application of *monl*, e.g.

$$\dfrac{\dfrac{\overset{[a:\neg A]}{\Pi_1} \qquad \Pi_2}{c{:}\perp\!\!\!\perp \quad c \sqsubseteq b}\ monl}{\dfrac{b{:}\perp\!\!\!\perp}{a^*{:}A}\ \perp\!\!\!\perp Ec} \quad \rightsquigarrow \quad \dfrac{\overset{[a:\neg A]}{\Pi_1}}{\dfrac{c{:}\perp\!\!\!\perp}{a^*{:}A}\ \perp\!\!\!\perp Ec}$$

In the permutative reductions of *monl* with $\mathcal{M}^u E$ and $\mathcal{M}^e E$, *monl* is 'pushed'

to rwffs, i.e. it is replaced with an application of $monR(1)$:

$$\dfrac{\dfrac{\overset{\Pi}{a{:}\mathcal{M}^u A_1 \ldots A_u} \quad \overset{\Pi_0}{a \sqsubseteq b}}{b{:}\mathcal{M}^u A_1 \ldots A_u}\ monl \quad \overset{\Pi_1}{b_1{:}A_1} \cdots \overset{\Pi_{u-1}}{b_{u-1}{:}A_{u-1}} \quad \overset{\Pi_r}{R^u\, b\, b_1 \ldots b_u}}{b_u{:}A_u}\ \mathcal{M}^u E$$

$$\rightsquigarrow \quad \dfrac{\overset{\Pi}{a{:}\mathcal{M}^u A_1 \ldots A_u} \quad \overset{\Pi_1}{b_1{:}A_1} \cdots \overset{\Pi_{u-1}}{b_{u-1}{:}A_{u-1}} \quad \dfrac{\overset{\Pi_r}{R^u\, b\, b_1 \ldots b_u} \quad \overset{\Pi_0}{a \sqsubseteq b}}{R^u\, a\, b_1 \ldots b_u}\ monR(1)}{b_u{:}A_u}\ \mathcal{M}^u E$$

$$\dfrac{\dfrac{\overset{\Pi}{a{:}\mathcal{M}^e A_1 \ldots A_e} \quad \overset{\Pi_0}{a \sqsubseteq b}}{b{:}\mathcal{M}^e A_1 \ldots A_e}\ monl \quad [b_1{:}A_1] \cdots [b_e{:}A_e]\ [R^e\, b\, b_1 \ldots b_e] \quad \overset{\Pi_1}{c{:}C}}{c{:}C}\ \mathcal{M}^e E$$

$$\rightsquigarrow \quad \dfrac{\overset{\Pi}{a{:}\mathcal{M}^e A_1 \ldots A_e} \quad [b_1{:}A_1] \cdots [b_e{:}A_e] \quad \dfrac{[R^e\, b\, b_1 \ldots b_e] \quad \overset{\Pi_0}{a \sqsubseteq b}}{R^e\, a\, b_1 \ldots b_e}\ monR(1) \quad \overset{\Pi_1[a/b]}{c{:}C}}{c{:}C}\ \mathcal{M}^e E$$

(the substitution in the permutative reduction of *monl* with $\mathcal{M}^e E$ is allowed by the proviso on $\mathcal{M}^e E$).

A similar situation occurs when we permute *monl* with itself; i.e. we exploit the transitivity of the partial order, which is an instance of $monR(n)$:

$$\dfrac{\dfrac{\overset{\Pi_1}{a{:}A} \quad \overset{\Pi_2}{a \sqsubseteq b}}{b{:}A}\ monl \quad \overset{\Pi_3}{b \sqsubseteq c}}{c{:}A}\ monl \quad \rightsquigarrow \quad \dfrac{\overset{\Pi_1}{a{:}A} \quad \dfrac{\overset{\Pi_2}{a \sqsubseteq b} \quad \overset{\Pi_3}{b \sqsubseteq c}}{a \sqsubseteq c}\ monR(n)}{c{:}A}\ monl$$

We are now in a position to state our normalization results. We first define:

Definition 24 *A derivation is in* normal form *(is a* normal derivation*) iff it contains no maximal lwffs and no permutable lwffs.*

Then we consider the three systems in Figure 1. For $\mathcal{M}L$ and $\mathcal{J}L$ we have that:

Lemma 25 *Every derivation of $a{:}A$ from Γ, Δ in $\mathcal{M}L$ or $\mathcal{J}L$ reduces to normal form.*

First, note that derivations in Horn relational theories $\mathcal{T}$ cannot introduce maximal or permutable lwffs. The lemma then follows by a straightforward modification of the well-known (long and technical) proof for minimal and intuitionistic logic given originally by Prawitz in [42], but also found in many textbooks,

e.g. [52, 55]. The proof, which for space reasons we omit here, relies on the identification of particular sequences of formulae (which Prawitz calls *threads*, *branches*, *paths*, and *segments*) to show that each application of proper and permutative reductions reduces a suitable well-ordered measure on derivations. Thus, the reduction process must eventually terminate with a derivation free of maximal and permutable lwffs.

Before proving analogous results for $\mathcal{C}L$, let us perform a standard simplification: the 'classical' negation and $\perp\!\!\!\perp$ rules allow us to define for each existential modality $\mathcal{M}^e$, with associated relation R^e, a dual universal modality $\mathcal{M}^u_e$, with associated relation R^u_e, while retaining completeness (cf. the discussion on the possible incompleteness of positive modal logics in Section 3.3). In particular we define:[15]

$$\begin{array}{lll} a{:}\neg\mathcal{M}^u_e A_1 \ldots A_{e-1}\neg A_e & \text{iff} & a{:}\mathcal{M}^e A_1 \ldots A_e \\ R^u_e\, a^*\, a_1 \ldots a_{e-1}\, a^*_e & \text{iff} & R^e\, a\, a_1 \ldots a_e \end{array}$$

To show that this is correct, i.e. that $\mathcal{M}^e$ and $\mathcal{M}^u_e$ are really interdefinable, we take $\mathcal{M}^u_e$ as primitive and derive the rules for $\mathcal{M}^e$, e.g. for $\mathcal{M}^e E$:

$$\dfrac{\dfrac{a{:}\neg\mathcal{M}^u_e A_1 \ldots A_{e-1}\neg A_e \qquad \dfrac{\dfrac{[a_1{:}A_1]^2 \cdots [a_{e-1}{:}A_{e-1}]^2 \quad [a_e{:}A_e]^1 \quad [R^u_e\, a^*\, a_1 \ldots a_{e-1}\, a^*_e]^2}{\begin{array}{c}\Pi\\ b{:}B\end{array}} \qquad \dfrac{}{[b^*{:}\neg B]^3}\,\neg E}{\dfrac{\dfrac{c{:}\perp\!\!\!\perp}{a^*_e{:}\neg A_e}\,\neg I^1}{a^*{:}\mathcal{M}^u_e A_1 \ldots A_{e-1}\neg A_e}\,\mathcal{M}^u_e I^2}\,\neg E}{\dfrac{d{:}\perp\!\!\!\perp}{b{:}B}\,\perp\!\!\!\perp Ec^3}$$

where, for brevity, we have identified a^{**} with a instead of explicitly using the rules $**i$ and $**c$. Hence we can safely replace $\mathcal{M}^e$ and R^e with $\mathcal{M}^u_e$ and R^u_e. Analogously, we can define disjunction in terms of conjunction and with these replacements we obtain the logic $\mathcal{C}L'$, which is adequate for representing a non-classical logic with a classical treatment of negation.

Considering this simplified language (with $\wedge$, $\supset$, $\mathcal{M}^u$, $\mathcal{M}^u_e$, $\neg$, $\perp\!\!\!\perp$) allows us to reduce applications of $\perp\!\!\!\perp Ec$ to instances where the conclusion is atomic, by showing that any application of $\perp\!\!\!\perp Ec$ with a non-atomic consequence can be replaced with a derivation in which $\perp\!\!\!\perp Ec$ is applied only to lwffs of smaller degree. For instance, again identifying a^{**} with a,

$$\dfrac{\dfrac{[a{:}\neg\mathcal{M}^u A_1 \ldots A_u]}{\begin{array}{c}\Pi\\ b{:}\perp\!\!\!\perp\end{array}}}{a^*{:}\mathcal{M}^u A_1 \ldots A_u}\,\perp\!\!\!\perp Ec$$

[15] Note that this is equivalent to defining $R^e\, a\, a_1 \ldots a_e$ iff $R^u_e\, a_e\, a_1 \ldots a_{e-1}\, a$ and adding *switching* rules for both R^e and R^u_e, e.g.

$$\dfrac{R^u_e\, a_e\, a_1 \ldots a_{e-1}\, a}{R^u_e\, a^*\, a_1 \ldots a_{e-1}\, a^*_e}$$

This is, for instance, the case in relevance logics, where fusion ($\circ$) and relevant implication ($\rightarrow$) are associated with the one and the same R and $a{:}A \circ B$ is shown equivalent to $a{:}\neg(A \rightarrow \neg B)$ by means of switching; see [14].

is replaced with:

$$\dfrac{\dfrac{[a_u^*{:}\neg A_u]^2 \quad \dfrac{[a^*{:}\mathcal{M}^u A_1 \ldots A_u]^1\,[a_1{:}A_1]^3 \cdots [a_{u-1}{:}A_{u-1}]^3\,[R^u\, a^*\, a_1 \ldots a_u]^3}{a_u{:}A_u}\,\mathcal{M}^u E}{c{:}\perp\!\!\!\perp}\,\neg E}{a{:}\neg\mathcal{M}^u A_1 \ldots A_u}\,\neg I^1$$

$$\dfrac{\dfrac{\Pi}{b{:}\perp\!\!\!\perp}}{a_u{:}A_u}\,\perp\!\!\!\perp Ec^2$$

$$a^*{:}\mathcal{M}^u A_1 \ldots A_u\,\mathcal{M}^u I^3$$

Therefore, in the case of $\mathcal{C}L'$ the only permutative reductions that need to be considered are those for *monl*, and, by analogy with Lemma 25, we have that:

Lemma 26 *Every derivation of $a{:}A$ from Γ, Δ in $\mathcal{C}L'$ reduces to normal form.*

One of the main advantages of normal derivations is that they possess a well-defined structure that has several desirable properties (a full account of the applications and consequences of normalization, as given by Prawitz, is out of the scope of this paper). In particular, in any of the three families of logics we considered, the two parts of the proof system are strictly partitioned: lwff judgements can depend on rwff judgements, but not vice versa. As a consequence, any derivation of an lwff is structured as a central derivation in the base logic 'decorated' with subderivations in the relational theory, which attach onto the central derivation through instances of $\mathcal{M}^u E$, $\mathcal{M}^e I$, or *monl*. Moreover, when in normal form, the structure of the central derivation in $\mathcal{B}$ can be further characterized by identifying particular sequences of lwffs (Prawitz's threads, branches, paths and segments), and showing that in these sequences there is an ordering on inferences. By exploiting this ordering, we can then show a subformula property for all three families of systems.

Definition 27 *A is a* subformula *of B iff (i) B is A, or (ii) B is $B_1 \wedge B_2$, $B_1 \vee B_2$, $B_1 \supset B_2$, $\neg B_1$, $\mathcal{M}^u B_1 \ldots B_u$, or $\mathcal{M}^e B_1 \ldots B_e$, and A is a subformula of one of the B_is. We speak loosely of $a{:}A$ being a subformula of $b{:}B$, meaning A is a subformula of B.*

Given a derivation $\Gamma, \Delta \vdash a{:}A$, let S be the set of subformulae of the formulae in $\{C \mid c{:}C \in \Gamma \cup \{a{:}A\}\}$, i.e. S is the set consisting of the subformulae of the assumptions Γ and the goal $a{:}A$.

We say that a derivation $\Gamma, \Delta \vdash a{:}A$ in $\mathcal{M}L$ or $\mathcal{J}L$ satisfies the subformula property *iff for all lwffs $b{:}B$ used in the derivation, $B \in S$. We say that a derivation $\Gamma, \Delta \vdash a{:}A$ in $\mathcal{C}L$ satisfies the* subformula property *iff for all lwffs $b{:}B$ used in the derivation, (i) $B \in S$, or (ii) B is an assumption $\neg D$ discharged by $\perp\!\!\!\perp Ec$ and $D \in S$, or (iii) B is an occurrence of $\perp\!\!\!\perp$ immediately below an assumption $\neg D$ discharged by $\perp\!\!\!\perp Ec$ and $D \in S$.*

Lemma 28 *Every normal derivation of $a{:}A$ from Γ, Δ in $\mathcal{M}L$, $\mathcal{J}L$ or $\mathcal{C}L'$ satisfies the subformula property.*

To summarize, our presentations have the following properties:

Theorem 29

(i) The deductive machinery is minimal: the proof systems formalize a minimum fragment of first-order logic required by the semantics of logics with Horn axiomatizable properties of the relations.

(ii) Derivations are strictly partitioned: the derivation of lwffs may depend, via rules for non-local connectives, on derivations of rwffs, but not vice versa.

(iii) Derivations normalize: the derivations of lwffs have a well-structured normal form that satisfies the subformula property.

For comparison, consider the semantic embedding approach, e.g. [27, 35, 36], in which a non-classical logic is encoded as a 'suitable' (e.g. intuitionistic or classical) first-order theory by axiomatizing an appropriate definition of truth: (i) a non-classical logic with Horn axiomatizable properties of the relations constitutes a theory of full first-order logic, as opposed to an extension of labelled propositional logic with Horn-clauses; (ii) all structure is lost as propositions and relations are flattened into first-order formulae; (iii) there are normal forms, those of ND for first-order logic, but derivations of lwffs are mingled with derivations of rwffs, as opposed to the separation between the base logic and the relational theory that we have enforced.

This separation is in the philosophical spirit of LDSs and it also provides extra structure that is pragmatically useful: since derivations of rwffs use only the resources of the relational theory, we may be able to employ theory-specific reasoners to automate proof construction.[16] However, in exchange for this extra structure there are limits to the generality of the formulation: the properties in Theorem 29 depend on design decisions we have made, in particular, the use of Horn relational theories. This, of course, places stronger limitations on what we can formalize than a semantic embedding in first-order logic. Consider, for instance, the relevance logic **RM**, the extension of **R** with the postulate

$$R\,a\,b\,c \text{ implies } (R\,G\,a\,c \text{ or } R\,G\,b\,c)\,, \tag{28}$$

which corresponds to the axiom $A \to (A \to A)$. We cannot formalize **RM** because (28) is not formalizable as a set of Horn rules. This is a design decision. Consider the alternatives. We can extend our deductive machinery by providing proof rules for a full first-order relational theory and explicitly add (28) as an axiom. However, if we then maintain (ii) of Theorem 29 we lose completeness (with respect to the semantics), since, by analysis of normal form proofs, we can show that $G{:}A \to (A \to A)$ is not provable. Alternatively, we can regain completeness by giving up (ii), by identifying *falsum* in the first-order relational theory with $\perp\!\!\!\perp$. However, the resulting system is then essentially equivalent to semantic embedding and we lose (i); see [5], where we investigated analogous problems for modal logics.

But there is another reason why this latter solution is not satisfactory: since it is based on the $\perp\!\!\!\perp$ rules, it does not apply to positive fragments. For these

[16] Then, to restrict further the structure of normal derivations, it is interesting to study the eliminability of *monl* from the systems.

(and also for full logics), we can regain completeness by again giving up (ii) to introduce rules similar to Simpson's geometric rules [50], e.g. we encode (28) with the 'relational' rule

$$\frac{R\,a\,b\,c \qquad \begin{array}{c}[R\,G\,a\,c]\\ \vdots\\ d{:}A\end{array} \qquad \begin{array}{c}[R\,G\,b\,c]\\ \vdots\\ d{:}A\end{array}}{d{:}A}$$

5 Related and Future Work

Gabbay has proposed LDSs as a general methodology for presenting logics [18], however the formal details of his development are different from what we discuss here. For example, the labelled modal logics presented in [18, 48] are based on a notion of diagrams and logic data-bases, which are manipulated by multiple conclusion rules. Further, the purely semantic view of logics taken in the work of [18, 48] means that it is closer to the semantic embedding approach to which we have compared our work above, and which we discuss in more detail in [5]. In [8] labelled tableaux for substructural logics based on algebraic semantics are proposed. The rules support automated proof search, but are not easy to recast as ordinary pure ND proof rules (e.g. the general closure rule they give depends on arbitrarily many formulae).

In [37, 38], Orlowska introduces tableaux-like relational proof systems for relevance, modal and intuitionistic logics, by first translating formulae into relations, which are then proven by a process of decomposition. Although the metalogic is different, relational logic instead of predicate logic, this method is comparable to semantic embedding, since formulae of the logic and relations from the Kripke semantics are treated in a uniform way as relations.

Our work is closely related to, and influenced by, the algebraic approach proposed by Dunn (see [14] and the references there), who introduces *gaggle theory* as an abstraction of Boolean algebras with operators [31], where n-ary operators are interpreted by means of $n+1$-ary relations. Gaggle theory yields a landscape of algebras where the standard Kripke semantics for a particular logic is obtained by manipulating the gaggle presentation at the level of the canonical model, as opposed to instantiating the appropriate relational theory as in our approach. For instance, an analysis of the canonical model shows how to reduce the ternary relation associated with the binary intuitionistic implication to the more customary partial order on possible worlds. This algebraic approach is extremely powerful, but does not lend itself well to direct implementation; however, with appropriate simplifications or by combination with Belnap's *display logic* [6], as in [44], this may be possible. Investigation of this remains as future work.

A Implementation and its Correctness

We have implemented the approach described in this paper in the Isabelle Logical Framework [40], which is based on a logical framework of minimal implicational logic with quantification over higher types [39]. Since the implementation issues are not significantly different from the simpler case for modal logic described in [5], we refer the interested reader there for more extensive details and we give only a brief overview here.

A.1 The Implementation

We call the framework logic of Isabelle $\mathcal{M}$, and write universal quantification and implication in 'machine readable' form as `!!` and `==>`.

A logic is encoded in Isabelle using a theory composed of a signature and axioms, which are formulae in the language of $\mathcal{M}$. The axioms are used to establish the validity of judgements, which are assertions about syntactic objects declared in the signature [26]. Then proving theorems in the encoded logic means simply proving theorems with these axioms in the metalogic. As an example, consider the theory $\mathcal{M}_{\mathbf{R}^+}$,

```
R+ = Pure +             (* R+ extends Pure (Isabelle's metalogic)  *)
types                   (* with the following signature and axioms *)
  l,o 0
arities
  l, o :: logic
consts
  G     :: "l"
  h     :: "[l,l,l,l,l] => l"
  g     :: "[l,l,l] => l"
  inc   :: "o"
  star  :: "l => l"             ("_*" [40] 40)

  (* Connectives *)
  and   :: "[o, o] => o"        (infixr 35)
  or    :: "[o, o] => o"        (infixr 30)
  imp   :: "[o, o] => o"        (infixr 25)

  (* Judgements *)
  L     :: "[l, o] => prop"     ("(_ : _)" [0,0] 100)
  R     :: "[l, l, l] => prop"  ("(R _ _ _)" [0,0,0] 100)
rules
  (* Base Logic *)
  conjI    "[| a:A; a:B |] ==> a: A and B"
  conjE1   "a: A and B ==> a:A"
  conjE2   "a: A and B ==> a:B"
  disjI1   "a:A ==> a: A or B"
  disjI2   "a:B ==> a: A or B"
  disjE    "[| a: A or B;  a:A ==> b:C;  a:B ==> b:C |] ==> b:C"
  impI     "[| !!b c. [| b:A; R a b c |] ==> c:B |] ==> a: A imp B"
```

```
  impE     "[| a: A imp B; b:A; R a b c |] ==> c:B"
  monl     "[| a:A; R G a b |] ==> b:A"

  (* Properties of R *)
  monR1    "[| R a b c; R G d a |] ==> R d b c"
  monR3    "[| R a b c; R G c d |] ==> R a b d"
  iden     "R G a a"
  ass1     "[| R a b e; R e c d |] ==> R b h(a,b,c,d,e) d"
  ass2     "[| R a b e; R e c d |] ==> R a c h(a,b,c,d,e)"
  idem     "R a a a"
  comm     "R a b c ==> R b a c"
  cont1    "R a b c ==> R a b g(a,b,c)"
  cont2    "R a b c ==> R g(a,b,c) b c"
end
```

which encodes the presentation of $\mathbf{R}^+$ given in Definition 8. The signature of $\mathcal{M}_{\mathbf{R}^+}$ declares two types `l` and `o`, for labels and (unlabelled) formulae. Constants and connectives are then declared as typed constants over this signature; e.g. `inc` (for incoherence, i.e. $\perp\!\!\!\perp$) of type `o`, and `imp` of type `o => (o => o)`. There are two judgements, encoded as predicates: first, `L a A`, for provable lwffs, which we abbreviate to `a:A`; second, `R a b c`, for provable rwffs.[17] The axioms for `L` and `R` correspond directly to the rules in Definition 8. Note that in the axioms, free variables are implicitly outermost universally quantified, comments are added between '`(*`' and '`*)`', and there is additional information present to fix notation and help Isabelle's parser.

We may now extend $\mathcal{M}_{\mathbf{R}^+}$ by adding axioms, which reflect the discussion in Section 2.4. The encoding of $\mathcal{J}\mathbf{R}$ is obtained by extending $\mathcal{M}_{\mathbf{R}^+}$ with axioms for an intuitionistic treatment of negation:

```
JR = R+ +
consts
  neg   :: "o => o"              ("~_" [40] 40)
rules
  negI     "(a*: A ==> b: inc) ==> a: ~A"
  negE     "[| a: ~A; a*: A |] ==> b: inc"
  incEi    "b: inc ==> a: A"
  anti     "R a b c ==> R a c* b*"
  stari    "R G a a**"
end
```

Then we can further add an axiom encoding the rule *int* to obtain intuitionistic logic **J** as in Proposition 9, or we can add 'classical' negation rules to obtain $\mathcal{M}_{\mathbf{R}}$, the encoding of **R**:

```
R = JR +
rules
  incEc    "(a: ~A ==> b: inc) ==> a*: A"
```

[17] Indexed judgements similar to ours have been also adopted in other encodings of some non-classical logics in logical frameworks, e.g. modal and dynamic logics in [3, 28, 50].

```
  starc    "R G a** a"
end
```

(Alternatively, we can encode $\mathbf{R}$ by directly extending $\mathcal{M}_{\mathbf{R}^+}$.) Using this encoding we can, e.g., prove `G : ~~A imp A` in $\mathbf{R}$ as follows, cf. the proof of $G{:}\neg\neg A \to A$ given in Section 2.4.

```
> goal R.thy "G : ~~A imp A";
G : ~~A imp A
 1. G : ~~A imp A
```

At the Isabelle prompt, '>', we specify a logic and the goal to be proved. Isabelle responds with the next 2 lines, which give the goal to be proved, and what subgoal(s) must be established to prove it. We begin by instructing Isabelle to apply implication introduction using resolution to the first (and only) subgoal.

```
> by (rtac impI 1);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c |] ==> c : A
```

We now apply `monl` and dispose of the second subgoal using the rule `starc`, given above in the declaration of the theory $\mathcal{M}_{\mathbf{R}}$.

```
> by ((rtac monl 1) THEN (rtac starc 2));
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c |] ==> c** : A
```

Next we apply rules as in the proof in Section 2.4 (`atac` proves a subgoal by assumption after solving for unknowns).

```
> by (EVERY [rtac incEc 1, rtac negE 1, atac 2]);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> c : ~~A

> by (rtac monl 1);
G : ~~A imp A
 1. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> ?a5(b, c) : ~~A
 2. !!b c. [| b : ~~A; R G b c; c* : ~A |] ==> R G ?a5(b, c) c
```

This leaves us with two subgoals, which are both proved by assumption, simplifying `?a5(b, c)` to `b`. Since no unproved subgoals remain, Isabelle reports that we are finished.

```
> by (REPEAT (atac 1));
G : ~~A imp A
No subgoals!
```

A.2 Correctness

By reasoning about our encoding and the metalogic $\mathcal{M}$ we can prove that, e.g., $\mathcal{M}_{\mathbf{R}}$ corresponds to the original $\mathbf{R}$. We do this in two parts, by showing

first *adequacy*, that any proof in $\mathbf{R}$ can be reconstructed in $\mathcal{M}_{\mathbf{R}}$, and then *faithfulness*, that we can recover from any derivation in $\mathcal{M}_{\mathbf{R}}$ a proof in $\mathbf{R}$ itself.

Adequacy is easy to show, because the rules of $\mathbf{R}$ map directly onto the axioms of $\mathcal{M}_{\mathbf{R}}$. A simple inductive argument on the structure of proofs in $\mathbf{R}$ establishes this, cf. [5, §5.2]. Faithfulness is more complex, since there is no such simple mapping in this direction: arbitrary derivations in $\mathcal{M}_{\mathbf{R}}$ do not map directly onto proofs in $\mathbf{R}$. Instead we use proof-theoretic properties of $\mathcal{M}$: any derivation in $\mathcal{M}$ is equivalent to another in a (expanded) normal form [5, 40, 43]. Thus, given a derivation in $\mathcal{M}_{\mathbf{R}}$ we can, by induction over its normal form, find a derivation in $\mathbf{R}$. This establishes faithfulness (again cf. [5, §5.2]). Moreover, this proof is constructive: it not only tells us that there is a proof in $\mathbf{R}$, it also provides an effective method for finding one.

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