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Children's Understanding of Successive Divisions in Different Contexts

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To meet the demands of high-school mathematics, one must understand that mathematical symbols, beyond being used to represent events and situations of the external world (signifier function), are also meaningful by themselves (signified function). In four experiments it was investigated under which conditions and at which age level elementary-school children (grades 4–6) would become aware of this conflict between practical and theoretical considerations in mathematics. To encourage children to mention this conflict, they were presented with mathematical story problems describing familiar situations. These situations were constructed so that they would adhere to abstract mathematical concepts of infinite divisions and limit because they required successive divisions of quantities. Solving these problems resulted in a conflict between practical and mathematical considerations. Results indicated that students from grades 4 and 5 did not mention this conflict because either they neglected the mathematical perspective of the problems or their mathematical solution was based on severe misconceptions regarding rational numbers. Only in grade 6 had most subjects overcome these misconceptions. About 30% of these subjects mentioned the conflict between practical requirement and mathematical principles.

Preschool and elementary-school mathematics is largely related to concrete everyday situations involving quantities and their relationships. Young children begin to deal with quantifying sets of objects, compare the number of their toys with those of others, and recognize that quantities can be operated on by per-

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forming additions and subtractions. Children also learn to share objects with other children in a fair manner, divide an object into equal parts, and make the transition from parts to a whole and vice versa. According to constructivistic approaches such as developed by Piaget (1953), mathematical competencies are developed from actions carried out in concrete, real-world situations. Several studies tend to support this assumption. Carraher, Carraher, and Schliemann (1985) showed that untrained children working in the market as vendors solved arithmetic problems embedded in a market context much better than problems embedded in a school-like context. The facilitating effects of situational contexts have also been documented by researchers who investigated how children solve arithmetic word problems in school settings. This research has shown that even some rather superficial changes in arithmetic word problems can make them much easier (Baranes, Perry & Stigler, 1989; Davis-Dorsey, Ross and Morrison, 1991; Hudson, 1983; Mevarech, Snir & Movshovitz-Hadar, 1992; Stern & Lehrndorfer, 1992). The benefit of presenting mathematical problems in situational contexts is not limited to young children. Resnick, Cauzinille-Marmèche, and Mathieu (1987) reported that students ages 11–15 understood algebra rules better when the problems were presented as word problems that dealt with familiar situations.

The given examples suggest that mathematical competencies are acquired by engaging in activities corresponding to real life and by drawing a structure mapping between the situational context and the numerical operations. Not all mathematical concepts, however, correspond to concrete, real-life experiences. Some mathematical concepts are so abstract that using everyday knowledge to construct a situational representation of the problem may not be possible. The concepts of infinity, infinite divisions, and limit provide examples of cognitive constructs that cannot be explained by simply referring to direct experience. In fact, these concepts may contradict practical, everyday experiences with finite objects and operations. This paper is concerned with the conceptual restructuring of elementary-school children's mathematical knowledge necessary to understand the abstract concepts of infinite divisions and limit.

CONCEPTUAL RESTRUCTURING IN MATHEMATICS: UNDERSTANDING THE DUAL FUNCTION OF MATHEMATICAL LANGUAGE

Conceptual restructuring describes cognitive development that goes beyond the continuous growth of a cognitive skill. During conceptual restructuring, the most important features of a concept may become relatively irrelevant or even disappear from the concept definition altogether. The restructured concept may then be based on more abstract principles, while the previously used concrete features have disappeared (Carey, 1985; Keil, 1989). The restructured concepts may then become close enough to natural language for easy translation into a statable, communicable form (Karmiloff-Smith, 1992). With respect to mathematics, conceptual restructuring in elementary-school children is characterized by extending the function of numbers and by becoming aware of the peculiarities of the mathematical symbol system.

Extending the Function of Numbers

Research has shown that we must attribute to the human infant some innate processes that are sensitive to number-relevant information (Gelman & Gallistel, 1978; Wynn, 1992). These innate processes guide the acquisition of early mathematical competencies. Without receiving systematic instruction, children learn to use natural numbers to denominate the size of discrete sets by counting (Gelman & Greeno, 1989; Wynn, 1990). To understand more advanced mathematical concepts such as rational numbers, it must be learned that numbers are not only used for counting (Gelman, 1991; Resnick, 1986). Rational numbers are not used for counting but rather to describe the relations between sets. Understanding rational numbers requires one to give up several principles that guided the understanding of natural numbers. While every natural number has a successor, this is not true for rational numbers. For natural numbers, there is a referent for the phrase ‘‘the next number after one.” However, there is no referent for the phrase ‘‘the next number after one half.” There is a smallest natural number but no smallest rational number. All natural numbers but not all rational numbers lying between two numbers can be enumerated.

Becoming Aware of the Peculiarities of Mathematical Language

According to Vygotsky (1962), the transition from intuitive concepts to scientific ones presupposes a change in understanding symbol systems. With development, children can no longer simply think about language as ways of communicating meaning, but rather language expressions themselves become *objects* of communication. This approach is especially suitable for explaining the development of mathematical understanding. Understanding abstract mathematical concepts requires one to divorce mathematical symbols from concrete contexts. In order to meet demands of high-school mathematics, one must deal with symbols having no referential meaning. The algebra system, for example, allows extensive manipulation of relationships among variables within a well-specified system without referring to the external world. The principles underlying the number-symbol system allow the development of new concepts because numbers and symbols of operations are meaningful even if they do not refer to real-world situations (Resnick et al. 1987). To understand abstract mathematical concepts, one must be aware of the *dual function of mathematical language:* Mathematical symbols can be used as an instrument of reasoning to describe real-world situations (signifier function), and they can be used as an object of reasoning because they possess a meaning by themselves (signified function).

Understanding the abstract mathematical concepts of infinite divisions and limit requires the described conceptual restructuring. One must overcome the restrictions of natural numbers. Only by referring to rational numbers can one understand that endlessly dividing a number will always result in another number very close to zero but not exactly zero. Similarly, endlessly adding halfs of halfs will result in numbers that will never meet the number one. Moreover, understanding infinite divisions and limit also requires an understanding of the signified function of numbers because both concepts do not correspond to events that can be simulated in the concrete world. By dividing mass very frequently, it will lose its function; from a practical point of view it will basically disappear. Also by very frequently adding halfs of halfs of a particular mass, one will get an amount of mass which corresponds to a whole from a practical point of view. One must understand that rules based on formal mathematical language can function independently of the ‘‘real” objects represented by the symbols.

Yet, children’s understanding of this dual function of the mathematical language has been a neglected issue in research on the development of mathematical cognition. Elementary-school children, of course, cannot be expected to give statements about the philosophical foundations of mathematics. However, children of this age might be able to verbalize the conflict between practical and theoretical considerations resulting from the dual function of mathematical language. In this paper, we take an exploratory first step toward understanding under what conditions elementary-school children mention this conflict. To encourage subjects to express this issue, they are presented with *paradox mathematical problems.* These problems are paradox because abstract and pure mathematical concepts are embedded in highly familiar context stories and therefore the outcome of the mathematical solution does not make sense with respect to practical requirements. In four experiments, students from grades 4–6 were presented with paradox mathematical problems which differed in the cover story. However, all cover stories used the same underlying mathematical concepts of infinite divisions and limit.

EXPERIMENT 1

The purpose of Experiment 1 was to investigate how children respond to problems where abstract mathematical concepts are not in accord with a real-life situational context. Problems involving the concepts of infinite divisibility and limit were presented in the context of money. Money provides a clear example of a familiar, everyday context that is often used in mathematics classrooms. Even a quick glance at arithmetic textbooks shows that the example of money is embedded in almost every unit. As early as the first grade, children learn to count coins and bills, they practice addition and subtraction by checking accounts and giving change, and they solve word problems relating to shopping and saving. Later, they learn about decimals and percents by solving more complex ‘‘money problems” involving discounts and profits. In addition, empirical studies have indicated that mathematical problems involving money are relatively easy to solve. For example, Carraher et al. (1985) reported that elementary-school children who had not learned fractions in school knew all terms of the currency and were familiar with transforming money into smaller units. Thus, there is reason to assume that children would be able to use their mathematical knowledge in a money context.

On the other hand, the context of money is not in accord with the concept of limit. For example, in the process of successive divisions, infinite divisibility creates a conflict when the smallest unit of a currency (e.g., 1 pfennig, 1 cent, 1 agora, etc.) is reached. How do children cope with this conflict? What do they do when they have to successively divide the smallest unit? Do they realize that there is a conflict between practical and theoretical solutions? Do they ignore the practical constraints and solve such problems from a mathematical point of view, or vice versa? Experiment 1 was designed to address these questions.

Method

Subjects. Fourteen fourth-graders (9 girls, 5 boys) randomly selected from three Munich after-school centers participated in the study. The mean age was 9 years, 9 months. The three centers, serving a population of four elementary schools, were randomly selected from all centers in the city. The experiment was run at the beginning of the school year.

Materials. The following problem was presented to all children in an individual interview:

> Peter received 32 marks pocket money each month. One month he wanted to buy a football as well as many other things. The football cost 32 marks. Because Peter wanted to buy a lot of other things too, he could not buy the football at that time. Peter decided to start saving. In the first month, he saved half of his pocket money.

Question 1 was asked: ‘‘How much money did he save in the first month?”

The story continued:

> In the second month, Peter again wanted to buy a lot of other things. Therefore, he could only save half of the sum that he saved in the first month.

Then two questions were asked—Question 2a: ‘‘How much money did Peter save in the second month?” Question 2b: ‘‘How much money had Peter saved up to now?”

The story continued:

> In the third month Peter was only able to save half of the money he had saved in the previous month.

Questions 3a and 3b were similar to Questions 2a and 2b. The story continued in the same manner. The children were asked for the saving rates and the sum of the savings for months 4 (saving rate 2 DM), 5 (saving rate 1 DM), 6 (saving rate 50 pfennigs), and 7 (saving rate 25 pfennigs). The final question was: ‘‘If Peter continued like this for a long time, would he ever be able to save enough money to buy the football?” If the answer was ‘‘yes”, the interviewer asked: ‘‘When would it happen and why?” If the answer was ‘‘no”, the interviewer asked: ‘‘Why?”

Procedure. Children were tested individually in a quiet room in the after-school center. A pretest was given 1 week prior to the beginning of the study to ensure that all children: (1) knew the fraction terms ‘‘half’’ and ‘‘half of,’’ (2) were able to switch between monetary units (from marks to pfennigs and vice versa), and (3) understood the concept of successive division. In the pretest the children were asked in an individual interview: ‘‘How much is half of 32 DM, 16 DM,’’ etc. *All* children correctly answered these questions. We also asked ‘‘How much is half of one mark?’’ (correct answer 50 pfennigs) and ‘‘How much is a quarter of one mark?’’ *All* children gave the correct answers. These findings correspond with those of a previous study showing that German children in fourth grade understand the expressions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ (Stern & Mevarech, 1991).

To test children’s understanding of successive divisions in a physical context, they were shown a circle drawn on a paper and were told: ‘‘Imagine that this circle is an apple. I can cut this apple into two halves.’’ The experimenter cut the ‘‘apple’’ into two halves and showed the parts to the child. Then she took one half and continued: ‘‘Now, I can cut this part of the apple into two equal halves.’’ She cut it and showed the two parts to the child. This procedure was repeated for $\frac{1}{8}$ and . Then the interviewer asked: ‘‘If I continue like this for a very long time, that is, cutting every half part of the apple into two halves, do you think that there will be a time when nothing of the apple remains?’’ The children had to answer and to explain their answers. All children answered that something would always remain. Their explanations included ‘‘things cannot disappear’’ or ‘‘at least a molecule will remain.’’ These results correspond with the results of previous studies showing that children at this age have an intuitive understanding of the concept of limit (Lappan and Montague-Wheeler, 1987; Fischbein, Tirosh, & Hess, 1979).

The main problem was presented orally and the children could use a paper and pencil to assist them in calculations. Because the problem includes a series of questions, each depending on the solution to the previous question, children first said the answer orally and then wrote it in a table describing the particular month, the amount of money saved in that month, and the total amount of money saved up to and including that month.

Results and Discussion

Overall, 21% of the subjects answered that Peter would get the money for the football ‘‘if he will save long enough.’’ Others (79%) answered that Peter would never get enough money. However, a detailed analysis of the data indicated that *none* of the children had realized the conflict between a practical and a mathematical point of view. The analysis showed that although 85% of the children could correctly divide in half 32, 16, 8, 4, and 2 DM, only 5 children (36%) could divide in half one mark or smaller sums than one mark. Among the rest, 9 children (64%) answered: ‘‘Half of one mark is *zero.* Therefore, Peter didn’t save any money in the fifth month. He could not buy the football.’’ Two other children stopped dividing when they reached 50 pfennigs. One of those children said that half of 50 pfennigs is *zero.* The other child said, ‘‘You can’t divide a 50 pfennig coin.’’ (In German currency, there is no ‘‘quarter.’’ There are coins for 50, 10, 5, 2, and 1 pfennig(s).) Finally, three children reached 25 pfennigs. Two of them said that half of this amount was zero. One child said: ‘‘At some point you will reach a pfennig and half of a Pfennig is zero.”

The results of Experiment 1 clearly provide a useful methodological caution: The children’s judgments would be very misleading if they were interpreted without considering the children’s explanations. The situational context provided in Experiment 1 did not induce cognitive conflicts. Rather, Experiment 1 clearly shows that even available knowledge about dividing money was not accessed in a familiar context. Children who in the pretest had shown that they knew how to divide money, including sums that were smaller than one (half and quarter), did not use this knowledge in the context of buying and saving. Thus, most children were not even confronted with the conflict of the limit, but rather got stuck much earlier when they had to convert units from marks to pfennigs. The results indicate that children faded out the situational context and referred to their knowledge about natural numbers: 1 is smallest number that cannot be further divided. Starting the division process by presenting a rather large natural number like 32 might have encouraged this solution process. To prevent subjects from using their knowledge about natural numbers, in Experiment 2 no numbers were mentioned.

EXPERIMENT 2

Experiment 2 focused on the concept of infinite divisions. Do children understand that dividing quantities successively will not lead to zero? Two types of stories were developed. Either the described story was derived from a familiar social situation, suggesting a pragmatic solution (*pragmatic condition*), or it was embedded in a more artificial, laboratory-like context (*laboratory-like condition*). Which of the two contexts will encourage children to mention the conflict between practical and theoretical considerations or even mention the dual function of the mathematical language? There are arguments in favor of the pragmatic context as well as in favor of the laboratory-like context.

In the pragmatic context, the conflict between practical events and pure mathematical concepts is quite evident. There are two different ways in which children may react. Children may, on the one hand, become aware that with the help of the mathematical language they can describe events that are impossible or paradoxical from a practical point of view. Therefore, the more pragmatic and familiar the context, the earlier children might recognize the dual function of mathematical symbols. On the other hand, the children could also construct a mental model of the situation that would match the described context while ignoring the logico-mathematical schemata. Thus, children might treat the problem in a nonmathematical way.

When faced with the laboratory-like context, children might be inclined to activate their logico-mathematical knowledge and also recognize the conflict between practical events and principal requirements, but in a quite different manner.

Method

Subjects. Forty-eight fourth-graders (29 girls, 19 boys) from three Munich after-school centers participated in the study. Subjects were randomly selected from the same sample as in Experiment 1, but no child participated in both experiments. Participants' mean age was 10 years, 4 months. The experiment was run at the end of the school year.

Materials. Two contextual conditions—''pragmatic context'' and ''laboratory-like context''—were designed. Under the laboratory-like context condition, we presented the following problem:

> Suppose there is a very large number of glasses, all of the same size. All the glasses are empty. Water has to be poured into the glasses. The first glass is completely filled with water. Then, the second glass is filled with half as much water as was poured into the first glass. The third glass is filled with half as much water as was poured into the second glass. If this procedure continues for a very long time, and a glass is always filled with half as much water as was poured into the glass before, will there ever be a glass in which there is no water?

Under the pragmatic context condition, we used a birthday-party script that mentioned the goal-related action of serving orange juice:

> Peter had a birthday party. He invited many children and wanted to serve them orange juice. There was a very large container of orange juice in the refrigerator and many glasses, all of the same size. The first child came to the party. Peter gave him a full glass of orange juice. Then, a second child came in. Peter worried that there might not be enough orange juice for all the children. Therefore, he gave the second child only half as much juice as he gave to the first child. Then, a third child came in. Peter got even more worried that there would not be enough juice for all the children. Therefore, he gave the third child only half as much juice as he had given to the second child. Peter continued to serve the juice in this way. That is, he gave each child half as much juice as he gave to the child before.

For half of the children, the story was ended with the ''abstract'' question:

> If Peter continues like this, will there ever be a glass in which there is no juice? Why?

For the other half, the story was ended with the ''pragmatic'' question:

> If Peter continues like this, will there ever be a child who gets no juice? Why?

Whereas the first question is identical to the question in the laboratory-like context condition, the second question is more related to familiar situations and might lead to reason in a more pragmatic way.

Procedure. Subjects were assigned randomly to the three conditions: 17 subjects to the laboratory-like context, 15 subjects to the pragmatic, lifelike context with the abstract question related to the liquid in the glass, and 16 subjects to the pragmatic context with the pragmatic question related to children getting juice. All children were individually tested in a quiet room in their after-school center. As in Experiment 1, the problems were presented orally to the children.

Results and Discussion

Children's answers were categorized, depending upon whether they said that (1) water [juice] *will not* disappear by successive divisions (''always some liquid in the glass'') or (2) that water [juice] *will* at some point be used up (''no liquid in the glass''). In the laboratory-like condition 14 children (82%) fell into Category (1) and three subjects (18%) fell into Category (2). In the pragmatic, lifelike condition *all* subjects fell into Category (2). The way the question was formulated (abstract vs pragmatic) had no effects on subjects' answers. Fisher's exact tests revealed that the differences between the two contextual conditions was significant ($p < .001$). A detailed analysis of children's explanations indicated that none of the subjects addressed the conflict between practical and theoretical considerations.

Similar to Experiment 1, most of the children under the pragmatic, lifelike context condition believed that successive division is finite. The children's explanations indicate that although no natural numbers were used in the ''juice task,'' children returned to the unit of ''1'' and mentioned that this unit cannot further be divided. Children further argued that dividing this ''basic unit'' would yield zero. A typical explanation was: ''At the end there will be a drop and half of a drop is zero.'' In contrast, the children who believed that there would always be some liquid in the glass explained that ''there will be still a drop'' or ''a molecule'' or stated that ''mass can't disappear by division.''

Although none of the subjects raised the conflict between practical and theoretical considerations, Experiment 2 suggested that children access different kinds of knowledge for solving problems embedded in different context stories. Under the laboratory-like context condition, children referred to intuitive knowledge that mass cannot disappear by successive divisions, whereas under the pragmatic context condition they mentioned the fact that successively divided mass loses its former purpose. Thus, the results indicate that a pragmatic context will encourage subjects to treat problems from a practical point of view and therefore seems not to be suitable for promoting children's abstract mathematical understanding. However, also under the laboratory-like context, none of the subjects mentioned the conflict between practical and theoretical aspects of mathematics. The way the problems were presented might have encouraged superficial treatment of the problems. As the subjects only had to answer a single question, no deeper analysis of the problem structure was stimulated.

Another limitation of Experiment 2 is that the problems presented under the pragmatic and laboratory-like conditions were not completely isomorphic. Under the pragmatic context condition, the story began by mentioning the ''big container'' from which the orange juice was poured into the glasses. No such container was mentioned under the laboratory-like context. Thinking about this container, in which the amount of juice is continually decreasing, may support children's assumption that after a while there will remain nothing in the container. Although we did not ask for the juice in the container, but for the juice in the glasses, the children might have been misled by the very existence of the container. This small semantic difference between the two stories created two different mathematical structures: one involving subtraction (pragmatic context) and the other not (laboratory-like context). It is possible, therefore, that the difficulty children faced under the pragmatic context resulted from the inclusion of the subtraction operation.

EXPERIMENT 3

The purpose of Experiment 3 was to overcome the described limitations of Experiment 2. Therefore, problems with an isomorphic mathematical structure were presented under the laboratory-like context and under the pragmatic context. To encourage a deeper analysis of the problem structure, subjects were asked three questions instead of one.

Experiment 2 revealed a clear context effect: the pragmatic context encouraged subjects to treat the problem from a practical rather than from a mathematical point of view. To research the long-term effects of the first context presentation, a repeated measurement design was chosen. Half of the subjects were presented with the problem at first under the pragmatic context story and then under the laboratory-like story, while the other half of the subjects were presented with the problems in reversed order.

Method

Subjects. Thirty-three fourth-graders (19 girls, 14 boys, mean age 10 years, 2 months) from three Munich after-school centers participated in the study. Subjects were randomly selected from the same after-school centers as in the first two experiments. This experiment was conducted more than a year after the first two experiments were conducted. None of the children who participated in these experiments had participated in the previous studies. The experiment was run at the beginning of the school year.

Problem presentation was randomized for each child. Thus, about half of the children ($n = 17$) were initially presented with the problem under the laboratory-like context condition; the remaining half ($n = 16$) were initially presented with the problem under the pragmatic context condition. In the second task administration, the problems were reversed.

Materials and procedure. Under the pragmatic context condition we presented the same story as used in Experiment 2. Under the laboratory-like context condition the story started:

> There is a big container with a very large amount of water and many glasses, all of the same size. Water from the container is poured into the glasses. . . .

From here on, the story continued as described in Experiment 2.

Subjects were asked the following three questions:

Question 1: ‘‘If one continues to pour water/juice from the container into the glasses, will the container ever be empty?” Question 1 was asked to check whether children treated the problem in a mathematical way and therefore immediately recognized that only little water/juice was taken from the container, or whether they only focused on the frequent take-away action but did not consider the small amounts taken away.

Question 2: ''Overall, How many full glasses of juice/water will be taken from the container?" Question 2 required the ability to consider the concept of limit. In order to answer this question correctly, children must build a mathematical model of the situation which is based on the inequality ''$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots < 2$." Only children who build a mathematical model of the problem are able to recognize that no more than two glasses will be taken from the container.

Question 3: ''Will there ever be a glass in which there will be no juice/water?" By asking Question 3, we wanted to encourage subjects to discuss the conflict between practical and theoretical considerations caused by the dual function of mathematical language. Question 3 was also considered to bring to light the misconception of division obtained already in Experiments 1 and 2.

The questions for each task were presented orally in a fixed order. The problems were presented in two different sessions with 7–10 days in between. As in Experiments 1 and 2, children were tested individually. For each session, one task was presented orally, and children were asked to answer the questions and to explain their solutions.

Results and Discussion

For each of the three questions, a categorizing scheme was developed for classifying the children's answers. Special care was taken so that the categorization scheme would be equally appropriate for both context conditions. Table 1 shows the different categories together with an example of children's explanations for each category.

For statistical analyses, an overall score was built by considering the explanations given for each of the three questions. The points assigned to each type of explanation are mentioned in the following.

The explanations of Question 1 (If one continues to pour water/juice from the container into the glasses, will the container ever be empty?) were assigned to two main categories depending on whether children said that the box would not become empty (Category 1.1, 1 point) or would become empty (Category 1.2, 0 points).

For Question 2 (Overall, how many full glasses of juice/water will be taken from the container?), it was considered whether children said that less than 2 glasses were taken from the container (Category 2.3, 0 points) or whether they recognized that no more than two glasses were taken out. An analysis of the explanations indicated that only some students gave mathematically appropriate explanations (Category 2.1, 1 point) while others did not (Category 2.2, 0 points). Explanations given by these children were based on the misconception of division already obtained in the prior experiments. Subjects stated that at some point

TABLE 1
Children's Explanations in Experiment 3

Question 1: Will the box ever become empty?

(*1.1*) *Answering that the box will not become empty*

Laboratory-like context condition

''The amount taken away is no more than two glasses; this is less than was in the container.''

''Because he always takes so little from the container it can't get empty.''

Pragmatic context condition

''When the first child gets one (glass) and the second a half, this is one and a half. The third child gets a quarter; this is one and a half and a quarter. This is less than two. The fourth child will get only half of a quarter. Taken altogether this is still less than two glasses.''

''Because every child gets so little, it can't become empty.''

(*1.2*) *Answering that the box will become empty*

Laboratory-like context condition

''It will be less and less; then there will be a drop in the container and half of a drop is nothing.''

''At one point everything will become empty.''

''If one always takes out and out, it must become empty.''

''To answer this question I have to know how many glasses he takes out.''

Pragmatic context condition

''More and more children will get something and there will be one sip in the container. When the last child has got this sip, nothing will be left.''

''Because he invited so many children, the container must become empty.''

''If more and more children come and all drink, it must be empty at some point.''

''If he takes something out and takes out, nothing can remain.''

''It depends on how many children come to the birthday party.''

Question 2: How many glasses will be taken from the container?

(*2.1*) *Less than 2 glasses, considering appropriate mathematical knowledge*

Laboratory-like context condition

''One and a half and a half of the half and a half of a half of a half—that is more than one but less than 2.''

''It will not be two at all because always something will miss; also it is very small.''

''At first he takes one (glass), and a then half, this makes together one and a half. Then he takes a quarter, makes together one and three quarters. Then the added quantity becomes very small, but it is still something. It is so small that at all it will not be more than 2.''

Pragmatic context condition

''Because he gives each child so little juice, it will not be two glasses at all.''

(*2.2*) *Less than 2 glasses, considering inappropriate mathematical knowledge*

Laboratory-like context condition

''At first there is one glass; then there is a half glass. Then there is a quarter glass and then $\frac{1}{8}$ glass and then $\frac{1}{16}$ glass. Half of $\frac{1}{16}$ is zero. Therefore, it will not be two at all.''

''It is one glass plus half of this, that is a half, and if you always take a half, you have one thousand glass, and half of this is zero. Therefore, always something is missing.'' ''If one takes always half of a half, the one has $\frac{1}{1000}$ and half of this is zero.''

Pragmatic context condition

''The first child gets one full glass; the second child gets half a glass. The third child gets only a quarter and the fourth child gets nothing because half of a quarter is nothing. Therefore, only one glass plus half a glass plus a quarter of a glass will be taken out, which is less than two.''

TABLE 1—*Continued*

(*2.3*) *More than 2 glasses*

Laboratory-like context condition

‘‘More than 30. If one takes out and takes out and takes out, it will be very many glasses.”

‘‘Maybe 200. One glass and a half glass are one and a half. Half of a half is a quarter. You can’t divide a quarter. Therefore, he will always take out many quarters.”

Pragmatic context condition

‘‘10,000 perhaps. Because so many children are coming he takes many glasses.”

‘‘One and a half and half of a half and then always the half, that will be very many, maybe fifty.”

‘‘Because he always takes half of the half. This must be very many if you continue, at least one hundred.”

Question 3: Will there ever be a glass with no water?

(*3.1*) *Saying that there will always be something in the glass by mentioning the conflict between practical and theoretical aspects*

Pragmatic context condition

‘‘Everything has an end but numbers do not.”

‘‘There will be something in the glass but it will become so small, you can’t see it.”

‘‘There will be no empty glass, because there are infinite numbers, billions of billions; it will be always something. It is very small but it is always something.”

(*3.2*) *Saying that there will always be something in the glass by referring to principles of division*

Laboratory-like context condition

‘‘Even a drop is something.”

‘‘The water can’t disappear. There will be at least a molecule.”

Pragmatic context condition

‘‘You have a mini-mini sip but this is still something.”

‘‘Something divided by 2 can’t become zero.”

‘‘You can’t get zero by dividing by 2.”

(*3.3*) *Saying that there will be nothing in the glass by referring to the misconceptions of division*

Laboratory-like context condition

‘‘At the end there will be one drop and half of that drop is zero.”

‘‘At some point you will have a drop. Then you will have a half drop and then you will have a quarter drop. Half of a quarter drop is zero.”

(*3.4*) *Saying that there will be nothing in the glass by referring to the misconceptions of division*

Pragmatic context condition

‘‘Maybe yes, maybe no. Depends on how much he pours.”

‘‘If there are too many children, the juice will run out.”

there would be a very small quantity and dividing this quantity by 2 would result in zero.

For Question 3 (Will there ever be a glass in which there will be no juice/ water?), four categories were developed. Category 3.1 was developed for children who mentioned the conflict between practical and theoretical aspects (2 points). Answers were assigned into Category 3.2 if children said that there will always be something in the glass and based their explanations on the principles of division (1 point).

Subjects who said that there will be a glass with nothing were assigned to Category 3.3 if they mentioned the misconceptions of division (0 points) and to Category 3.4 if they referred to the frequency of the pouring action or constructed irrelevant events (0 points). All answers were assigned to the categories by two independent raters. The inter rater agreement in categorizing the explanations was 88%. Disagreements were discussed and then assigned to a single category. Table 2 presents the frequencies of responses in each category and the means of the overall scores.

The overall score was the dependent variable in a 2×2 analysis of variance. One independent, repeated-measurement variable was Context (pragmatic context, laboratory-like context). The other independent variable was Subgroup: Subjects who were presented with the pragmatic context problem in the first trial and with the laboratory-like context in the second trial were assigned to Subgroup 1; subjects who were presented with the problems in reversed order were assigned to Subgroup 2. The analysis revealed a significant effect for context, $F(1,31) = 13.79$, $p < .001$, and a significant interaction between subgroup and context, $F(1,31) = 6.18$, $p < .05$. Post-hoc tests indicated that the score revealed under the pragmatic context condition when presented in the first trial was lower than the three other scores ($p < .05$). The significant interaction effect shows that when being presented with the pragmatic context condition in the first trial, subjects are inclined to treat the problem from a nonmathematical view. If, however, subjects were presented with the laboratory-like problem in the first

TABLE 2
Percentage of Children Who Were Assigned to the Explanation Categories for Questions 1–3

Run:	First		Second	
Context:	Laboratory	Pragmatic	Laboratory	Pragmatic
Question 1: Will the container be empty once?				
1.1. Box will not become empty	70.6	25.1	69.1	58.9
1.2 Box will become empty	29.4	43.8	31.6	41.2
Question 2: How many glasses will be taken from the container?				
2.1 Less than 2 glasses, appropriate mathematical knowledge	11.8	6.3	6.3	11.8
2.2 Less than 2 glasses, inappropriate mathematical knowledge	41.2	12.5	56.3	41.2
2.3 More than 2 glasses	47.1	81.3	38.6	47.1
Question 3: Will there be an empty glass once?				
3.1 Saying that there will be always something in the glass by mentioning the conflict between practical and theoretical considerations	0.0	12.5	0.0	17.6
3.2 Saying that there will be always something in the glass by mentioning the principles of division	70.6	12.5	56.3	41.2
3.3 Saying that there will be a glass with nothing by referring to frequency of division	0.0	75.4	0.0	42.2
3.4 Saying that there will be a glass with nothing by mentioning the misconceptions of division	29.4	0.0	43.8	0.0
Overall score				
M	1.9	.7	1.8	1.7
s	1.3	1.2	1.3	1.4

trial and with the pragmatic problem in the second trial, the latter was also treated in a mathematical way. Thus, there was an impact of the laboratory-like problem on the way the pragmatic problem was treated, but no reverse effect was found. Subjects who had treated the pragmatic problem from a practical point of view in the first trial did not transfer this kind of treatment to the laboratory-like problem but rather accessed mathematical knowledge.

However, even if under the laboratory-like condition a problem was treated from a mathematical point of view, in most cases the accessed mathematical knowledge was based on misconceptions of division already obtained in Experiments 1 and 2. The children knew that there were some numbers between zero and one, but they did not know that there are endless numbers between zero and one. Rather, they said that there will at one point be a fraction that will result in zero if it is divided by 2. Some of these children said that this will be $\frac{1}{8}$ or $\frac{1}{16}$; other children mentioned fractions such as $\frac{1}{100}$, $\frac{1}{1000}$, or $\frac{1}{10000}$. The conflict between practical and theoretical considerations was mentioned only by very few children under the pragmatic condition. The small *n* did not allow statistical analysis.

Summing up, the results show that under the laboratory-like context condition, the majority of the children were inclined to solve the problem from a mathematical point of view. They related numbers to the amount of water taken from the container and added these numbers. However, most of the children misleadingly assumed that there is a basic unit that if divided further results in zero. If the pragmatic context condition was presented in the first run, most of the subjects referred to the situational context of the story. They considered that many children would come to the party and that all of them want to have a glass with orange juice.

EXPERIMENT 4

Experiments 1–3 indicated that fourth-graders' limited understanding of non-natural numbers prevented them from discussing the conflict between practical and theoretical considerations resulting from the dual function of mathematical language. The lack of formal numerical understanding prevented subjects from understanding the conflict expressed in the context stories. In German schools, fractions are introduced in the beginning of grade 5 and computing with fractions is the main focus of mathematical instruction in grade 5 and grade 6. Experiment 4 is concerned with the way students from grades 5 and 6 are dealing with paradox problems.

Method

Subjects. Subjects were 164 (51% male) children participating in the longitudinal study LOGIC (Longitudinal Genesis of Individual Competencies) conducted at the Max Planck Institute for Psychological Research in Munich (Weinert & Schneider, 1985). When presented for the first time with the paradox problems, subjects were in fifth grade and their mean age was 11 years, 7 months.

When presented the last time with the paradoxical problems, subjects were in sixth grade and their mean age was 12 years, 7 months.

Measures. The paradox problems and the instructions were similar to those presented in Experiment 3.

Procedure. The longitudinal sample was investigated three times a year in 2-h sessions. The problems were presented together with other measures not discussed here by research assistants in individual testing sessions. When the subjects were in fifth grade, the pragmatic context problem but not the laboratory-like problem (it had not been developed at this time) was presented in the last third of the school year. When the subjects were in sixth grade, the pragmatic context problem was presented in the middle of the school year and the laboratory-like problem was presented three months later at the end of the school year.

Results

Children's answers were classified according to the category system described in Experiment 3 by two raters (94% agreement). Table 3 depicts the percentage

TABLE 3
Percentage of Subjects in the Pragmatic Condition in Grade 5 and the Pragmatic and the Laboratory-Like Conditions in Grade 6 Who Were Assigned to the Explanation Categories Described in Table 1

	Pragmatic grade 5	Pragmatic grade 6	Laboratory grade 6
Question 1: Will the container be empty once?			
1.1 Box will not become empty	56.9	74.4	86.6
1.2 Box will become empty	43.9	25.6	13.4
Question 2: How many glasses will be taken from the container?			
2.1 Less than 2 glasses, appropriate mathematical knowledge	9.1	47.0	59.1
2.2 Less than 2 glasses, inappropriate mathematical knowledge	3.8	2.5	17.4
2.3 More than 2 glasses	87.1	50.5	23.5
Question 3: Will there be an empty glass once?			
3.1 Saying that there will be always something in the glass by mentioning the conflict between practical and theoretical considerations	9.1	42.7	11.6
3.2 Saying that there will be always something in the glass by mentioning the principles of division	15.5	13.4	44.5
3.3 Saying that there will be a glass with nothing by referring to frequency of division	67.2	41.0	24.8
3.4 Saying that there will be a glass with nothing by mentioning the misconceptions of division	7.2	2.9	19.1
Overall Score			
M	1.2	1.7	1.3
s	.09	.13	.10

of answers assigned to the categories. Similar to Experiment 3, an overall score was developed. We decided not to compare the results of Experiments 3 and 4 statistically because the samples were not completely comparable. In Experiments 1–3, only children attending an after-school center were considered, which is more typical for middle- and lower-class children than for higher-class children. The longitudinal sample considered in Experiment 4, however, was representative of the Munich area; therefore, higher-class children were also considered. Moreover, in Experiments 1–3, the problems were presented in the familiar after-school center, while in Experiment 4 the problems were presented in the Max Planck Institute, which was visited only three times a year by the children.

In the analyses, we focused on two questions: (1) How do fifth- and sixth-graders deal with the paradox problems? (2) Do sixth graders differ in the way they treat the pragmatic and the laboratory-like problems?

The answers were analyzed in the same way as the answers of Experiment 3 and an overall score was developed for statistical analysis. The frequencies of answers assigned to each category and the mean overall scores are depicted in Table 3. A *t*-test revealed a significant difference between fifth- and sixth-graders in the overall score, $t(163) = 13.77$, $p < .0001$. The results indicate that most fifth-graders showed severe deficits when dealing with the paradox problems. They either treated the problem from a practical rather than from a mathematical point of view or showed the misconception of division also found in the previous experiments. In grade 6, however, these kinds of inappropriate treatment decreased considerably. Only a minority of the subjects showed misconceptions of division. The majority of the subjects treated the problem from a mathematical point of view.

A comparison between the two problem types within sixth grade revealed that the sum-score was significantly higher under the pragmatic context condition than under the laboratory-like condition, $t(193) = 2.37$, $p < .01$. The results depicted in Table 3 indicate that the higher value under the pragmatic condition was related to the frequency of answers assigned to Category 3.1 (mentioned conflict between practical and theoretical considerations). Under the pragmatic context condition, significantly more answers were assigned to this category than under the laboratory-like condition, $\chi^2 = 48.5$, $p < .0001$. This difference between the context conditions is of special interest. Subjects who say that there will always be something in the glass differ in their explanations, dependent on the story context. Under the laboratory-like condition, most of the subjects gave rather short explanations either by considering physical reasons, such as “If you divide mass you always have something,” or by considering mathematical reasons such as “Dividing by 2 can’t lead to zero.”

While under the laboratory-like condition, short explanations were typical, the pragmatic context condition encouraged deeper analysis of the problem context. Discussing the conflict between practical and theoretical considerations was usual. Typical explanations were: “If you always take half, the juice will disappear. There will be something, an atom or something smaller, but it is too small to drink it," or "First there is a sip, then a mini-sip, and then a mini-mini sip. This is practically nothing. But you can also divide a mini-mini-sip."

Under the pragmatic context condition a considerable number of subjects (19.3% of the whole sample) discussed the dual function of mathematical language, such as:

"Nothing means, it would be zero. However, you can't get zero by dividing a number by another number. Practically it will be zero, because after having divided the juice that often, there will be nothing."

"Yes and no. You can't drink it, because it will not get into your mouth. However, it is still something, and there is always a number for it."

"After, let's say dividing the juice for 16 times, it will be nothing you can drink. But, of course, at first you have a half, then a quarter, then one eighth, and so on. You can divide the numbers as long as you want."

Under the laboratory-like condition, only 3.9% of the subjects gave similar explanations.

Altogether, Experiment 4 showed that in fifth grade, although the subjects learned fractions at school, they made mistakes similar to those made by fourth-graders. Only in sixth grade was progress in mathematical understanding obtained. Moreover, the results showed that embedding problems referring to abstract mathematical concepts into a pragmatic context encourages sixth-graders to discuss the dual role of mathematical language. They understand that statements described with the help of mathematical language are not constrained by the same principles as language statements referring to natural events. In contrast, under the laboratory-like context condition, only very few subjects mentioned this conflict spontaneously.

GENERAL DISCUSSION

We have taken an exploratory first step toward understanding an issue that has been quite unexplored in the past. To determine what obstacles children must overcome before they can understand abstract mathematical concepts, we presented them with paradox problems. The mathematical solution to these problems yielded a result that contradicted practical requirements. Four experiments revealed different effects of paradox problems at different age levels. For students in fourth and fifth grades, the paradox problems revealed severe mathematical misconceptions that prevented children from more abstract mathematical understanding. In sixth grade, about half of the subjects addressed the conflict between practical and theoretical considerations.

Mathematical Misconceptions

Fourth- and fifth-graders, although they understand rational numbers such as "half," "quarter," and "third" as words and as fraction symbols, hardly ever mentioned the conflict between practical and theoretical considerations when being presented with paradox problems. In Experiment 1 it was shown that all fourth-graders understand that the mass of an apple cannot disappear by dividing it. However, the same children got stuck in a successive division task dealing with the very familiar context of money. The children did not apply the very common procedure of converting money to a smaller unit (i.e., pfennigs) but considered the larger unit (i.e., mark) to be the smallest possible quantity. Experiments 2–4 support the findings of Experiment 1. Children who knew about numbers smaller than “1” nevertheless thought that there would exist some smallest quantities and therefore they thought that half of a half, a quarter, or one-eighth would be zero because it would be smaller than that hypothetical quantity. In fourth and fifth grades, children’s understanding of rational numbers is still confined to a limited repertoire of fraction names, while the principles that guided understanding of natural numbers were transferred to rational numbers. This is even true for all fifth-graders although they had already practiced using fractions and decimals for more than half a year at school. As long as children do not understand the principles underlying rational numbers they cannot understand that these principles do not correspond to real-life situations. Therefore they are not able to recognize the conflict between practical and theoretical considerations which is caused by the dual function of the mathematical language.

Effects of the Situational Problem Context

All experiments suggest that the situational context in which the paradox problem is embedded has a strong impact on the accessed knowledge. Those problem stories which clearly suggested a mathematical solution revealed the misconceptions of division most clearly, particularly in fourth- and in fifth-graders. This was the case in Experiment 1, where the problem story contained numerical information, and under the laboratory-like context condition in Experiments 2–4, which was not that much connected to personal experience. If the context story, however, made contact with personal experience, like the pragmatic context story of Experiments 2–4, fourth- and fifth graders were inclined to treat the problem from a practical point of view and did not build a mathematical model. As long as children are impeded by misconceptions of division, a pragmatic context condition leads to a one-sided problem analysis. However, after these misconceptions have been overcome, which is the case for most children in the sixth grade, children are especially encouraged by the pragmatic context condition to discuss the conflict between practical and theoretical aspects of mathematics. They may even mention the dual function of the mathematical language. Under the pragmatic context condition, the conflict between practical requirement and mathematical principles appears all too obvious to be neglected.

REFERENCES

Baranes, R., Perry, M., & Stigler, J. W. (1989). Activation of a real-world knowledge in the solution of word problems. *Cognition and Instruction,* **6,** 287–318.

Carey, S. (1985). *Conceptual change in childhood.* Cambridge, MA: MIT Press.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology,* **3,** 21–29.

Davis-Dorsey, J., Ross, S. M., & Morrison, G. R. (1991). The role of rewording and context personalization in the solving of mathematical word problems. *Journal of Educational Psychology,* **83,** 61–68.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics,* **10,** 3–40.

Gelman, R. (1991). Epigenetic foundations of knowledge structures: Initial and transcendent constructions. In S. Carey, & R. Gelman (Eds.), *The epigenesis of mind: Essays on biology and cognition* (pp. 293–322). Hillsdale, NJ: Erlbaum.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Cambridge, MA: Harvard University Press.

Gelman, R., & Greeno, J. G. (1989). On the nature of competence: Principles for understanding in a domain. In L. B. Resnick (Ed.), *Knowing, learning, and instruction,* (pp. 125–186). Hillsdale, NJ: Erlbaum.

Hudson, T. (1983). Correspondences and numerical differences between disjoint sets. *Child Development,* **54,** 84–90.

Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science.* Cambridge, MA: Massachusetts Institute of Technology.

Keil, F. (1989). *Concepts, kinds, and cognitive development.* Cambridge, MA: MIT Press.

Lappan, G., & Montague-Wheeler, (1987). Children's understanding of zero and infinity. *Arithmetic Teacher,* **35,** 42–44.

Mevarech, Z. R., Snir, N., & Movshovitz-Hadar, N. (1992). Is more always better? The separate and combined effects of a computer and video programme on mathematics learning. *British Journal of Educational Psychology,* **62,** 106–116.

Piaget, J. (1953). *The child's concept of number.* London: Routledge & Kegan Paul.

Resnick, L. B. (1986). The development of mathematical intuition. In M. Perlmutter (Ed.), *The Minnesota Symposia on Child Psychology,* Vol. 19, *Perspectives on intellectual development* (pp. 159–194). Hillsdale, NJ: Erlbaum.

Resnick, L. B., Cauzinille-Marmèche, E., & Mathieu, J. (1987). Understanding algebra. In J. A. Sloboda, & D. Rogers (Eds.), *Cognitive processes in mathematics* (pp. 169–203). Oxford, UK: Oxford University Press.

Stern, E., & Mevarech, Z. R. (1991). *When children do and do not access familiar knowledge in understanding fractions* (Res. Rep. No. 4/1991). München: Max-Planck-Institut für psychologische Forschung.

Stern, E., & Lehrndorfer, A. (1992). The role of situational context in solving word problems. *Cognitive Development,* **7,** 259–268.

Weinert, F. E., & Schneider, W., (Eds.) (1986). *First report on the Munich longitudinal study on the genesis of individual competencies (LOGIC).* München: Max-Planck-Institut für psychologische Forschung.

Wynn, K. (1990). Children's understanding of counting. *Cognition,* **36,** 155–193.

Wynn, K. (1992). Addition and subtraction by human infants. *Nature,* **358,** 749–750.

Vygotsky, L. S. (1962). *Thought and language.* Cambridge, MA: MIT Press.

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