CHAPTER 6

ABSTRACT REASONING WITH MATHEMATICAL CONSTRUCTS

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Abstract

Mathematical constructs have a dual role because they can be used as instruments to model real world situations and events, but they can also become an object of reasoning. Mathematics is a particularly abstract domain because the affordances and constraints underlying the use of mathematical constructs may be different from the affordances and constraints in real-world situations. We argue that this makes the acquisition of quantitative schemata a difficult task but also accounts for the potential to extend our understanding of the world by mathematical means. We refer to developmental, educational, and experimental studies supporting the view that new understandings and powerful ways of reasoning become possible on the basis of culturally mediated mathematical constructs.

Although students of all age levels often experience learning mathematics as an end in itself, mathematical symbols are nevertheless often used to describe and predict states and events highly relevant in everyday life. Money transactions require an understanding of the situation in terms of quantification. Understanding the payment of a loan for a house requires highly sophisticated modeling with mathematical symbols. The technical development in our century would have been inconceivable without mathematics. In psychology, for example, the psychometric concept of intelligence is based on the mathematical concepts of mean, variance, and normal distribution.

Schema theories are considered to describe the knowledge underlying such reasoning processes. Schema models generally share characteristics (Rumelhart & Norman, 1985; Thorndyke, 1984). Schemata are thought to be active recognition devices and structures that control the use of concepts and the production of actions. They can be embedded hierarchically within another schema and consist of stable parts and variables whose instantiations are constrained by conditions. If variable slots cannot be filled with available information, default values allow for predictions. Specific schema theories have been developed to model the cognitive structures that guide quantitative reasoning in elementary mathematics, such as the knowledge underlying understanding and solving of arithmetic and algebra word problems (e.g., Kintsch & Greeno, 1985; Reed, 1993; Reusser, 1990; Riley & Greeno, 1988).

Research has shown that even preschool children can be attributed with having addition and subtraction schemata to a certain extent. Long before children can reason numerically, they are able to draw non-numeric inferences about quantities of physical objects (e.g., making comparisons such as *more*). These inferences are based on what Resnick (1989, 1992) termed protoquantitative schema. Only through schooling, however, do children gradually develop quantitative schemata that enable them to flexibly use numbers and equations in a broad variety of situations. With respect to mathematics beyond the elementary level, Ohlsson (1993) states that educated people operate with abstract quantitative schemata that include at least function, correlation, distribution, matrix, and three dimensional space.

Generality: The Aristotelian View of Abstract Schemata

Most people would agree that mathematics is more abstract than geography. The concepts in the domain of geography refer to real things, such as rivers and volcanoes, and to concrete and perceptible events, such as floods. In contrast, the "objects" dealt with in mathematics are symbols that do not refer to specific objects or events in the real world. Such differences between content domains may correspond to differences in the respective knowledge structures: Representing and processing knowledge may be more abstract or require more abstraction in the domain of mathematics than in the domain of geography. However, any communication about the world by means of some language involves abstraction. For example, the relationship between the statement "The River Rhine is contaminated" and the River Rhine, as it exists in the real world, is abstract in the sense that the verbally denoted fact does not exhaustively describe or capture the "object" to which it refers. In addition to contamination, there are innumerable other features that characterize the River Rhine. Any representation of real world situations by means of symbols is general and selective.

Abstraction is involved in similar ways when describing situations in terms of quantitative aspects. Consider, for example, the following word problem. "John has already caught 5 trout. The cook has asked him to bring a total of 12 trout. How many trout does John still need to catch?" This situation can be modeled by the equation $12-5=?$ The equation retains only the specific quantitative information of the situation asked for in the question and discards everything else. The very same equation is also an appropriate formal model for the following situation: "When Rachel came back from the beach, she had 12 pebbles. This morning she gave 5 beautiful pebbles to her little brother. How many pebbles did Rachel keep?" An equation does not display features of the modeled situation, other than the quantitative relationship on which its derivation focused. This provides for the generality of pure mathematical notations: The same equation is potentially useful for an indefinite number of real-world situations in which the focus is set on identical quantitative relationships. An even wider range of applications of a formula is possible by using variables instead of specific numbers, for example, $a+b=c$, where a and b are part-sets, and c is the whole-set.

Most content domains presented in school settings deal with more or less explicitly defined concepts represented in language rather than with the real objects of the domain. In geography, the River Rhine constitutes an exemplary object that can be referred to and analyzed. Even particular research questions focusing on the River Rhine are embedded within the abstract concept of river. Comparing generic statements in geography, such as, "Rivers leaving densely populated agglomerations tend to be contaminated," with generic statements expressed in mathematical language, such as the second law of Newton in physics, may lead to the conclusion that there is a gradual difference in the degree of abstractness. Generic mathematical statements allow for a wider application than generic statements in geography. The mathematical procedures used to figure out the total sum resulting from interest and compound interest can also be used to figure out the final speed given a particular degree of acceleration. The multiplicative relationship expressed in the second law of Newton corresponds to numerous multiplicative relationships, for example, intensive quantities such as Area=width × length.

The characterization of abstraction thus far provided reflects the aspect of generality in terms of reference. When considering the hierarchical structure of concepts, *animal* is a more general concept than *dog*, which is again more general than *Doberman*. The higher the degree of abstractness the fewer features of a concept are specified and the larger is the number of particular instances that constitute the extension of the concept. The degree of generality with respect to the range of (potential) real world references is one aspect of the abstract nature of quantitative schemata; but, as is argued below, generality is not the core aspect that makes mathematical constructs especially abstract.

The Dual Meaning of Mathematical Constructs

In recent years, research has shown that mathematics is a privileged domain in the sense that humans are biologically prepared for processing quantitative information. Within the first four years of life, most children learn to count sets containing up to ten objects without being given systematic instruction, and they know how to add and subtract small numbers (see Wynn, 1992, for an overview). The languages of all known human cultures, including those that are not literate, have an oral designation for the size of sets containing up to 20 objects (Damerow, 1988). However, although humans are endowed with sensibility for cardinal aspects of quantification, school mathematics is difficult to learn and hard to teach. The main reason might be that advanced mathematics, that is, reasoning within symbol systems, is a product of cultural development. Not all cultures have developed specific written signs to denote numbers and the existing symbol systems differ considerably in their underlying principles. Not all number systems developed throughout history allowed for an infinite extension of numbers or the use of non-natural numbers. Only around 800 AD, when a symbol for zero was transferred from India via the Arab countries to Europe, did an infinite extension of numbers become possible (Ifrah, 1985). Describing fractions as decimal numbers was not invented until the sixteenth century.

Mathematical inventions that took mankind many centuries to complete can be grasped and further developed by members of the following generations only through particular efforts of schooling. To acquire advanced mathematical competencies, a fundamental change in the understanding of numerical symbol systems is necessary: Not only is mathematical language a way of communicating about quantitative aspects of the world, but also the mathematical symbols themselves become objects of reasoning and communication. The dual role of formal mathematical language is emphasized by Resnick, Cauzinille-Marmeche, and Mathieu (1987): In its role as "signifier," mathematical language is used as instrument of reasoning to describe real-world situations; in its role as "signified," it is itself the object of reasoning. According to Ohlsson (1988)

> (...) there are two factors that imbue mathematical constructs with meaning. First, a construct acquires meaning from the mathematical theory in which it appears. The axioms and theorems of the theory function as meaning postulates that specify the mathematical meaning of the construct. Second, a construct acquires meaning from its applications in the real world (p. 61).

In the following sections it is argued that consequences resulting from the signified meaning of mathematical symbols are at the core of what makes this content domain particularly abstract.

Affordances and Constraints in Mathematical Reasoning

According to the theory of ecological perception developed by Gibson (1979), actions can be understood as interaction between an individual's capabilities and his environment. In environments, some objects provide the affordance to be moved and grasped, whereas others serve as a base to walk over, and so on. Development can thus be characterized as the process of discovering and learning to use new affordances provided by the environment. The reverse side of affordances is constraints (Greeno, 1995). The external world not only provides the resources for actions but restricts the kinds of actions that are possible in the empirical world. Elephants cannot be cuddled as cats can, and because of the gravitational pull of the earth, objects thrown in the sky will fall down. Greeno, Smith, and Moore (1993) suggest that the view of ecological cognition is also useful to conceptualize reasoning based on representational tools, for example, symbols. Learning to operate with mathematical symbols requires learning to perform transformational activities afforded by the respective numerical system and following the constraints set by the syntax of the corresponding notation. For example, when adding or multiplying numbers, their order does not need to be considered, whereas subtracting or dividing numbers is constrained by the necessity of keeping the order.

Within a specified number-symbol system, new constructs may be defined without the new constructs refering to the empirical world. For instance, the concept of prime numbers is a purely mathematical concept that is only meaningful in the context of dividing whole numbers. Although prime numbers do not have a reference in the external world, they are based on rules that allow the identification of yet unknown entities of this category of numbers. Place-value systems of numbers allow for the endless enlargement of numbers without ever reaching a limit. The restricted applicability of certain mathematical operations within culturally transmitted number systems led to the invention of new number systems with expanded affordances of mathematical reasoning. Subtracting a larger number from a smaller one is not defined in the system of natural numbers; therefore, the system of integers was invented, which includes negative numbers. Only in a few cases does dividing a whole number by another whole number result in another whole number. In order to allow for all kinds of divisions, rational numbers had to be invented. Because rational numbers allow infinitely repeated division without ever reaching zero, they provide the basis for developing concepts of infinity, infinite divisions, and limit.

Abstraction Beyond Direct Modeling: Exploiting Affordances of Formal Representational Systems

Although mathematical constructs can be treated in ways not possible with entities in the physical world, they can serve to model aspects of the real world. What makes mathematics particularly abstract is not its generality with respect to reference but rather that principles underlying the use of mathematical constructs may be different from principles underlying the affordances and constraints in real-life situations. These differences allow for the power of mathematical reasoning, but they also have consequences with respect to the acquisition of flexible schemata. In the following sections three perspectives that are pivotal for an understanding of the nature of reasoning with mathematical constructs and its importance for understanding and modeling aspects of the world are discussed.

Substitution of Actions Through Numerical Reasoning

Actions that manipulate and relate quantified objects can be fully substituted through quantitative reasoning. Physical tools, such as measuring units of 1 cm in length, can be used to determine the length of a particular stick through direct modeling: The stick's physical extension is matched with the necessary measuring units and the number of units is counted. Of course, using an embodied mathematical tool such as a ruler makes a measuring task much easier. If the goal is to determine the length of two sticks put together, one 172 cm long and the other 122 cm long, one way of computing is 100+100+72+22. Given that a 100 cm ruler is available, a corresponding action in the physical world could be to mark the two sticks into pieces of 100 cm and 72 cm and of 100 cm and 22 cm, respectively, and then to perform the corresponding additions. Given that the lengths of the two sticks are already known, however, performing this action in the concrete world is not necessary to figure out the combined length of the sticks. Numbers can be manipulated according to principles of the base-ten system. The mathematical computation described is abstract because the numbers referring to the length of the tangible sticks are treated in ways in which the physical objects themselves do not need be treated. Based on the principle of additive decomposition, the decimal number system affords many different strategies of how to determine mentally the sum of two given quantities. Most of these strategies are more efficient than using (non-electronic) physical tools. With large numbers, the economical advantage of computations exploiting the base-ten system in flexible ways becomes even more obvious.

Correspondence and Differences Between Principles Underlying Actions and Principles Underlying Numeric Reasoning

Some principles underlying actions of combining and exchanging sets of objects correspond to fundamental principles underlying elementary arithmetic operations. For instance, associativity and commutativity underlie the addition of numbers as well as many concrete actions. The fact that all numbers are additive compositions of other numbers not only provides the conceptual basis of much of elementary arithmetic, but also relates fundamental properties of the number system to an intuitive basis deeply rooted in knowledge acquired from actions in the everyday world (Resnick, 1992). These correspondences are related to the epistemological questions of why actions with quantified objects can be modeled through purely numerical reasoning.

In other important ways, however, principles underlying concrete actions with quantified objects differ from purely quantitative reasoning. In mathematical operations such as addition and multiplication, for example, the principle of commutativity has to be followed in all circumstances, but it also affords flexibility with respect to the specific computing strategy used. In contrast, in the world of concrete actions, the specific goals determine whether commutativity does in fact also constitute a degree of freedom in the way the sub-actions are to be hierarchically structured. For example, with respect to the goal of having "two covers around my foot," it is irrelevant whether I first put on the shoes and then the stockings, or the other way around. With respect to comfort and the goal of keeping warm feet, however, it is definitively not irrelevant. Concrete actions always depend on affordances and constraints of external and internal resources as well as on the intentions of the actors involved. Someone who has learned to operate flexibly within a number system can do so without further consideration of external constraints.

Mathematical Constructs: A Basis to Extend our Understanding of the World

The application of mathematical constructs to concrete situations and events allows us to capture structural relations that are impossible to describe effectively otherwise. For instance, we frequently talk about the price of a pound of tomatoes, even though, because of the varying size and weight of tomatoes, a combination of tomatoes rarely results in a weight of exactly one pound. Because we have an understanding of quantifying weight, we have no problems understanding the price of one pound of tomatoes. Intentionally transferring the constraints and affordances underlying the use of mathematical constructs to other content domains allows us to exploit familiar mathematical structures in construcing models and reasoning schemata that not only make communication more efficient but also may extend and deepen the understanding of the domain.

Time and distance, for example, are two concepts referring to independent dimensions of the experienced world. However, based on methods of measurement, numbers can be matched to each dimension; and by using the rules of division to relate these two variables, the concept of *speed* is constructed. If the goal is to measure the speed of a vehicle, and the unit of measurement is kilometers per hour, it is not necessary to measure the distance after driving exactly one hour. The principle of equivalence (which means that the relation between the denominator and numerator of a fraction is not affected by multiplicative operations) provides infinite possibilities of expressing the same rational number and, therefore, allows for the measuring of constant speed in all conceivable situations.

Percent and *rate* provide further examples of quantitative schemata developed from rational numbers, which can be used to model a huge variety of events and situations in the real world. The German Chancellor, Helmut Kohl, well known for his stylistic howlers, promised: "By 1998 we will halve unemployment." A satiric comment asked how he would halve unemployment: "vertically, horizontally, or like an apple?" Despite the joke, we all know that Chancellor Helmut Kohl intended to halve the unemployment rate, that is, the percentage of people of legal working age who have no job but are looking for one. By building on mathematical constructs, the meaning of a verb such as *to halve*, which in its original meaning only refers to actions with physical objects, can be extended to complex conceptual entities.

In a very profound way, the mathematical concept of zero has extended our understanding of the world. In everyday language, *zero* and *nothing* are often used as synonyms. However, the word *nothing* is used to describe a status rather than to describe an action. Although *nothing* cannot be added or subtracted, zero can be treated like other numbers according to algebraic rules. Saying "there is nothing" makes sense only if the referent of *nothing* is clear. *Nothing*, *no*, or *nobody* used in everyday language means that one can neglect the entity. However, neglecting zero in the base-ten system leads to severe mistakes. Once a concept of zero has been established, it can be used in the sense of the empty set and thus restructures the meaning of the word *no* and *nothing* in natural language.

Concepts such as *infinity* and *limit* have no direct reference in our personal experience. Nonetheless, making use of these mathematical constructs may be helpful when reasoning within a content domain. For instance, the concept of infinity developed in dealing with natural numbers may be helpful in trying to grasp the endlessness of the universe. The concept of repeated divisions provides an understanding of the concept of the half-life of radioactive elements.

The concept of psychometric intelligence allows for quantification of the capacity underlying the outcome in intelligence tests, even though the mental capacity itself cannot be quantified. The number of correctly solved items on an intelligence test allows conclusions based on an ordinal scale only. However, by presupposing normal distribution in performance on intelligence tests, a standardized IQ test allows intelligence to be quantified on an interval-scale. The construct of normal distribution, however, did not arise from induction based on empirical observations; rather, empirical observations were adapted to the abstract mathematical concept of normal distribution.

Acquisition and Development of Quantitative Schemata

In the previous sections, it was argued that cultural progress in mathematics prepared the ground for developing many abstract concepts and reasoning schemata. The integration of quantitative schemata in content domains transformed these into more abstract domains. The question of cultural development, however, has to be distinguished from the question of how individuals acquire and develop abstract reasoning schemata. In the following sections two views on how individuals acquire and develop quantitative schemata are differentiated: From everyday situations to formal mathematics and from formal mathematics to new understandings of situations. Since in the developmental literature, the first view is prevalent, the second view, shall be dealt with more extensively. This reflects the belief — related to our understanding of the nature of reasoning with mathematical constructs presented in this chapter — that future research should focus more on how culturally mediated formal representations affect the development of mathematical cognition.

Building on Intuitive Knowledge: From Everyday Situations to Formal Mathematics

Among developmental psychologists, the view that abstract knowledge structures arise from more concrete knowledge related to actions and perceptions is prevalent. Despite theoretical differences, for example, Piaget and Bruner share the belief that conceptual understanding develops from the concrete to the more abstract. For the domain of mathematical cognition, Resnick has developed such a view in great detail (Resnick, 1989, 1992, 1994; Resnick & Singer, 1993), arguing that the foundations of mathematical cognition are based on universally developed protoquantitative schemata emerging from innate preparedness and interaction with the everyday world. Protoquantitative reasoning is non-numerical reasoning about relationships between sets: for example, by concluding that "As daddy-bear is larger than mummy-bear, the bed of daddy-bear has to be larger than the bed of mummy-bear."

Long before children receive systematic instruction in school, they can compare sets at least qualitatively by stating which one is bigger; they can reason about increases and decreases of sets; and they possess a protoquantative part-whole schema that allows them to understand that quantities can be composed of each other. As children integrate their protoquantitative reasoning with the separately developed competency to count objects, they begin to operate with quantitative schemata. Finally, given that children encounter situations that encourage participation in corresponding discourse, children also learn to reason about numbers and operators without immediate reference to actions and physical quantities. In discourse about numbers and their relations, numbers obtain the status of purely conceptual entities.

Acquiring Symbolic Schemata: From Formal Mathematics to New Understandings of Situations

In contrast to the view that mathematical schemata arise from reasoning in and about concrete situations involving quantification, there is an alternative view. In order for formal mathematical knowledge to develop, it is necessary to introduce mathematical concepts and ways of reasoning through more knowledgeable others. These two views are not necessarily mutually exclusive (Resnick, 1992, 1994).

Symbolic Reasoning in Young Children

The learning of principles underlying numerical reasoning must not be a conscious process that immediately leads to metalinguistically accessible knowledge. It can be suggested, however, that the development of purely mathematical principles is not so much a question of age, but rather a question of opportunities to use mathematical symbol systems. Even preschool children can acquire purely mathematical constructs from reasoning with and about mathematical symbols.

Infinity

The affordance of number systems, such as the base-ten system, to be endlessly extended cannot be understood based on an analogy to the concrete world. Based on personal experience, all activities have an end. Stern (1994b) has shown that some preschool children and most first graders can explain why there is no biggest number: "Because if you have a very big number, you can always add 1 to this number." Similar results are reported by Gelman and Evans (1981).

Zero

For an understanding of the base-ten system, one has to understand that zero is a number that underlies the same constraints and affordances as other numbers. Although preschoolers very often interpret zero as nothing, after a few months of mathematical instruction in school, children learn to treat zero as a number that can be used in mathematical operations such as 4+0=4 (Wellman & Miller, 1986). Understanding the concept of zero requires familiarity with a mathematical symbol system, which goes beyond personal experience with situations referring to nothing.

Early understanding of fractions

Children's initial concept of integers is based on the idea that one gets numbers when one counts entities. When learning about rational numbers, most children have the tendency to generalize inappropriately what they learned about integers to fractions. Gelman (1994) found that few preschool children and elementary school children nonetheless did well in interpreting fractions. Detailed analyses suggested that these children showed special abilities in working correctly with the symbolic representations of the entities, operations, and principles of arithmetic.

Infinitely repeated division

How do children come to understand that endlessly dividing a number will always result in a number very close to zero but not exactly zero? Everyday experience shows that successively dividing an object or liquid will cause it to lose its function; from a practical point of view, it will disappear. Stern and Mevarech (1996) showed that children's understanding of infinite divisions was closely related to symbolic reasoning. In Grade 6, most of the children understood the denominator of fractions as a divisor and applied their knowledge about infinity of numbers: By increasing the denominator, the value of the fraction will decrease, but it will never reach zero.

Fostering Knowledge for Mathematization by Focusing on Mathematical Principles

The following results suggest that reasoning about basic principles underlying the use of numerical systems fosters the ability to mathematize situations, which requires one to go beyond direct modeling of actions with quantified objects.

Quantitative comparison

Young children's difficulties with arithmetic word problems dealing with the comparison of sets is well documented (Cummins, Kintsch, Reusser, & Weimer, 1988; Riley & Greeno, 1988). The problem, "John has 5 marbles. Peter has 8 marbles," was easily solved by all first graders when it ended with the question "How many marbles must John get in order to have the same amount of marbles as Peter?". In contrast, only about 30% of the same subjects solved the problem when it ended with the question "How many fewer marbles does John have than Peter?"

(Riley, Greeno, & Heller, 1983). It has been emphasized that the two problem versions differ not only in the linguistic form of the question, but also in being confounded in the denoted underlying situation model (Staub & Reusser, 1995; Stern & Lehrndorfer, 1992; Stern, 1993).

The first question, an equalize problem, induces an action-related representation of the relationship between the two sets, and the mentioned numbers can be understood in a cardinal sense. The second question refers only to an abstract static situation, with a problem situation model that consists of nothing but two disjoint sets of entities. To solve compare problems of this second type, one must represent and relate the quantities based on a part-whole schema. A training study (Stern, 1994a) revealed that teaching children to reformulate comparison problems into equalize problems improved performance in solving some types of one-step problems, but not in solving more complex problems. However, two studies showed that familiarizing children with mathematical principles, such as the inverse relationship between addition and subtraction, improved children's performance on comparison problems considerably (Renkl & Stern, 1994; Stern, 1994a). According to results from Lewis (1989), teaching children to mark the extension of sets on the number line — which is a purely mathematical construct — especially enhances performance on comparison problems.

Speed

The acquisition of the formal concept of speed provides difficulties because it is an intensive quantity. Levin and Gardosh (1994) investigated under what conditions the everyday concept of speed was transformed into a formal concept and the formal concept was used to predict linear as well as rotational speed. The authors concluded:

> Although children relate distance and time when considering speed even without learning to do so in school, they need the school's input to grasp that speed is measured by a mathematical relation (multiplication/division) between them (p. 198).

The subjects were presented with a problem that required the transformation of rotational speed into linear speed or vice versa. The results showed that, although most subjects were never taught about angular speed, most schooled subjects produced the relevant formula when faced with problems based on rotational speed.

Statistical rules

Fong, Krantz, and Nisbett (1986) investigated training effects on the use of statistical rules in real-world problems related to questions of probability. They found that training covering formal aspects of the law of large numbers produced substantial effects on the frequency and the quality of statistical answers. The results indicate that statistical solutions to problems can be made more likely through teaching the abstract constructs. The law of large numbers is a mathematical construct that is not based on personal experience, but it can be used to model potential outcomes of concrete situations.

Meaningful Mathematics Beyond Everyday Intuition

Mathematics teaching in schools has been severely criticized in the past decades. Too often, students acquired skills in manipulating symbol systems without learning to use mathematical constructs as instruments of reasoning in contexts of applied and realistic problem solving. From an educational point of view, quantitative schemata should play a role in two worlds: in the realm of the numeric system as well as in real-world situations. Knowledge that enables computations only without providing the basis to model real-world situations remains "blind" and vacuous.

Researchers and teachers have contributed much toward enhancing children's understanding of mathematics by making its developmental roots in infancy and early childhood explicit and by demonstrating the relevance of quantification in real-life contexts. The emphasis in this chapter on acquiring symbol systems and principles related to their use is not intended to diminish efforts to make mathematics meaningful by building on intuitive prior knowledge. Rather, learning to reason with culturally mediated mathematical constructs, which allow people to go beyond prior ways of understanding situations, may initiate cognitive development in two ways: first, with respect to new understandings of the situations in which the constructs are introduced, and second, as cognitive tools that may generate new understandings in other situations over the whole life span.

Research in many domains has shown that only through specific instructional efforts is knowledge learned in one situation utilized in another situation. Individuals at all age levels cannot be expected to spontaneously use specific symbolic representations to solve problems, unless they have developed situation specific habits of representing and reasoning in such ways or have the clear intention, for whatever reason, to tackle a problem by using specific formal means. In the previous section, training studies from Fong et al. (1986) and Stern (1994a) showing that abstract rule training improved content-specific reasoning were cited. However, this is only half the truth. In both studies, best performance was achieved by combining abstract rule training with worked-out examples of application.

With respect to the power of mathematization, it is important to note that attempts to model aspects of the world by means of mathematical constructs do not necessarily extend and deepen our understanding. With the example of the psychometric concept of intelligence, the construct is problematic. The power of mathematics should not distract from the fact that inferences derived within a particular numerical system are only valid as part of a model of an intended domain if there is a method of measurement allowing us to represent relevant empirical distinctions in the content domain through a homomorphic relation with numbers in the numerical system. Where easily definable and publicly accessible objects of some sort are to be identified and counted, this seems trivial. But, in cases where aspects of mental abilities are to be mathematized, such questions become critical and challenging.

Learning and discovering patterns and principles by reasoning with purely mathematical constructs can be fascinating. Learning in which situations which mathematical constructs can be used to extend our understanding of relevant aspects of the world may be even more exciting.

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