Towards abstract and executable multivariate polynomials in Isabelle

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A cultural „gap“ between two communities.

- **Theorem proving:**
  - Sound formal development of theories on top of a small trusted kernel.
  - Computations reduced to logical inferences.
  - Correct but inconvenient to use and painfully slow.

- **Computer algebra:**
  - Elaboration of mathematics by paper-and-pencil or TP software.
  - Separate implementation in mathematical software systems.
  - Convenient to use and reasonably fast but highly untrustworthy.

How can we bridge this gap?
Our Starting Point: Polynomial Algebra

An Isabelle package in which the working mathematician can develop

- **Mathematical theories** based on an abstract view of polynomials.
  - Type-checked definitions and theorems.
  - Computer-supported/mechanically verified proofs.

- **Algorithms** based on the defined mathematical notions.
  - Executable with „reasonable“ efficiency (rapid prototyping).
  - Formal specification and computer-supported verification.

A single computer-supported formal framework for proving and computing with (multivariate) polynomials.
What is the polynomial written as $2x^3 - 5x + 7$?

- **Traditional:** the symbolic expression itself.

$$
(\sum_{i=0}^{n} a_i x^i) \cdot (\sum_{j=0}^{m} b_j x^j) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i \cdot b_j\right) \cdot x^k
$$

- **Computer science:** an array [7, 5, 0, 3]

```java
int[] mult(int[] a, int[] b) {
    int m = a.length-1; int n = b.length-1;
    int[] c = new int[m+n+1];
    for (int i = 0; i <= m; i++)
        for (int j = 0; j <= n; j++)
            c[i+j] += a[i]*b[j];
    return c;
}
```

Two representations of a more fundamental concept.
Polynomials

The more fundamental concept is the modern view of polynomials.

- **Polynomial**: a function $[0 \mapsto 7, \ldots, 3 \mapsto 3, 4 \mapsto 0, 5 \mapsto 0, \ldots]$

  Let $R$ be a ring. A (univariate) polynomial over $R$ is a mapping $p: \mathbb{N}_0 \to R, n \mapsto p_n$, such that $p_n = 0$ nearly everywhere, i.e., for all but finitely many values of $n$.

- **Elegant mathematics**:

  $\cdot \cdot \cdot : (\mathbb{N}_0 \to R) \times (\mathbb{N}_0 \to R) \to (\mathbb{N}_0 \to R)$

  $a \cdot b := k \in \mathbb{N}_0 \mapsto \sum_{i+j=k} a_i \cdot b_j$

- **Polynomial ring**: $R[x]$

  The set of polynomials with (+) and (·); variable $x$ just denotes the polynomial $[0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0, 3 \mapsto 0, \ldots]$.

See e.g. [Winkler, 1996].
What is a polynomial $3x^2y + 5yz$ in variables $x, y, z$?

- **Polynomial**: a function $[(1, 1, 0) \mapsto 3, (0, 1, 1) \mapsto 5, (0, 0, 0) \mapsto 0, \ldots]$.

  An $n$-variate polynomial over the ring $R$ is a mapping $p : \mathbb{N}_0^n \to R, (i_1, \ldots, i_n) \mapsto p_{i_1,\ldots,i_n}$, such that $p_{i_1,\ldots,i_n} = 0$ nearly everywhere.

- **Polynomial ring**: $R[x_1, \ldots, x_n]$

  The set of all $n$-variate polynomials over $R$; variable $x_i$ denotes the polynomial $[\ldots, i \mapsto 1, \ldots]$.

- **Isomorphism**: $R[x_1, \ldots, x_n] \simeq (R[x_1, \ldots, x_{n-1}])[x_n]$.

  Recursive algorithms may be devised for many (not all) computational problems on multivariate polynomials.

- Polynomial division is defined on $K[x]$ where $K$ is a field.
- But $K[x_1, \ldots, x_{n-1}]$ is only a ring.
- Multivariate polynomials thus only support “pseudo-division“.


Prune mapping:
- Represent only exponents/monomials with non-zero coefficients.

Univariate polynomial representations:
- Dense: coefficient sequence $[c_0, \ldots, c_n]$
- Sparse: exponent/coeff. sequence $[(e_0, c_0), \ldots, (e_r, c_r)]$ with $e_i < e_{i+1}$.

$n$-variate polynomial representations:
- Recursive: univariate polynomial whose coefficients are $(n-1)$-variate polynomials (represented densely or sparsely).
- Distributive: monomial/coefﬁcient sequence $[(m_0, c_0), \ldots, (m_r, c_r)]$ (typically represented sparsely).
  - Total order on monomials required for unique representation.

Algorithmic efficiency:
- Recursive algorithms based on isomorphism operate most efﬁciently with recursive representation.
- Buchberger’s Gröbner bases algorithm processes terms in any given “admissible” order and proﬁts from distributive rep. in that order.
General Approach

abstract type

representations

<table>
<thead>
<tr>
<th></th>
<th>dense</th>
<th>sparse</th>
</tr>
</thead>
<tbody>
<tr>
<td>recursive</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>distributive</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
General Approach

**abstract type** ———— Map from monomials to coefficients

- Elegantly define basic operations
- Conveniently express algorithms, theorems, and proofs

**representations**

- dense: ✓
- sparse: ✓

- recursive: ✓ ✓
- distributive: ✓ ✓
General Approach

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**refinement**

- Theorems are preserved
- Representation values can instantiate variables of abstract type.

**representations**

- dense
- sparse
- recursive
- distributive

Lochbihler, Schreiner
General Approach

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representations

- dense ✓ sparse ✓
- recursive ✓ distributive ✓

-executable code
 generation
General Approach

abstract type ———— Map from monomials to coefficients
- Elegantly define basic operations
- Conveniently express algorithms, theorems, and proofs

refinement ————
- Theorems are preserved
- Representation values can instantiate variables of abstract type.

Every abstract algorithm is executable with every representation type.

representations
- dense ✓ sparse ✓
- recursive ✓
- distributive ✓

executable code
generation
typedef 'a ⇒0 'b = { f :: 'a ⇒'b | almost-everywhere-zero f}
\textbf{Implementation in Isabelle}

\textbf{typedef} \quad \forall a \rightarrow_0 b = \{ f :: \forall a \rightarrow b | \text{almost-everywhere-zero } f \}\}

\textbf{'}a mpoly}

\textbf{'}a poly-rec}

\textbf{'}a poly-distr}
typedef ′a ⇒_0 ′b = \{ f :: ′a ⇒ ′b \mid \text{almost-everywhere-zero } f \}\}

\begin{align*}
(nat ⇒_0 nat) ⇒_0 ′a & \cong ′a \text{ mpoly} \\
′a \text{ poly-rec} & \quad ′a \text{ poly-distr}
\end{align*}
**Implementation in Isabelle**

```isabelle
typedef 'a ⇒_0 'b = { f :: 'a ⇒ 'b | almost-everywhere-zero f }
```

```isabelle
(nat ⇒_0 nat) ⇒_0 'a ≃ 'a mpoly
```

```isabelle
datatype 'a poly-rec = Coeff 'a
                  | Rec (nat ⇒_0 'a poly-rec)
```

```isabelle
'datatype 'a poly-distr = 'a poly-rec
```
```
typedef ′a ⇒0 ′b = { f :: ′a ⇒′b | almost-everywhere-zero f}

(nat ⇒0 nat) ⇒0 ′a

≅

′a mpoly

datatype ′a poly-rec

= Coeff ′a

| Rec (nat ⇒0 ′a poly-rec)

′a poly-distr

≅

′a mpoly × monom-order
**Implementation in Isabelle**

\texttt{typedef } 'a ⇒₀ 'b = \{ f :: 'a ⇒ 'b | almost-everywhere-zero f \}

\[(\text{nat} ⇒₀ \text{nat}) ⇒₀ 'a \cong 'a \text{ mpoly}\]

\texttt{datatype } 'a poly-rec

\[= \text{Coeff } 'a \quad \mid \quad \text{Rec } (\text{nat} ⇒₀ 'a \text{ poly-rec})\]

\[
\quad \cong \quad 'a \text{ mpoly} \times \text{monom-order} \\
\quad \downarrow \quad \text{Rep} \\
((\text{nat} ⇒₀ \text{nat}) \times 'a) \text{ list} \times \text{monom-order}
\]
typedef 'a ⇒₀ 'b = { f :: 'a ⇒ 'b | almost-everywhere-zero f}
typedef \( 'a \Rightarrow_0 'b = \{ f :: 'a \Rightarrow 'b \mid \text{almost-everywhere-zero } f \} \)

\[(\text{nat} \Rightarrow_0 \text{nat}) \Rightarrow_0 'a \cong 'a \text{ mpoly} \]

**datatyper**
\[ 'a \text{ poly-rec} = \text{Coeff 'a} \]
\[ \mid \text{Rec (nat} \Rightarrow_0 'a \text{ poly-rec)} \]

\[ 'a \text{ poly-distr} \cong 'a \text{ mpoly} \times \text{monom-order} \]

\[ (\text{rec}_{\text{nat}}) \times 'a \text{ list} \times \text{monom-order} \]

factor out dense & sparse implementations
Design choice: number of variables

Should the number of variables show up in the type?

\[ \text{'}a \text{ mpol} \quad \text{vs.} \quad \text{'}a \text{ poly poly ... poly} \quad \text{vs.} \quad ('a, 7) \text{ mpol} \]
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- Algorithms change number of variables dynamically.
- No computation on types

\[ ('a, 4 + 3) \text{ mpoly} \neq ('a, 2 + 5) \text{ mpoly} \]
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- Algorithms change number of variables dynamically.
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\[ ('\texttt{a, 4 + 3}) mpol \not\equiv ('\texttt{a, 2 + 5}) mpol \]

- Polynomials over an unbounded number of variables

Derived notion \textbf{variable number}:
the highest index of a variable with non-zero coefficient.

\textbf{Implicitly} extend polynomials as needed.
Exploit the Representation in Abstract Algorithms

Example:

- Gröbner bases algorithm depends on a **monomial order**.
- Efficiency relies on fast access to leading monomial in that order.

\[ 'a \text{ mpoly} \quad \text{vs.} \quad 'a \text{ mpoly } \times \text{ monom-order} \]
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- Algebraic type classes require uniqueness of polynomials.
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\[ \text{'a mpoly} \quad \text{vs.} \quad \text{'a mpoly \times monom-order} \]

- Algebraic type classes require uniqueness of polynomials.
- Algorithm receives representational details as parameter.
- If polynomial’s representation fits to the parameter, execution is **fast**. Otherwise, convert polynomial . . . or search . . .
- No static checks, no efficiency guarantees!
Open Problem: Controlling Representations

- What happens when we combine two polynomials?

  \[
  \begin{array}{ccc}
  + & Rec & Distr. \\
  Rec & Rec & ??? \\
  Distr & ??? & Distr \\
  \end{array}
  \]

  How can we make contextual information available?
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Rec & Rec & ??? \\
Distr & ??? & Distr \\
\end{array}
\]

How can we make contextual information available?

- How can the user specify the representations?

\[
\text{value} \ (2 :: \text{int} \ poly) \times 3
\]

Recursive or distributive? Dense or Sparse? Which monomial order?

In CAS, the user declares his choice as a configuration option. Can we mimick this in Isabelle?
The Ubiquitous Type Class `zero`

typedef `'a ⇒_0 'b = { f :: 'a ⇒ 'b :: zero | almost-everywhere-zero f }`

There is no map function for `'b` that satisfies

$$map f \circ map g = map (f \circ g)$$
The Ubiquitous Type Class zero

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\[ map f \circ map g = map (f \circ g) \]

BNF \Rightarrow_0 \text{ is not a BNF!}
Must construct 'a poly-rec manually

Lifting Quotient theorem only for relations that respect zero
No parametrised correspondence relations

Transfer Transfer rules must restrict function space
===> is too weak

Library Re-implement finite maps with the invariant \(0 \notin \text{ran } m\)
How can we improve reuse?
Current state of multivariate polynomials in Isabelle:

- Design seems good
- Prototype of abstract and representation types with minimal set of operations
- Lemmas and algorithm implementations are still missing

Up for discussion:

- User-friendliness/convenience for the working mathematician.
- Control of representations
- Better integration with Isabelle packages
Summary

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