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Formal Methods and Functional Programming

Exercise Sheet 10: Small Step Semantics

Submission deadline: May 16th, 2011

Please submit your solution before **9:15am** on the submission date specified above. Solutions can be submitted via e-mail or by using the boxes to the left of **RZ F1**. Make sure that the first page (and preferably each sheet) always contains your name, the exercise sheet number as well as your tutor's name and the weekday (Tuesday or Wednesday) of your exercise group. Don't forget to staple your pages if you submit more than one page.

Notation: As in slide 105 of the lectures, we use the vector notation $\sigma[\vec{y}\mapsto\vec{v}]$, as a shorthand for a (possibly empty) sequence of state updates. i.e., for some $m \geq 0$, $\sigma[\vec{y}\mapsto\vec{v}]$ abbreviates $\sigma[y_1\mapsto v_1][y_2\mapsto v_2]\ldots[y_m\mapsto v_m]$, for some sequences of variables $\vec{y} = y_1, y_2, \ldots, y_m$ and corresponding values $\vec{v} = v_1, v_2, \ldots, v_m$. For empty sequences (i.e., $m = 0$), then $\sigma[\vec{y}\mapsto\vec{v}]$ is just σ.

Assignment 1 - implementing SOS

You find a solution of this assignment in the literate Haskell file `simpi.lhs`.

Assignment 2 - states and state updates

(i) **Proof:** We need to show that, $\forall y \in \mathrm{Var}, \sigma[x\mapsto v_1][x\mapsto v_2](y) = \sigma[x\mapsto v_2](y)$. For arbitrary $y \in \mathrm{Var}$, we have (using the definition of state update):

$$\begin{array}{rcl} \sigma[x\mapsto v_1][x\mapsto v_2](y) & = & \begin{cases} v_2 & \textit{if } y = x \\ \sigma[x\mapsto v_1](y) & \textit{otherwise} \end{cases} \\ & = & \begin{cases} v_2 & \textit{if } y = x \\ \sigma(y) & \textit{otherwise} \end{cases} \\ & = & \sigma[x\mapsto v_2](y) \end{array}$$

(ii) **Proof:** We need to show that, if $x \neq y$, then $\forall z \in \mathrm{Var}, \sigma[x{\mapsto}v_1][y{\mapsto}v_2](z) = \sigma[y{\mapsto}v_2][x{\mapsto}v_1](z)$. For arbitrary $z \in \mathrm{Var}$, we have (using the definition of state update):

$$\begin{array}{rcl}
\sigma[x{\mapsto}v_1][y{\mapsto}v_2](z) & = & \begin{cases} v_2 & \textit{if } z = y \\ \sigma[x{\mapsto}v_1](z) & \textit{otherwise} \end{cases} \\
& = & \begin{cases} v_2 & \textit{if } z = y \\ v_1 & \textit{if } z = x \textit{ (and } z \neq y \textit{ but we know } x \neq y) \\ \sigma(z) & \textit{otherwise} \end{cases} \\
& = & \begin{cases} v_1 & \textit{if } z = x \\ v_2 & \textit{if } z = y \textit{ (and } z \neq x \textit{ but we know } x \neq y) \\ \sigma(z) & \textit{otherwise} \end{cases} \\
& = & \begin{cases} v_1 & \textit{if } z = x \\ \sigma[y{\mapsto}v_2](z) & \textit{otherwise} \end{cases} \\
& = & \sigma[y{\mapsto}v_2][x{\mapsto}v_1](z)
\end{array}$$

Note that the reordering of the cases in the function definition only works because we assumed $x \neq y$ - this condition is necessary (and indeed, the result isn't true otherwise).

(iii) **Proof:** By strong induction on m (the sequence length). Let $\vec{y}$ and $\vec{v'}$ be arbitrary sequences of length m, and let σ be an arbitrary state (we use a different name from the quantified σ'). Then we need to show, $\sigma[x{\mapsto}v_1][\vec{y}{\mapsto}\vec{v'}][x{\mapsto}v_2] = \sigma[\vec{y}{\mapsto}\vec{v'}][x{\mapsto}v_2]$, and we can assume (our induction hypothesis) that the corresponding property holds for any sequences of smaller (than m) length, and for all states σ'.

We consider two cases:

$(m = 0)$**:** Then we need to show $\sigma[x{\mapsto}v_1][x{\mapsto}v_2] = \sigma[x{\mapsto}v_2]$. This follows from part (i) of the question.

$(m > 0)$**:** Consider the first variable y_1. We consider two (sub-)cases:

$(y_1 = x)$**:** Then, by part 1 of the question, $\sigma[x{\mapsto}v_1][y_1{\mapsto}v'_1] = \sigma[y_1{\mapsto}v'_1]$, and so we have $\sigma[x{\mapsto}v_1][\vec{y}{\mapsto}\vec{v'}][x{\mapsto}v_2] = \sigma[\vec{y}{\mapsto}\vec{v'}][x{\mapsto}v_2]$ as required.

$(y_1 \neq x)$**:** Then, by part 2 of the question, $\sigma[x{\mapsto}v_1][y_1{\mapsto}v'_1] = \sigma[y_1{\mapsto}v'_1][x{\mapsto}v_1]$. Now, what we need to show is equivalent to: $(\sigma[y_1{\mapsto}v'_1])[x{\mapsto}v_1][y_2{\mapsto}v'_2]\ldots[y_m{\mapsto}v'_m][x{\mapsto}v_2] = (\sigma[y_1{\mapsto}v'_1])[y_2{\mapsto}v'_2]\ldots[y_m{\mapsto}v'_m][x{\mapsto}v_2]$. Since the sequences $y_2, \ldots, y_m$ and $v'_2, \ldots, v'_m$ are of smaller length than m, this follows from our induction hypothesis, taking σ' to be $(\sigma[y_1{\mapsto}v'_1])$.

Assignment 3 - last year's exam

Let `s'` be the body of the loop.

$$
\begin{aligned}
\langle \mathtt{s}, \sigma \rangle &\to_1 \langle \mathtt{if\ n\#\ 0\ then\ s';\ s\ else\ skip\ end}, \sigma \rangle \\
&\to_1 \langle \mathtt{a\ :=\ a+n;\ b\ :=\ b*n;\ n\ :=\ n-1;\ s}, \sigma \rangle \\
&\to_1 \langle \mathtt{b\ :=\ b*n;\ n\ :=\ n-1;\ s}, \sigma[\mathtt{a} \mapsto 2] \rangle \\
&\to_1 \langle \mathtt{n\ :=\ n-1;\ s}, \sigma[\mathtt{a}, \mathtt{b} \mapsto 2, 2] \rangle \\
&\to_1 \langle \mathtt{s}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 2, 2, 1] \rangle \\
&\to_1 \langle \mathtt{if\ n\#\ 0\ then\ s';\ s\ else\ skip\ end}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 2, 2, 1] \rangle \\
&\to_1 \langle \mathtt{a\ :=\ a+n;\ b\ :=\ b*n;\ n\ :=\ n-1;\ s}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 2, 2, 1] \rangle \\
&\to_1 \langle \mathtt{b\ :=\ b*n;\ n\ :=\ n-1;\ s}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 1] \rangle \\
&\to_1 \langle \mathtt{n\ :=\ n-1;\ s}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 1] \rangle \\
&\to_1 \langle \mathtt{s}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 0] \rangle \\
&\to_1 \langle \mathtt{if\ n\#\ 0\ then\ s';\ s\ else\ skip\ end}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 0] \rangle \\
&\to_1 \langle \mathtt{skip}, \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 0] \rangle \\
&\to_1 \sigma[\mathtt{a}, \mathtt{b}, \mathtt{n} \mapsto 3, 2, 0]
\end{aligned}
$$

Assignment 4 - composing executions

Proof: By strong induction on number of steps k.

The case ($k = 0$) leads to a contradiction, since we assume our derivation sequence ends in a final state. The case ($k = 1$) is equivalent to the first rule of the structural semantics for sequential composition, and therefore follows easily.

Let's consider the case where $k \geq 2$. Recall that our induction hypothesis lets us assume that the proposition holds for $n < k$, i.e. for any $n < k$ and for all statements p, q and states τ, τ':

$$\langle p, \tau \rangle \to_1^n \tau' \ \Rightarrow\ \langle p; q\ ,\ \tau \rangle \to_1^n \langle q, \tau' \rangle \quad \text{IH}$$

Our assumption from the question is also that

$$\langle s_1, \sigma \rangle \to_1^k \sigma' \quad \text{A1}$$

We want to prove that

$$\langle s_1; s_2\ ,\ \sigma \rangle \to_1^k \langle s_2, \sigma' \rangle$$

From A1 and the fact that $k \geq 2$, we have that there is a configuration $\langle s_A, \sigma_A \rangle$ such that

$$\langle s_1, \sigma \rangle \to_1 \langle s_A, \sigma_A \rangle \to_1^{k-1} \sigma' \quad \text{A2}$$

which, by IH (instantiate: $\tau = \sigma_A$, $\tau' = \sigma'$, $p = s_A$ and $q = s_2$), becomes

$$\langle s_A; s_2\ , \sigma_A \rangle \to_1^{k-1} \langle s_2, \sigma' \rangle$$

and therefore what remains to be proven is:

$$\langle s_1; s_2, \sigma \rangle \to_1 \langle s_A; s_2\ , \sigma_A \rangle$$

This is proven by A2 and the second rule of the structural semantics for sequential composition.

Assignment 5 - executing in similar states

(EDIT 04/06: In the single-step lemma below, and in the general result proved over the length of derivation sequences, we need the extra information in the second of the two cases, that the resulting statement doesn't contain any new free variables compared with the original statement; i.e., $FV(s') \subseteq FV(s)$ below. Without the stronger induction hypothesis which this provides, we can't make the eventual argument work, at point $(*)$ below).

As is typical for interesting results about small-step semantics, we first need to prove the analogous result for single-step derivations, and then generalise to derivation sequences. Here is the single-step version, as a lemma: For all states σ, σ', statements s and configurations γ, if $\forall y \in FV(S), (\sigma(y) = \sigma'(y))$ and also $\langle s, \sigma\rangle \to_1 \gamma$, then there exist variables $\vec{x}$ and corresponding values $\vec{v}$ such that the $\vec{x}$ are all free variables of s (i.e., $\{\vec{x}\} \subseteq FV(s)$) such that *either*:

1. $\gamma = \sigma[\vec{x}\mapsto\vec{v}]$ and $\langle s, \sigma'\rangle \to_1 \sigma'[\vec{x}\mapsto\vec{v}]$, *or*
2. there exists a statement s' such that $\gamma = \langle s', \sigma[\vec{x}\mapsto\vec{v}]\rangle$ and $FV(s') \subseteq FV(s)$ and $\langle s, \sigma'\rangle \to_1 \langle s', \sigma'[\vec{x}\mapsto\vec{v}]\rangle$.

Proof We prove the result by induction on the derivation of $\langle s, \sigma\rangle \to_1 \gamma$, considering cases for each possible last rule applied in the derivation:

Skip$_{SOS}$ Then $s = \texttt{skip}$ and $\gamma = \sigma$. The result follows (taking empty sequences $\vec{x}$ and $\vec{v}$), since we can derive $\langle \texttt{skip}, \sigma'\rangle \to_1 \sigma'$ by the same rule.

Ass$_{SOS}$ Then, for some x and e we have $s = x := e$ and $\gamma = \sigma[x \mapsto \mathcal{A}[\![e]\!]\sigma]$. Note that, by the same rule, we can derive $\langle x := e, \sigma'\rangle \to_1 \sigma'[x \mapsto \mathcal{A}[\![e]\!]\sigma']$. By our previous result (Sheet 8, question 3), since $FV(e) \subseteq FV(s)$, we can deduce that $\mathcal{A}[\![e]\!]\sigma = \mathcal{A}[\![e]\!]\sigma'$. The result follows, taking singleton sequences ($x_1 = x$ and $v_1 = \mathcal{A}[\![e]\!]\sigma = \mathcal{A}[\![e]\!]\sigma'$).

Seq1$_{SOS}$ Then $s = s_1 ; s_2$ for some s_1, s_2, and for some σ'' we have $\langle s_1, \sigma\rangle \to_1 \sigma''$ and $\gamma = \langle s_2, \sigma''\rangle$. By induction hypothesis (applied to the sub-derivation of $\langle s_1, \sigma\rangle \to_1 \sigma''$), we obtain (from case 1, since σ'' is a final state) that $\sigma'' = \sigma[\vec{x}\mapsto\vec{v}]$ for some $\vec{x}$ in $FV(s_1)$ and some corresponding $\vec{v}$, and furthermore, that $\langle s_1, \sigma'\rangle \to_1 \sigma'[\vec{x}\mapsto\vec{v}]$ (this follows from the induction hypothesis, because $FV(s_1) \subseteq FV(s)$, and so we know that $\forall y \in FV(s_1), (\sigma(y) = \sigma'(y))$).

We aim to show that case 2 of our result holds (since γ is not a final state). Note that, since $FV(s_1) \subseteq FV(s)$ we also know that the $\vec{x}$ are also free variables of s, and note that we also know $FV(s_2) \subseteq FV(s)$. We can apply the rule $\textsc{Seq}2_{SOS}$ to the fact that $\langle s_1, \sigma'\rangle \to_1 \sigma'[\vec{x}\mapsto\vec{v}]$, to conclude $\langle s, \sigma'\rangle \to_1 \langle s_2, \sigma'[\vec{x}\mapsto\vec{v}]\rangle$ as required.

Seq2$_{SOS}$ Then $s = s_1 ; s_2$ for some s_1, s_2, and for some σ'' and s_3 we have $\langle s_1, \sigma\rangle \to_1 \langle s_3, \sigma''\rangle$ and $\gamma = \langle s_3 ; s_2, \sigma''\rangle$. By induction hypothesis (applied to the sub-derivation of $\langle s_1, \sigma\rangle \to_1 \langle s_3, \sigma''\rangle$), we obtain (from case 2, since $\langle s_3, \sigma''\rangle$ is not a final state) that $FV(s_3) \subseteq FV(s_1)$ and, for some $\vec{x}$ in $FV(s_1)$ ($\subseteq FV(s)$) and some corresponding $\vec{v}$, $\sigma'' = \sigma[\vec{x}\mapsto\vec{v}]$, and furthermore, that $\langle s_1, \sigma'\rangle \to_1 \langle s_3, \sigma'[\vec{x}\mapsto\vec{v}]\rangle$ holds.

We aim to show that case 2 of our result holds (since γ is not a final state). Since $FV(s_3) \subseteq FV(s_1)$, we have $FV(s_3 ; s_2) \subseteq FV(s_1 ; s_3)$. Using our knowledge that $\langle s_1, \sigma'\rangle \to_1$

$\langle s_3, \sigma'[\vec{x} \mapsto \vec{v}] \rangle$ holds, and applying the rule SEQ2_{SOS}, we can derive $\langle s_1 ; s_2, \sigma' \rangle \rightarrow_1 \langle s_3 ; s_2, \sigma'[\vec{x} \mapsto \vec{v}] \rangle$ as required.

IfT$_{SOS}$ Then $s =$ `if` b `then` s_1 `else` s_2 `end` for some b, s_1, s_2, and we know $\gamma = \langle s_1, \sigma \rangle$ and $\mathcal{B}[\![b]\!]\sigma = tt$. By the generalisation of Sheet 8, question 3 to boolean expressions, we must also have that $\mathcal{B}[\![b]\!]\sigma' = tt$ (since *FV*$(b) \subseteq$ *FV*(s), and so we know that $\sigma(y) = \sigma'(y)$ for all y in *FV*(b)). Therefore, by applying the same derivation rule, we can derive $\langle s, \sigma' \rangle \rightarrow_1 \langle s_1, \sigma' \rangle$ as required.

IfF$_{SOS}$ Analogous to the case for IFT_{SOS}.

While$_{SOS}$ Then $s =$ `while` b `do` s_1 `end` for some b, s_1, and $\gamma = \langle$`if` b `then` s_1 `;while` b `do` s_1 `end else skip end`$, \sigma \rangle$. Note that *FV*(`if` b `then` s_1 `;while` b `do` s_1 `end else skip end`) = *FV*$(b) \cup$ *FV*(s_1) = *FV*(s). By applying the same rule, we can derive $\langle s \rangle \sigma' \rightarrow_1 \langle$`if` b `then` s_1 `;while` b `do` s_1 `end else skip end`$, \sigma' \rangle$ as required.

Having proved our result for single-step reductions, we generalise it to reduction sequences: (EDIT 04/06: compared with the question, we prove the stronger result which includes the condition *FV*$(s') \subseteq$ *FV*(s) in the second cases, i.e., we prove:
for all states σ, σ', statements s and configurations γ, if $\forall y \in$ *FV*$(S), (\sigma(y) = \sigma'(y))$ and also $\langle s, \sigma \rangle \rightarrow_1^* \gamma$, then there exist variables $\vec{x}$ and corresponding values $\vec{v}$ such that the $\vec{x}$ are free variables of s (i.e., $\{\vec{x}\} \subseteq$ *FV*(s)) such that *either*:

1. $\gamma = \sigma[\vec{x} \mapsto \vec{v}]$ and $\langle s, \sigma' \rangle \rightarrow_1^* \sigma'[\vec{x} \mapsto \vec{v}]$, *or*
2. there exists a statement s' such that $\gamma = \langle s', \sigma[\vec{x} \mapsto \vec{v}] \rangle$ and $\langle s, \sigma' \rangle \rightarrow_1^* \langle s', \sigma'[\vec{x} \mapsto \vec{v}] \rangle$ and *FV*$(s') \subseteq$ *FV*(s)

)

Proof By induction on the number of steps (let's call this l) of the derivation sequence justifying $\langle s, \sigma \rangle \rightarrow_1^* \gamma$.

Base case ($l = 0$): Then our initial and final configurations are the same, and we have nothing to prove (we just take empty sequences $\vec{x}$ and $\vec{v}$, and we satisfy the second case of our desired result easily.

Inductive case ($l = k + 1$): Then, for some intermediate configuration $\langle s_k, \sigma_k \rangle$, we have $\langle s, \sigma \rangle \rightarrow_1^k \langle s_k, \sigma_k \rangle$ and $\langle s_k, \sigma_k \rangle \rightarrow_1^1 \gamma$. By induction, we obtain that (*) *FV*$(s_k) \subseteq$ *FV*(s), and that for some $\vec{x}$ in *FV*(s), and $\vec{v}$, we have $\sigma_k = \sigma[\vec{x} \mapsto \vec{v}]$ and $\langle s, \sigma' \rangle \rightarrow_1^k \langle s_k, \sigma'[\vec{x} \mapsto \vec{v}] \rangle$. Now, we apply the one-step version of our result to $\langle s_k, \sigma_k \rangle \rightarrow_1^1 \gamma$, and we obtain that, for some $\vec{x'}$ in *FV*(s_k), and some corresponding $\vec{v'}$, one of two possible cases occur (as stated in our lemma):

1. (Case 1 : $\gamma = \sigma_k[\vec{x'} \mapsto \vec{v'}]$ and $\langle s_k, \sigma'[\vec{x} \mapsto \vec{v}] \rangle \rightarrow_1 \sigma'[\vec{x} \mapsto \vec{v}][\vec{x'} \mapsto \vec{v'}]$). Then, by (*) above, we have that $\vec{x'}, \vec{x}$ are all in *FV*(s), and that $\langle s, \sigma' \rangle \rightarrow_1^* \sigma'[\vec{x} \mapsto \vec{v}][\vec{x'} \mapsto \vec{v'}]$ as required.

2. (Case 2: $\gamma = \langle s', \sigma_k[\vec{x'} \mapsto \vec{v'}] \rangle$ for some s' with $FV(s') \subseteq FV(s_k)$ and $\langle s_k, \sigma'[\vec{x} \mapsto \vec{v}] \rangle \rightarrow_1 \langle s', \sigma'[\vec{x} \mapsto \vec{v}][\vec{x'} \mapsto \vec{v'}] \rangle$). Then we have:
$\sigma_k[\vec{x'} \mapsto \vec{v'}] = \sigma[\vec{x} \mapsto \vec{v}][\vec{x'} \mapsto \vec{v'}]$, and we satisfy the second case of our result, by (*) (which justifies that $\vec{x'}, \vec{x}$ are all in $FV(s)$ and that $FV(s') \subseteq FV(s)$ and that we can derive $\langle s, \sigma' \rangle \rightarrow_1^* \langle s', \sigma'[\vec{x} \mapsto \vec{v}][\vec{x'} \mapsto \vec{v'}] \rangle$ as required.

Assignment 6 - Headache of the week: reordering programs

Assume $FV(s_1) \cap FV(s_2) = \emptyset$, and let σ_1, σ_2 be arbitrary states such that $\langle s_1; s_2, \sigma_1 \rangle \rightarrow_1^* \sigma_2$ holds. Then, we need to show that $\langle s_2; s_1, \sigma_1 \rangle \rightarrow_1^* \sigma_2$ holds.

Firstly, by the lemma from slide 132 in the lectures, we obtain (ignoring the exact number of steps involved in the derivation sequences) that there exists a state σ_3 such that both $\langle s_1, \sigma_1 \rangle \rightarrow_1^* \sigma_3$ and $\langle s_2, \sigma_3 \rangle \rightarrow_1^* \sigma_2$ hold. By applying Lemma 1 stated in the question to the statement $\langle s_1, \sigma_1 \rangle \rightarrow_1^* \sigma_3$, we obtain that $\sigma_3 = \sigma_1[\vec{x} \mapsto \vec{v}]$ for some $\vec{x}$ which are free variables of s_1, and for some $\vec{v}$. Similarly, applying to the Lemma 1 to $\langle s_2, \sigma_3 \rangle \rightarrow_1^* \sigma_2$, we have that $\sigma_2 = \sigma_3[\vec{y} \mapsto \vec{v'}]$ for some $\vec{y}$ which are free variables of s_2, and for some $\vec{v'}$. Note that, since we assumed that $FV(s_1)$ and $FV(s_2)$ are disjoint, the two sequences $\vec{x}$ and $\vec{y}$ must also be disjoint.

Now, consider the states σ_1 and $\sigma_3 = \sigma_1[\vec{x} \mapsto \vec{v}]$. Since all of the variables $\vec{x}$ are in $FV(s_1)$, we know that, $\forall z \in FV(s_2).(\sigma_1(z) = \sigma_3(z))$. Therefore, we can apply the result proved in the previous question, using the statement $\langle s_2, \sigma_3 \rangle \rightarrow_1^* \sigma_3[\vec{y} \mapsto \vec{v'}]$, to obtain instead $\langle s_2, \sigma_1 \rangle \rightarrow_1^* \sigma_1[\vec{y} \mapsto \vec{v'}]$.

Similarly, since it holds that $\forall z \in FV(s_1).(\sigma_1(z) = \sigma_1[\vec{y} \mapsto \vec{v'}](z))$, then we can apply the result of the previous question to the statement $\langle s_1, \sigma_1 \rangle \rightarrow_1^* \sigma_1[\vec{x} \mapsto \vec{v}]$ to obtain instead that $\langle s_1, \sigma_1[\vec{y} \mapsto \vec{v'}] \rangle \rightarrow_1^* \sigma_1[\vec{y} \mapsto \vec{v'}][\vec{x} \mapsto \vec{v}]$. Note that, by Lemma 2, this final state $\sigma_1[\vec{y} \mapsto \vec{v'}][\vec{x} \mapsto \vec{v}] = \sigma_1[\vec{x} \mapsto \vec{v}][\vec{y} \mapsto \vec{v'}] = \sigma_2$. Therefore, we actually have, $\langle s_1, \sigma_1[\vec{y} \mapsto \vec{v'}] \rangle \rightarrow_1^* \sigma_2$.

Returning to our knowledge that $\langle s_2, \sigma_1 \rangle \rightarrow_1^* \sigma_1[\vec{y} \mapsto \vec{v'}]$ holds, we can now apply the result proved in question 4, to obtain $\langle s_2; s_1;, \sigma_1 \rangle \rightarrow_1^* \langle s_1, \sigma_1[\vec{y} \mapsto \vec{v'}] \rangle$. Combining this information with $\langle s_1, \sigma_1[\vec{y} \mapsto \vec{v'}] \rangle \rightarrow_1^* \sigma_2$ we obtain that $\langle s_2; s_1;, \sigma_1 \rangle \rightarrow_1^* \sigma_2$ as required.