ETH
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Formal Methods and Functional Programming

Exercise Sheet 8: Structural Induction

Submission deadline: May 2nd, 2011

Please submit your solution before **9:15am** on the submission date specified above. Solutions can be submitted via e-mail or by using the boxes to the left of **RZ F1**. Make sure that the first page (and preferably each sheet) always contains your name, the exercise sheet number as well as your tutor's name and the weekday (Tuesday or Wednesday) of your exercise group. Don't forget to staple your pages if you submit more than one page.

Assignment 1

A chocolate bar consists of a number of squares $n \geq 1$ arranged in a rectangular pattern. You split the bar into small squares always breaking along the lines between the squares. Prove using strong induction that the minimum number of breaks in order to split the bar into small squares is $n - 1$.

Assignment 2

In the parlour game Nim, there are two players and two piles of matches. At each turn, a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins.

The second player applies the following strategy: when the opponent removes m matches from one pile, he removes m matches from the other pile.

Prove that if, when it is the first player's turn, the two piles contain the same number of matches, then the second player always wins.

Note: You should be careful to explicitly state the property of interest, and that the use of the induction hypotheses is justified.

Assignment 3

Let a be an arithmetic expression in $Aexp$. Let σ and σ' be two states such that $\forall x \in FV(a) : \sigma(x) = \sigma'(x)$. Prove that $\mathcal{A}[\![a]\!]\sigma = \mathcal{A}[\![a]\!]\sigma'$, or more explicitly that $\forall a \in Aexp \cdot \forall \sigma, \sigma' \in State \cdot ((\forall x \in FV(a) \cdot \sigma(x) = \sigma'(x)) \Rightarrow \mathcal{A}[\![a]\!]\sigma = \mathcal{A}[\![a]\!]\sigma')$

Assignment 4

Define a substitution function $b[y \mapsto e]$ for Boolean expressions that replaces all occurrences of the variable y in the Boolean expressions b with the arithmetic expression e. Prove that your definition satisfies the equality

$$\mathcal{B}[\![b[y \mapsto e]]\!]\sigma = \mathcal{B}[\![b]\!]\big(\sigma[y \mapsto \mathcal{A}[\![e]\!]\sigma]\big) \, ,$$

for all Boolean expressions b, all arithmetic expressions e, all variables y, and all states σ (where the second $\mapsto$ is the notation defined in p.55 of the introduction of the second part of the course).

Assignment 5

Implement using the programming language Haskell the syntax of the **IMP** language as algebraic (Haskell) data types (where `Aexp` and `Bexp` are the datatypes representing arithmetic and boolean expressions respectively). Implement also the semantics of boolean and arithmetic expressions (note that you do not have to implement a parser for **IMP**, but only the data types representing its syntax, and the semantics has to deal with these data types). The signature of these functions is `evalBexp :: Bexp -> State -> Bool` and `evalAexp :: Aexp -> State -> Integer` respectively (`State` is the datatype representing states).

Please mail your solution of this assignment to your tutor. The email addresses of the tutors are:

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Assignment 6 - Headache of the week

Let V be the set of all finite subsets of natural numbers.

1. Prove that the proper-subset relation $\subsetneq$ is well founded over V, and state the corresponding well founded induction principle over V for $\subsetneq$. **Hint:** you may use strong induction over the size of the set.

2. Prove by induction that every $S \in V$ has $2^{|S|}$ subsets, where $|S|$ denotes the cardinality of S.