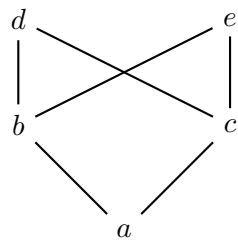


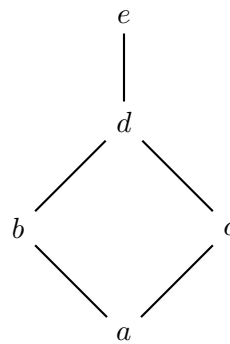
## Exercise 9

### Exercise 1

Are (a) and (b) complete lattices?



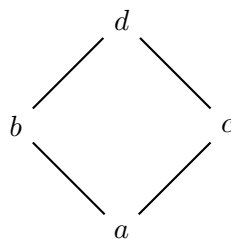
(a)



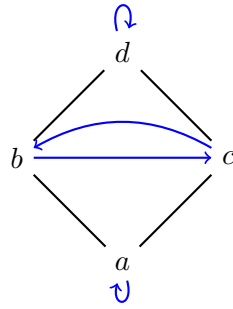
(b)

### Exercise 2

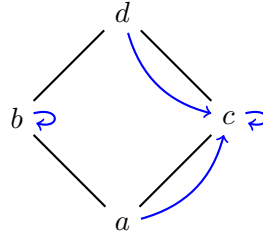
Consider the lattice  $L = (A, \sqsubseteq)$ , where  $A = \{a, b, c, d\}$ . The partial order  $\sqsubseteq \subseteq A \times A$  is depicted in the Hasse diagram below.



1. List the elements of  $\sqsubseteq$ .
2. Consider the following functions  $f, g : A \mapsto A$



Function  $f$

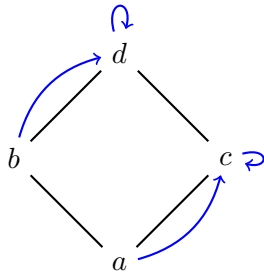


Function  $g$

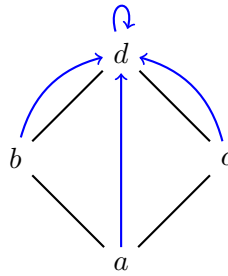
- Is  $f$  monotone? Is  $g$  monotone?
- List the set  $Fix(f)$  of fixpoints of  $f$ , and the set  $Red(f)$  of post-fixpoints of  $f$ .
- List the sets of fixpoints/post-fixpoints of the function  $g$ .

### Exercise 3

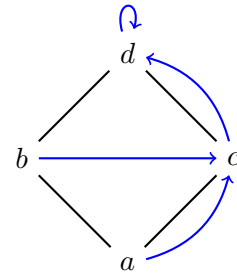
1. Consider the following three functions:  $f, g, h : A \mapsto A$ , defined below:



Function  $f$



Function  $g$



Function  $h$

- Does  $g$  approximate  $f$ ?
  - Does  $h$  approximate  $f$ ?
2. Let  $R = \mathbb{R} \cup \{-\infty, +\infty\}$  and  $Z = \mathbb{Z} \cup \{-\infty, +\infty\}$ , where  $\mathbb{R}$  is the set of rational numbers and  $\mathbb{Z}$  is the set of integers.  
 $(R, \leq)$  and  $(Z, \leq)$  are complete lattices.  
Let  $\alpha : R \mapsto Z$  as  $\alpha(x) = \lceil x \rceil$ . (Here  $\lceil x \rceil$  rounds-up  $x$  to the nearest integer.)  
Let  $\gamma : Z \mapsto R$  as  $\gamma(x) = x$ .  
Consider the function  $f : R \mapsto R$  defined as  $f(x) = x^2$ .

- Give two functions  $g, h : Z \mapsto Z$  that approximate  $f$ . Which one is more precise?
- Give a function  $k : Z \mapsto Z$  that approximates any function  $f : R \mapsto R$ .

## Exercise 4

Recall that an interval transformer for an action  $a$  is defined as:

$$\llbracket a \rrbracket_i : (Var \mapsto L^i) \mapsto (Var \mapsto L^i)$$

where  $L^i = \{[x, y] \mid x, y \in \mathbb{Z}^\infty, x \leq y\} \cup \{\perp_i\}$  are the interval domain's elements (see slide 17 from the lecture).

1. Consider the interval maps:

$$m_1 = x \mapsto [-3, 8], y \mapsto [0, 5]$$

$$m_2 = x \mapsto [-3, 8], y \mapsto \perp_i$$

The interval transformer for  $\leq$  is defined on slide 33. Apply the transformer to compute the result of:

$$\begin{array}{ll} \llbracket x \leq y \rrbracket(m_1) = & \llbracket x \leq y \rrbracket(m_2) = \\ \llbracket 3 \leq 5 \rrbracket(m_1) = & \llbracket 3 \leq 5 \rrbracket(m_2) = \\ \llbracket 5 \leq 3 \rrbracket(m_1) = & \llbracket 5 \leq 3 \rrbracket(m_2) = \end{array}$$

2. Define the interval transformer for assignment:

$$\llbracket x := a \rrbracket(m) =$$

3. Define the multiplication expression for interval elements:

$$\langle a_1 * a_2, m \rangle \Downarrow_i ?$$

4. Define the interval transformer for equality:

$$\llbracket x = y \rrbracket(m) =$$

## Exercise 5

Consider the following program:

```
foo (int x) {  
1   y := 2  
2   if (x <= y)  
3       z := 3 * x  
   else  
4       z := y  
5   z := y * z  
6 }
```

- Give two concrete traces  $t_1$  and  $t_2$  of the program.
- Apply the interval abstraction function  $\alpha^i$  given on slide 19 from the lecture on the set  $\{t_1, t_2\}$ .
- Compute the least fixpoint  $\text{lfp} F^i$  of the program using the interval domain abstraction.
- Give a concrete trace  $t \in \gamma^i(\text{lfp} F^i)$  that is not a valid trace. Here  $\gamma^i$  is the concretization function; see slides 19-20 from the lecture.