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der Versuchsanstalt für Wasserbau,
Hydrologie und Glaziologie
an der Eidgenössischen Technischen Hochschule Zürich
Herausgegeben von Prof. Dr. D. Vischer

On the fundamental equations of floating ice

Kolumban Hutter

Zurich, June 1973

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Preface

In the past, the analyses of floating ice plates were based on the theory of elastic plates under the assumption of an elastic foundation. The dynamic interaction with the sublaying flowing and standing water is only seldom accounted for. Furthermore, nonuniform distribution of the temperature, thermal stresses and viscoelastic behavior of the ice are usually not treated.

It therefore seemed to be appropriate to reexamine the theme "floating ice-plates" in the sense of a basic study. The aim was to derive the governing dynamic equations, which would furnish the basic tools to a broad variety of physical questions. The present report therefore emphasizes clarity in the exposition of the basic assumptions, however, nevertheless exemplifies the calculations to such an extent that questions motivated by practice oriented engineers and geophysicists may easily be attacked. It is our intention to treat such problems in a further report.

Professor Dr. D. Vischer

Abstract

In the past, the analyses of floating ice plates were based on the theory of thin homogeneous elastic plates on elastic foundation. The dynamic behavior was only rarely investigated and more careful investigations, which would examine the reliability of the governing equations, are generally lacking. In this report therefore, an attempt is made, to examine the entire matter in its essentials and to present it in utmost generality, however with the intention of later practical applicability.

Chapter 1 reviews the physical problem and, very briefly, also touches the mathematical methods used. The important literature which has had considerable impact on the approach used here is quoted. It has been tried, to the authors best knowledge to mention all important research articles, however completeness is not intended.

In Chapter 2 the governing equations for the fluid layer bearing the ice plate are developed. Many of the results in this Chapter are known; in the method used here, which is based upon a regular perturbation approach, however, special emphasis is observed with regard to the boundary conditions valid at the ice water interface. These are of particular importance if the interaction of ice and water is taken into account.

In Chapter 3 and all the subsequent derivations, essentially new results are obtained. Based on a three dimensional theory of nonlinear elasticity, starting from thermodynamic considerations, various plate theories are developed for the case that the temperature distribution in the ice plate may be nonuniform . Apart from deformations

in the plane of the plate, also various bending theories are obtained, such as what could be called a generalized Reissner and von Kàrmàn theory. Shear deformation and rotatory inertia and the nonuniformity of the temperature field are included in the theory. The well known Kirchhoff-Love hypothesis, according to which line elements (directors) perpendicular to the undeformed middle plane maintain this orthogonality during the course of deformation, is abrogated. Accordingly, in these plate theories all boundary conditions can be satisfied without encountering the boundary singularities of the Kirchhoff-Love theory.

In Chapter 4, finally, the viscoelastic plate theories, analogous to the ones presented in Chapter 3 are developed, under the same assumption of nonuniform temperature distribution. Based on methods used in the theory of high polymers, the viscoelastic plate theories are developed by assuming that ice satisfies the postulates of thermorheologically simple materials.

Practical results will be dealt with in a later report.

Zusammenfassung

Die Berechnung schwimmender Eisplatten ist bis anhin mittels der Theorie homogener elastischer Platten auf elastischer Unterlage erfolgt. Nur selten wird auf dynamisches Verhalten eingegangen und eingehendere Untersuchungen, welche die Zuverlässigkeit der verwendeten Gleichungen überprüfen, fehlen fast vollständig. Im vorstehenden Bericht wird daher der Versuch unternommen, den ganzen Fragenkomplex von Grund auf zu überprüfen und in möglichster Allgemeinheit, jedoch mit der Absicht späterer praktischer Verwendung, darzustellen.

Kapitel 1 gibt eine allgemeine Uebersicht über die physikalische Problemstellung und zeigt in Stichworten die verwendeten mathematischen Methoden auf. Soweit dem Autor bekannt, wird auf die vorhandene Literatur, welche im betreffenden Abschnitt von Bedeutung ist, eingegangen. Anspruch auf Vollständigkeit wird selbstverständlich nicht erhoben.

Kapitel 2 behandelt die Grundgleichungen der dem Eis unterliegenden Flüssigkeit. Die meisten der in diesem Kapitel hergeleiteten Resultate sind bekannt, die hier dargestellte Methode, welche von der Störungsrechnung Gebrauch macht, nimmt jedoch speziell Bezug auf die an der Eis - Wasser Grenze vorhandenen Randbedingungen welche im Rahmen einer dynamischen Interaktion von Eis-Wasser von Wichtigkeit sind.

Kapitel 3 und alle folgenden Untersuchungen hingegen enthalten wesentlich neue Resultate. Es werden, von der 3 dimensionalen Elastizitätstheorie ausgehend, mit den Mitteln der Thermodynamik, verschiedene Plattentheorien hergeleitet für den für schwimmendes Eis wichtigen Fall, dass das Temperaturfeld im Eis nicht uniform verteilt ist. Neben einer Scheibentheorie wird auch eine verallgemeinerte Reissner- und von Kàrmàn-Theorie entwikkelt. Scherverformungen und Rotationsträgheit, wie auch die durch das

Temperaturfeld erzeugte Inhomogenität sind also berücksichtigt in dieser Theorie. Die bekannte Kirchhoff - Lovesche Hypothese, gemäss welcher materielle Linienelemente, welche senkrecht auf der undeformierten Mittelebene der Platte stehen, unter Verformung senkrecht auf dieser (verformten) Ebene bleiben, wird daher nicht getroffen. Dementsprechend können in dieser Plattentheorie alle Randbedingungen ohne die in der Kirchhoffschen Theorie bekannten Randsingularitäten erfüllt werden.

Kapitel 4 gibt schliesslich die Herleitung der im Kapitel 3 aufgestellten elastischen Plattentheorie für den viskoelastischen Fall. Aufgrund von Methoden der Polymerphysik werden die dem Kapitel 3 analogen Theorien viskoelastischer Platten mit nicht uniformem Temperaturfeld hergeleitet, indem angenommen wird, dass Eis die Voraussetzungen thermorheologisch einfacher Materialien erfülle.

Auf praktische Probleme wird in einer weiteren Mitteilung eingegangen.

Résumé

Le calcul des plaques de glace flottantes se faisait jusqu'a présent à l'aide de la théorie des plaques élastiques homogènes reposant sur un milieu élastique. On ne s'est intéressé que rarement a leur comportement dynamique et les expériences qui confirmeraient la validité des équations utilisées n'ont pratiquement jamais été réalisées. L'étude faisant l'objet du présent rapport s'est attachée a vérifier depuis la base, un vaste ensemble de questions et a le décrire le plus généralement possible sans perdre de vue les applications pratiques .

Le chapitre 1 donne une vue générale de l'aspect physique du problème et décrit brièvement la méthode mathématique utilisée. L'auteur y mentionne toutes les références bibliographiques dont il a connaissance et s'étend sur les ouvrages qui ont leur place dans ce chapitre. La bibliographie citée n'a cependant pas la prétention d'être complete.

Le chapitre 2 traite les équations fondamentales du liquide sur lequel repose la glace. La plupart des résultats auquels on aboutit dans ce chapitre sont connus. Toutefois la méthode décrite ici qui est basée sur la méthode des perturbations, tient compte spécialement des conditions aux limites existant sur la surface de contact glace-eau car celles-ci jouent un role important lors d'une interaction dynamique glace-eau.

Le chapitre 3 et toutes les déductions qui suivent contiennent en revanche des résultats essentiellement nouveaux. A partir de la théorie de l'élasticité à 3 dimensions et en utilisant les moyens de la thermodynamique, différentes théories des plaques ont pu être établies dans le cas important pour la glace flottante où le champ de températures n'est pas réparti uniformément dans la glace. Parallelement a une théorie

des parois, une théorie de Reissner et von Kàrmàn généralisée est aussi développée. Les déformations de cisaillement ainsi que l'inertie de rotation et l'inhomogénéité engendrée par le champ de températures sont également prises en considération dans cette théorie. On a pu laisser de côté l'hypothèse bien connue de Kirchhoff-Love d'après laquelle les éléments matériels unidimensionnels situés perpendiculairement au plan moyen non-déformé de la plaque restent dans cette position après la déformation de celle-ci. Ainsi dans cette théorie des plaques, toutes les conditions aux limites peuvent être remplies sans qu'apparaîssent les singularités aux limites qui interviennent dans la théorie de Kirchhoff.

Le chapitre 4 fait finalement apparaître des résultats analogues à ceux du chapitre 3 obtenus pour les plaques élastiques, mais trouvés cette fois pour le cas viscoélastique. A l'aide de méthodes utilisées par la physique des polymères, des théories analogues à celles du chapitre 3 sont élaborées; elles sont adaptées aux plaques viscoélastiques dont le champ de températures n'est pas uniforme. Pour cela on admet que la glace remplit les conditions thermorhéologiques des matériaux simples.

Une publication ultérieure traitera les problemes pratiques.

Обобщение

Расчеты плавучих полос льда проводились до сих пор с помощью теории однородных упругих плит на упругой подложке. Вопросы динамического поведения освещались только изредка и более подробные исследования, проверяющие правильность применяемых уравнений почти совершенно отсутствукт. В представленном сообщении предпринята поэтому попытка полностью проверить весь комплекс вопросов и представить его по мере возможности в его совокупности, не упуская однако из виду последующее практическое применение.

Глава 1 дает общий обзор физической проблематики и предельно кратко излагает примененные математические методы. Упоминается существующая литература, имеющая значение для этого раздела, поскольку она известна автору, который, разумеется, не выдвигает притязаний на исчерпывающий список трудов.

Глава 2 рассматривает основные уравнения расположенной под льдом жидкости. Большинство выведенных в этой главе результатов уже известны. Описанный метод, который положен в основу метода возмущений, особо ссылается на краевые условия, существующие на грани лед-вода, которые имеют значение в рамках динамического взаимодействия льда и воды.

Глава 3 и все последующие исследования содержат, напротив, существенно новые результаты. Исходя из трехмерной теории размерности теории упругости, и с помощью средств термодинамики, выведены различные плиточные теории в связи с важным для плавучего льда случаем, что температурное поле неравномерно распределено во льду. Наряду с дисковой теорией развита также обобщенная Рейснера и фон Кармана теория. Деформации сдвига и инерция вращения, а также вызванная температурным полем разнородность, тоже учтены в этой теории. Известная Кирхгоффа-Лове гипотеза, согласно которой материальные линейные элементы, которые вертикально расположены в недеформированной центральной плоскости плиты, остаются при деформации в вертикальном положении в этой (деформированной) плоскости, следовательно, не выдвигается. Соответственно, эта плиточная теория позволяет выполнить все краевые условия, без известных по Кирхгоффа теории граничных особенностей.

Глава 4 приводит затем вывод выдвинутой в главе 1 упругой плиточной теории для случая вязкоупругости. На основе методов полимерной физики выведены аналогичные главе 3 теории вязкоупругих плит с неравномерным температурным полем, исходя из предположения, что лед выполняет предпосылки термореологически простых материалов.

Практические проблемы рассмотрены в последующем сообщении.

Notation

In this article symbolic and Cartesian tensor notation is used. Accordingly, Latin and Greek indices assume the values 1, 2, 3 and 1, 2, respectively. Einstein's summation convention is used according to which summation is understood over doubly repeated indices. Commas indicate differentiation with respect to a space variable, while dots denote total (material) time derivatives.

Subsequently a list of symbols is given.

A	Cross sectional area per unit width
A_i	Displacement vector potential
$A^{(0)}, A^{(1)}$	Abbreviations connected with the normal strains perpendicular to the plate surface
$\mathcal{A}_{(n)}$	Banach space of dimension n
a	Sound speed
$\underset{s=0}{\overset{\infty}{\mathfrak{A}}}{}_{ij}(\cdot)$	Strain functional under constant stress
$B^{(0)}, B^{(1)}$	Abbreviations connected with the normal strains perpendicular to the plate reference plane
C_{ij}	Right Cauchy-Green deformation tensor
$C_{ij}^{(m)}$	Right Cauchy-Green deformation tensor of order m
$\mathcal{C}_{ijkl}$	First order elastic constants of three dimensional anisotropic linear elasticity
$\mathfrak{C}_{ijkl}^{(m)}$	First order elastic constants of order m of two dimensional anisotropic finite linear elasticity
D	Plate bending rigidity of the generalized Reissner Theory
$\mathfrak{D}^{(0)}, \mathfrak{D}^{(1)}$	Zeroth and first order plate rigidities

$\mathfrak{D}^{(0)}(t)$, $\mathfrak{D}^{(1)}(t)$	Zeroth and first order extensional relaxation functions
$\mathfrak{D}^{(0)}$, $\mathfrak{D}^{(1)}$	Zeroth and first order flexural rigidities
$\mathfrak{D}^{(0)}(t)$, $\mathfrak{D}^{(1)}(t)$	Zeroth and first order flexural relaxation functions
D_{ij}	Stretching tensor
E	Modulus of elasticity
E_{ij}	Elongation tensor or Lagrangian strain tensor
$E_{ij}^{(m)}$	Lagrangian strain tensor of order m
$\tilde{E}_{ij} = u_{(i,j)} =$	$\frac{1}{2}(u_{i,j}+u_{j,i})$
$\overset{\infty}{\underset{s=0}{\mathfrak{E}_{ij}}}(\cdot)$	Strain functional
$\overset{\infty}{\underset{s=0}{\overset{*}{\tilde{\mathfrak{E}}}_{X\,ij}}}(\cdot)$	Isothermal strain functional at the particle X
$\overset{\infty}{\underset{s=0}{\mathfrak{E}}}(\cdot)$	Functional for the internal energy
F	Stress function
$F, {}_{f}f, {}_{p}f, {}_{g}f$	Functions describing bounding surfaces
F_{ij}	Displacement gradient
$\overset{\infty}{\underset{s=0}{\mathfrak{F}}}(\cdot)$	Functional for a typical variable f
f_i	Body force per unit mass
$F_i^{(m)}$	Body force of order m (per unit area)
g	Gravity constant
$G_{ijkl}(t)$	Stress-strain relaxation function of anisotropic viscoelasticity.

$\mathfrak{G}_{ijkl}(t)$	Stress-strain relaxation function of anisotropic viscoelasticity
H	Water depth
h	Thickness of the plate
$\mathfrak{H}$	Constant in "macroscopic plane stress" situation
i	imaginary unit
$I^{(p)}$	Moment of inertia of order p
J	det F_{ij}, Jacobian determinant of the motion $\underset{\sim}{\chi}(\cdot)$
$\mathfrak{I}(\cdot)$	Functional, of which an extremum is sought
$\mathfrak{K}$	Density of kinetic energy
L	Reference length
$\mathfrak{L}^{(p)}(t)$	Averaged stress strain relaxation function of order p in isotropic finite linear viscoelasticity plates
m, $\mathfrak{m}$	Mass densities
$M^{(p)}(t)$	Averaged shear relaxation function of order p in isotropic finite linear viscoelastic plates
$M^{(p)}$	Averaged shear modulus of order p in isotropic finite linear elastic plates
M_x, M_y	Bending moments
M_{xy}	Twisting moments
N_i	External normal vector in the reference configuration
n_i	External normal vector in the present configuration
N, $\mathcal{N}^{(p)}$, $\mathfrak{N}^{(p)}$	"Generalized Poisson's ratios"
$0^+_{ij}(s)$	Tensor valued function having value zero

$\mathfrak{O}_{ij}$	Orthogonal Transformation
p	Pressure
$\overset{\infty}{\underset{s=0}{\mathfrak{P}}}(\cdot)$	Functional for the free energy
$\mathfrak{P}^{*}_{X}(\cdot)$	Isothermal free energy function at the particle X
Q_i	Heat flux (Lagrangian)
q_i	Heat flux (Eulerian)
Q_x, Q_y	Shear forces
$\overset{\infty}{\underset{s=0}{\mathfrak{Q}}}{}_i(\cdot)$	Functional for the heat flux vector
$\tilde{R}_{ij} = u_{[i,j]}$	
s	Complex variable for the Laplace transforms
$\mathfrak{z}$	Surface area
$S_k^{(p)}$	Surface load of order p
$\mathfrak{S}$	Static moment
$\overset{\infty}{\underset{s=0}{\mathfrak{S}}}(\cdot)$	Functional for the entropy
t_i, t_i^*	Stress vector
T_{ij}	Extra stress (Deviation from equilibrium stress)
T_{ij_D}	Extra stress (Deviation from equilibrium stress)
$\overset{\infty}{\underset{s=0}{\mathfrak{T}}}{}_{ij}(\cdot)$	Functional for the first or second Piola-Kirchhoff stress tensor
u	velocity in the x_1-direction

u_i	Displacement vector
$\ddot{U}_i^{(m)}$	Acceleration vector of order m.
v	Velocity in the y-direction
v_i	Velocity vector
ϑ	Volume
V_0, V_1	Normalization velocities
w	Velocity component in z-direction
w_i	Velocity vector at the plate - ice interface
X, Y	Particle label
α_{ij}	Strain at constant stresses
Γ_x, Γ_y	Averaged shear modulus
δ_{ij}	Kronecker delta, unit tensor, $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$
ϵ	Perturbation parameter, internal energy
ϵ_{ijk}	Lévi-Civitá tensor, permutation symbol
$\eta = u_3^{(0)}$	Zeroth order displacement
η	Entropy
$\varphi = -m^{(1)}$	First order displacement in the x-direction
Φ	Velocity potential, Stress function
$\emptyset$	Empty set
$\chi(\cdot)$	Temperature time shift function
$\psi = -u_2^{(1)}$	First order displacement in the y-direction
ψ	Free energy
Ψ	Potential function for $\rho\underset{\sim}{v}$ or the displacement field

λ, μ	Lamé's constants, or material constants for a Navier-Stokes fluid, or isotropic relaxation functions in viscoelasticity
$\Lambda^{(p)}, M^{(p)}$	Averaged Lamé's constants of order p
ρ	Density of the fluid
ϑ	Absolute temperature
$\Theta^{(m)}, \theta^{(m)}$	Temperature resultants of order m
π	Pressure at the plate-water interface
τ	Nondimensional time, relaxation time
Σ_{ij}	Second Piola-Kirchhoff stress tensor
ω	Frequency
$\omega_{ij}, \omega\delta_{ij}$	Thermal expansion coefficient
Ω	Nondimensionalized frequency

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Chapter 1: Statement and Motivation of the Problem

In the past, the analyses of floating ice plates subjected to static and dynamic loads were based on the theory of thin homogeneous elastic plates, although in actual floating ice plates the material constants may vary strongly with depth. The reason for the variation of the material constants is chiefly due to a nonuniform distribution of the temperature with depth, but also due to a nonhomogeneity induced by the freezing process. In addition, the material behavior may be more complex than the simple linearly elastic response which is generally assumed in ordinary plate theories. In fact, it has been shown by Nevel [1] that moderately slow processes should be described by a material model which takes dissipation into account.

With regard to the elastic behavior A. Assur has suggested that the elastic plate theory may be applied in floating ice plates provided the plate constant (Young's modulus) is replaced by a quantity which is averaged over the depth of the ice plate. This conjecture has been substantiated by Kerr and Palmer [2], who on the basis of linear elasticity theory show that the usual plate theory emerges when the pertinent equations are averaged over the thickness of the plate. Kerr and Palmer's result, however, is based on the tacit assumption that Poisson's ratio does not vary with depth, an assumption nowhere justified in their article. Moreover, they do not take thermodynamic arguments into account. These, so we believe, are nontheless important, because the temperature varies from one material point to another.

A reinvestigation of the entire matter seems to be necessary, because, apart from the above reasons, various effects have not been studied in the past. First, it is not clear a priori that the nonuniformity of the temperature distribution can simply be interpreted as a nonhomogeneity of the material constants. In fact this is certainly not so in the viscoelastic theory. Second, dependent upon the local air temperature the corresponding temperature distribution in the plate will change and so do the local material constants and the averaged constants as well. Third, thermal stresses are induced through the temperature variation. These thermal stresses, in general, will cause deformations in the plane of the plate. A proper derivation thus should include thermal stresses. Fourth, the situation is not too rare where displacements are of the order of or larger than the thickness of the ice plate. It therefore also seems to be justified that the von Kàrmàn plate theory is investigated with regard to its validity in the situation of floating ice plates.

While for considerably fast processes it generally suffices to assume that the material response is elastic, this assumption must be abandoned for quasistatic processes and replaced by a more appropriate hypothesis, which takes dissipation into account. If the same generality as in the elastic theory is kept then a simple linear theory of viscoelasticity must fail, because it does not include the analogue to the von Kàrmàn theory. Moreover, the temperature variation of the relaxation or creep functions of linear viscoelasticity generally induce a nonlinearity in the governing equations. Because our interest is not in a full coupled theory of thermoviscoelasticity and we only search for a theory which takes a nonuniformity in the base temperature into account, the basic thermodynamic theory, which is very difficult, must be simplified. Fortunately, such a simplified theory is already available in the literature of high polymeric materials. The key idea is that the temperature dependence of the material constants is transformed into a time dependence; this behavior is known as the temperature time shift property and the material is referred to as being thermorheologically simple. By assuming that ice does satisfy the postulates of thermorheologically simple materials, reinvestigating existing theories of high polymers, it is then possible to carry over the calculations of elastic plates into those of viscoelastic plates. In this way we then arrive at corresponding plate theories valid for viscoelastic materials. As a side result of these calculations, the basic question as to whether the ordinary material constants (as determined by the material scientists) or other constants must be chosen, is naturally answered.

Thus far, only the behavior of the plates has been considered. An equally important part of the description of floating ice is the one of its underlying water, because it determines the boundary forces at the interface plate-water. While this interaction is generally only taken into account by assuming that the water pressure is proportional to the plate deflection, a moments thought shows that this is only correct in the static situation. Dynamic processes are much more complex. In particular there are various physical situations for which a different set of differential equations applies. Generally, lake and sea ice requires a different treatment of the fluid equations than ice on rivers. And even then other situations can arise as in the interaction of ice and water in the tidal motion. While for all the preceding situations the fluid sublayer may be assumed to be an incompressible viscous or inviscid material, compressibility must be taken into account when seismic studies on floating ice are performed.

The foregoing considerations thus make it desirable to list briefly the governing equations of the fluid together with a careful investigation of the boundary conditions on the plate-water interface and the free surface. Contrary to the plate equations, however, a temperature variation in the liquid layer needs not be taken into account. With regard to the physical situation, the following cases may be differentiated: (i) lake- and sea-ice: The motion of the water particles is confined to small perturbations about an equilibrium position. Viscosity is unimportant. (ii) Tidal waves: In this case the shallow water assumption is justified. (iii) Rivers: The water is in pronounced motion and viscosity must not be neglected.

Specifically, the paper is divided into four Chapters. Chapter 2 deals with the equations for the fluid. Starting with the basic balance laws of mass and linear momentum and the basic constitutive equations for a compressible and incompressible ideal or viscous fluid we develop the basic governing equations first for the compressible inviscid fluid under small perturbation flow. Contrary to the usual treatments (see any book on hydrodynamics [3],[4],[5]) we show how, in principle, an entire hierarchy of differential equations and boundary conditions can be obtained, by performing a regular perturbation expansion. We derive the potential equations of the first order which apply when the density is nonuniform and then turn our attention to the incompressible inviscid fluid; using a similar regular perturbation expansion, the well known small amplitude approximation, valid for lake- and sea-ice is developed. In contrast to the foregoing treatments the derivation of the governing equations of the shallow water theory, are obtained by a singular perturbation expansion. The procedure applied here follows Stoker [6] and it turns out that a reasonable set of equations, describing the interaction of large ice plates with the tidal motion is obtained. Chapter 2 ends with an investigation of the viscous equations valid in rivers when being covered by ice. In all the aforementioned calculations special care is observed with regard to the boundary conditions, valid at the plate-water interface.

Chapter 3 deals with the derivation of the basic dynamic equations of plates with nonuniform temperature distribution. In spite of various significant contributions in the recent literature a complete nonlinear theory of elastic (or viscoelastic) plates, which provokes further developments and enables one to make plausible assumptions is not currently available. An attempt is thus made to develop a fully nonlinear theory of plates within the framework of three dimensional elasticity and viscoelasticity.

The earliest works on plates may be traced back to Kirchhoff [7]. This theory is reported by Love [8]. These works are bound to linear elasticity and are formulated by means of ad hoc assumptions. In this respect the reader is reminded of the familiar Kirchhoff-Love hypothesis according to which planes normal to the middle surface maintain this orthogonality under deformation. Attempts to abandon this hypothesis are to be found in two different directions. One approach is based on the theory of oriented media and may be summarized as the theory of Cosserat surfaces (for an account see Truesdell and Toupin [9] and Truesdell and Noll [10]). This approach is axiomatic and we do not see how it could be generalized to deliver a theory for plates with nonuniform temperature distribution in a form which would determine this temperature dependence from the 3 dimensional constants. Another apparently more fruitful approach starts from the three dimensional theory of nonlinear dynamics. These equations are then averaged over the thickness of the plate, thereby reducing a three dimensional problem to a two dimensional one. This branch of research has been initiated by Mindlin and Medick [11]. Various authors such as Nigul [12], Kalinin [13] and Koiter [14] discuss the method. A comprehensive survey of the technique is given by Gol'denweizer [15]. The technique was applied to specific plate problems, using techniques such as the direct method, the asymptotic method and the series expansion method by various authors, see [16], [17], [18], [19], [20]. In our investigation the series expansion technique is used. The method involves the expansion of all variables in terms of the appropriate coordinates. These series are either in terms of orthogonal polynomials or ordinary Taylor series. The method converts three dimensional approximate equations of elasticity (or of more complicated material behavior) into a hierarchy of two dimensional approximate equations by the use of a variational principle or the method of integration.

It not only furnishes the governing two dimensional equations for the plate motion but also the characteristic constants (flexural and extensional rigidities) as functions of the base temperature distribution. The temperature variation with depth and its dependence upon the surrounding air temperature is important in floating ice plates. Its influence on all occuring constants in the various plate theories is thus investigated for the most important case that the temperature varies linearly with depth.

To summarize the results of Chapter 3, mention must be made of two different cases. If Poisson's ratio is not a function of the temperature, then, loosely speaking, all results known from the various classical elastic plate theories may be adopted, if only the material constants are replaced by appropriate averages over the thickness of the plate. This is, strictly speaking, however, only correct in the static situation. In the bending theory of elastic ice-plates with nonuniform temperature distribution, an acceleration term occurs which in spite of its numerical insignificance, makes the equivalence only approximate. We thus conclude that the validity of Assur's conjecture approximately goes far beyond the case investigated by Kerr and Palmer. It is, however incorrect when Poisson's ratio cannot be assumed to be independent of the temperature.

Chapter 4 gives an account on the viscoelastic theory of plates under the same general assumptions as in Chapter 3. Hereby, only the constitutive theory of finite viscoelasticity must be newly developed, because this is the only element of Chapter 3 which changes when a viscoelastic body is considered. Unfortunately the theory of viscoelastic materials with nonuniform temperature distribution is extremely complex. This is easily seen when one is investigating the temperature dependence of the relaxation or creep functions. The integral of these functions over the plate thickness generally again furnishes a function of time. Only in special cases (which may be considered unphysical)* does this function reduce to a constant function. If this property would exist then similar results to the purely elastic theory would be observed. But this is not so. Hence, in general, a viscoelastic plate theory with nonuniform base temperature is governed by a different set of equations from the corresponding isothermal (homothermal) plate theory. Since no systematic measurements of the temperature dependence of the relaxation functions are presently available and since no order of magnitudes of this influence does exist, we are unable to go further in neglecting terms which might become significant.

On the other hand it seems worthwhile to note that apart from an interest in floating ice, the above difference in the behavior of plates with uniform and nonuniform temperature distribution calls for an investigation of a standard example to be solved, which clearly would show limits under which situation the same set of equations could be used in both situations. This, among other things will be dealt with in a later article.

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* The only physical case occurs when the temperature distribution is symmetric which is a very rare situation for ice.

Chapter 2. Basic Equations of the Fluid Sublayer

Because the response of floating ice plates to external loads is basically described by an interaction of the plate with its underlying fluid layer, it seems to be natural that a proper description must take into account both the behavior of the fluid and of the plate as well. In this Chapter we present the basic equations for the liquid.

As it was already mentioned in the Introduction, there is a basic difference between sea-, lake- and river-ice. Accordingly, with regard to fluid motions one must differentiate between neighboring states of standing water and water in pronounced motion. To substantiate this statement consider the situation as illustrated in Fig. 2.1, that is, consider shallow water which to the left of the reference coordinate x = 0 possesses a free surface while being covered with an ice plate to the right of x = 0.

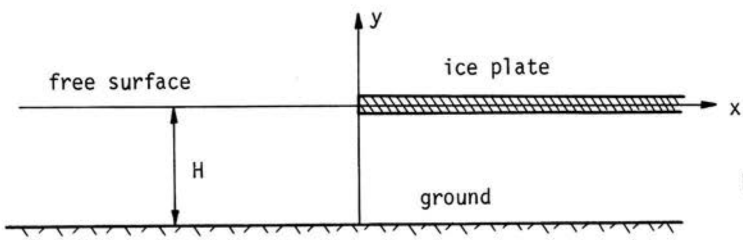

Fig 2.1

Loosely speaking, in standing water there is no difference in the basic physical behavior of the layer for $x \leq 0$ and for $x \geq 0$. Neither at the free surface nor at the fixed bottom surface is there any major boundary layer to be significant. In rivers, however, or more generally when horizontal velocities are large, it is known that boundary layers are important. In fact, for a rigid ice sheet the flow pattern far downstream must be the well known Couette flow. In short, while sea- and lake-ice interaction phenomena can be determined accurately enough from a theory based on an ideal fluid assumption, there is a definite need to treat the fluid as viscous in all those cases where horizontal drift velocities are substantial*. We therefore treat the fluid as viscous from which the ideal fluid can be easily obtained by specialization. Moreover we do not assume, in general, that the fluid be incompressible, because in so doing one loses the important property that dilational disturbances propagate with finite speed. Hence, the response of ice sheets and of the sublaying water to seismic waves would not be described. Nevertheless, it seems to be clear that the dynamics of ice plates under seismic pulses belong to another class of physical problems than the response of ice plates to oceanic waves which may be caused by wind, or the response of the entire system to external dynamic loadings. It therefore seems to be justified to divide the entire topic into three groups

(i) Compressible inviscid fluid - small perturbations under standing water conditions (propagation of seismic pulses)

(ii) Incompressible inviscid fluid (sea- and lake-ice)

(iii) Incompressible viscous fluid with large drifting velocities parallel to the undeformed plate (rivers)

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* The inviscid approximation as used in the airfoil theory, described by the Prandtl-Glauert equations, [22], cannot be considered to be a valid one, because it discards the fact that the flow pattern of a river is entirely due to the viscous boundary layer. The entire velocity profile is the boundary layer profile. The Prandtl-Glauert approximation would only be justified in deep water with pronounced drifting velocities.

To introduce the state of affairs it suffices to note that the dynamic equations are conservation of mass, balance of linear momentum, moment of momentum and energy. In general, these equations contain the temperature as a basic field variable. However, when the boundary conditions of place and traction are independent of temperature and the stress does not depend upon the temperature either, then the energy equation separates from the other equations. This is thus the simplifying assumption we shall make: The temperature does not enter the constitutive equation of stress and all boundary conditions of place and traction are temperature independent. For classical viscous fluids this assumption is tantamount to assuming that the viscosity be independent of the temperature.

With this last simplifying assumption the energy equation becomes redundant and the dynamical equations reduce to continuity:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\underset{\sim}{v} = 0 \quad , \tag{2.1}$$

linear momentum:

$$\rho\frac{d\underset{\sim}{v}}{dt} = \nabla\cdot\underset{\sim}{T} + \rho\underset{\sim}{f} \quad , \tag{2.2}$$

with the condition $\underset{\sim}{T}^T = \underset{\sim}{T}$, $\underset{\sim}{T}^T$ denoting the transpose of $\underset{\sim}{T}$. In the above equations, ρ denotes the density, $\underset{\sim}{v}$ the velocity, $\underset{\sim}{T}$ the Cauchy stress tensor and $\underset{\sim}{f}$ the body force per unit mass.

In the problems of interest to us, $\underset{\sim}{f}$ is the earth acceleration vector $\underset{\sim}{f} = \underset{\sim}{g}$. We shall restrict our considerations henceforth to this case.

It remains to establish a constitutive equation for $\underset{\sim}{T}$. To this end we decompose $\underset{\sim}{T}$ into its equilibrium and viscous parts $-p\underset{\sim}{1}$ and $\underset{\sim}{T}_D$, respectively:

$$\underset{\sim}{T} = -p\underset{\sim}{1} + \underset{\sim}{T}_D \tag{2.3}$$

where in equilibrium $\underset{\sim}{T}_D = \underset{\sim}{0}$. This is a thermodynamic consequence we shall not further investigate. For Navier-Stokes fluids

$$\underset{\sim}{T} = -p\underset{\sim}{1} + \lambda(\text{trace } \underset{\sim}{D})\underset{\sim}{1} + 2\mu\underset{\sim}{D} \quad , \tag{2.4}$$

where $\underset{\sim}{D}$ denotes the stretching tensor and is given by

$$\underset{\sim}{D} = \frac{1}{2}[\nabla\underset{\sim}{v}+(\nabla\underset{\sim}{v})^T] \quad . \tag{2.5}$$

Note, that the role of p is different for compressible and incompressible materials. For an incompressible material p is function of place and time only and its determination is part of the integration of the system (2.1) and (2.2). On the other hand, in a compressible material p is a function of the density ρ.

In accord with common usage in classical fluid dynamics we impose the Stokes condition, viz. trace $\underset{\sim}{T}_D = \underset{\sim}{0}$, thereby arriving at

$$(\lambda+(2/3)\mu) = 0 \quad . \tag{2.6}$$

With this equation we have arrived at the full set of equations describing the fields. Before specializing the equations for the groups listed above we consider the boundary conditions.

To this end notice that according to Fig. 2.1 there are three different types of boundary conditions to be satisfied, (i) at the free surface, (ii) at the impermeable wall and (iii) at the contact surface with the plate. In general, the free surface is not known a priori. The same is true for the contact surface of plate and liquid. To begin with let $y = f(x,z,t)$ be the equation of the ground, instantaneous free surface or instantaneous contact surface, respectively. Clearly, on the wall $f(x,z,t)$ is independent of time. For an inviscid fluid the boundary condition there apparently must be that the normal velocity be zero. Thus

$$\underset{\sim}{v}\cdot\nabla f = 0 \text{ on the ground.} \qquad (2.7)$$

On the free surface or the interface $F = y - f(x,z,t) \equiv 0$ for any particle and in particular

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \underset{\sim}{v}\cdot\nabla F = 0 \quad ;$$

hence

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_z \frac{\partial f}{\partial z} - v_y = 0 \quad , \qquad (2.8)$$

where we have set $\underset{\sim}{v} = (v_x, v_y, v_z)$. The relation (2.8) is the boundary condition for the interface and expresses, as is easily seen the fact that the fluid particles on the interface remain there during the course of the motion. For the free surface, however, (2.8) does not suffice to determine the boundary conditions, and must be complemented by a condition on the pressure. We shall work with relative pressure and assume that it be zero at the free surface. This is in sharp contrast to the single boundary conditions (2.7) and (2.8) for wall and plate, but it is not strange that two conditions should be satisfied in the case of the free surface since an additional unknown function f, the equation of the free surface, is involved in the latter case. Note further that both equations (2.8) for free surface and interface as well, are nonlinear, because f and $\underset{\sim}{v}$ are not known from the outset.

The boundary conditions for the viscous fluid case will be treated later.

In the following sections we shall list the basic hydrodynamic equations for the three subgroups mentioned above.

2.1) Compressible, Inviscid Fluid - Small Perturbation Flow

Compressible inviscid fluid flow is a suitable approximation for the description of seismic waves. Each particle undergoes reversible adiabatic changes of state. Consequently, if the entropy was initially uniform as is the case for the fluid at rest or in a uniform stream, it must remain constant and uniform throughout the flow. The dynamical equations (2.1) and (2.2) then assume the form

$$\frac{d\rho}{dt} + \rho\nabla\cdot v = \frac{1}{a^2}\frac{dp}{dt} + \rho\nabla\cdot v = 0 \; ;$$

$$\frac{d\underset{\sim}{v}}{dt} = -\frac{1}{\rho}\nabla p \quad , \tag{2.9}$$

where $\underset{\sim}{T}_D = \underset{\sim}{0}$ and where we have assumed that $p = \hat{p}(\rho)$ is invertible to yield explicitly $\rho = \hat{\rho}(p)$. The dilational velocity a is given by

$$\frac{1}{a^2} = \frac{d\rho}{dp} \; . \tag{2.10}$$

The equations (2.9) both are nonlinear and consequently are extremely difficult to solve.

The usual approximation is to restrict considerations to small perturbations. We assume the velocity and its derivatives to be small and the corresponding pressure and density to differ only slightly from the values far away from disturbances, at infinity, say. Mathematically this may be expressed by the following perturbation expansions

$$\rho = \sum_{\nu=0}^{\infty} \epsilon^{\nu} \rho_{\nu} \quad ; \quad \frac{\rho_{\nu+1}}{\rho_{\nu}} \to 0 \text{ as } \epsilon \to 0 \quad ;$$

$$p = \sum_{\nu=0}^{\infty} \epsilon^{\nu} p_{\nu} \quad ; \quad \frac{p_{\nu+1}}{p_{\nu}} \to 0 \text{ as } \epsilon \to 0 \quad ;$$

$$\underset{\sim}{v} = \sum_{\nu=1}^{\infty} \epsilon^{\nu} \underset{\sim}{v}_{\nu} \quad ; \quad \frac{\|\underset{\sim}{v}\|_{\nu+1}}{\|\underset{\sim}{v}\|_{\nu}} \to 0 \text{ as } \epsilon \to 0 \quad ; \tag{2.11}$$

$$\frac{1}{a^2} = \sum_{\nu=0}^{\infty} \epsilon^{\nu} \frac{1}{a_{\nu}^2} \quad ; \quad \frac{a_{\nu}^2}{a_{\nu+1}^2} \to 0 \text{ as } \epsilon \to 0 \quad ,$$

where ϵ is a small parameter (the amplitude of the reference disturbance). Clearly, because of the constitutive equation relating ρ and p, p_{ν}, ρ_{ν} and a_{ν}^2 are related also. Substituting (2.11) into (2.9) and collecting terms of like powers in ϵ results in the following system:

$$\nabla p_o = \underset{\sim}{g} \quad ;$$

$$\frac{\partial \underset{\sim}{v}_1}{\partial t} = -\frac{1}{\rho_o} \nabla(p_1) \quad ; \tag{2.12}$$

$$\frac{1}{a_o^2} \frac{\partial p_1}{\partial t} + \rho_o \nabla \cdot \underset{\sim}{v}_1 = 0$$

for the $O(\epsilon)$ terms and

$$\frac{\partial \underset{\sim}{v}_n}{\partial t} = -\nabla\left(\frac{p_n}{\rho_n}\right) + \underset{\sim}{\mathfrak{F}}_n$$

$$\frac{1}{a_o^2}\frac{\partial p_n}{\partial t} + \rho_o \nabla\cdot \underset{\sim}{v}_n = \mathfrak{G}_n \tag{2.13}$$

for the $O(\epsilon^n)$ terms. In the above equations $\underset{\sim}{\mathfrak{F}}_n$ and $\mathfrak{G}_n$ are functions involving the quantities ρ_j, v_j, p_j, a_j for $j < n$ only. They are known, once the system (2.13) is solved for indices smaller than n.

Before proceeding further we apply the same perturbation procedure to the boundary conditions. To this end, let ${}_g f(\cdot)$ and ${}_p f(\cdot)$ and ${}_f f(\cdot)$, respectively, be the functions describing the ground, the plate interface and the free surface:

$$y = {}_g f(x,z) \quad ; \quad y = {}_f f(x,z,t) \quad ; \quad y = {}_p f(x,z,t) \tag{2.14}$$

Because ${}_f f(\cdot)$ and ${}_p f(\cdot)$ are not known, they are subjected to the same perturbation expansions, the velocities have been (see (2.11)), explicitly

$${}_f f(\cdot) = \sum_{\nu=1}^{\infty} \epsilon^{\nu}\, {}_f f_{\nu}(\cdot) \quad ; \quad {}_p f(\cdot) = \sum_{\nu=o}^{\infty} \epsilon^{\nu}\, {}_p f_{\nu}(\cdot) \quad . \tag{2.15}$$

Notice that we have assumed in these expansions that the unperturbed free surface be described by $y = 0$, (${}_f f_o = 0$), but that $y = {}_p f_o(x,z)$ does not vanish. It represents the interface at equilibrium.

With these preliminaries, the boundary conditions for each stage of the perturbation approximation are obtained by substituting $(2.14)_1$ into (2.7) and (2.15) into (2.8) and collecting terms of like powers in ϵ.

The result of this substitution is

$$\underset{\sim}{v}_1 \cdot \nabla _g f = 0 \; ; \qquad \text{at } y = {}_g f(x,z) \; ;$$

$$\frac{\partial _f f_1}{\partial t} - v_1 = 0 \; ; \qquad \text{at } y = 0 \; ; \qquad (2.16)$$

$$\frac{\partial _p f_1}{\partial t} + u_1 \frac{\partial _p f_o}{\partial x} + w_1 \frac{\partial _p f_o}{\partial z} = v_1 \; ; \quad \text{at } y = f_o(x,z) \; ;$$

and

$$\underset{\sim}{v}_2 \cdot \nabla _g f = 0 \qquad \text{at } y = {}_g f(x,z) \; ;$$

$$\frac{\partial _p f_2}{\partial t} + u_2 \frac{\partial _p f_o}{\partial x} + w_2 \frac{\partial _p f_o}{\partial z} + u_1 \frac{\partial _p f_1}{\partial x} + w_1 \frac{\partial _p f_1}{\partial z} - v_2 - \frac{\partial v_1}{\partial y} {}_p f_1 = 0 \; ;$$

$$\text{at } y = {}_p f_o(x,z) \; ;$$

$$(2.17)$$

$$\frac{\partial _f f_2}{\partial t} + u_1 \frac{\partial _f f_1}{\partial x} + w_1 \frac{\partial _f f_1}{\partial z} - v_2 - \frac{\partial v_1}{\partial y} {}_f f_1 = 0 \; ;$$

$$\text{at } y = {}_f f_o(x,z) \; .$$

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In these formulas we have set $\underset{\sim}{v}_n = (u_n, v_n, w_n)$. As a major result of the small amplitude expansion the boundary conditions are to be satisfied on the unperturbed bounding surfaces. This theory therefore is a development in the neighborhood of the rest position of equilibrium of water and ice plate.

As we shall be dealing with the lowest approximation only, we shall henceforth discard the equations (2.13) and (2.17) and drop the subscripts of velocity v_1, pressure p_1, density ρ_o and sound speed a_o, respectively, because misinterpretation is unlikely to occur.

It follows from $(2.12)_1$ and the equation of state that the pressure p_o varies with depth. More precisely, for the coordinate system as chosen in Fig (2.1)

$$p_o = -gy \rightarrow \rho = \rho(y) \quad ; \quad a = a(y) \quad .$$

The coefficients of the equations $(2.12)_{2,3}$ are therefore functions of position. This fact is often overlooked*. In view of the initial condition of uniform rest, equation $(2.12)_2$ implies that there exists a potential for the product of density and velocity: $\rho v = \nabla\psi$. Equations $(2.12)_{2,3}$ can then be written as follows:

$$\begin{aligned} \frac{\partial\Psi}{\partial t} &= -p \quad ; \\ \frac{\partial^2\Psi}{\partial t^2} &= a^2\nabla^2\Psi - a^2\nabla\Psi\cdot\nabla(\ln\rho) \quad . \end{aligned} \tag{2.18}$$

Apart from the last term, these are the classical equations of sound propagation. The last term of $(2.18)_2$ accounts for a density variation in the equilibrium configuration. Its influence is usually neglected.

The boundary conditions follow now from (2.16). At the impermeable wall $(2.16)_1$ applies and this is easily written as

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* Normally one neglects gravity in which case p_o, ρ_o and a_o are uniform.

$$\frac{\partial \Psi}{\partial n} = 0 \quad ; \quad \text{on the ground,} \tag{2.19}$$

where $\partial(\cdot)/\partial n$ denotes the derivative normal to the wall. At the plate interface, $(2.16)_3$ must hold, which gives

$$\rho_o \frac{\partial\, {}_pf_1}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{\partial\, {}_pf_o}{\partial x} + \frac{\partial \Psi}{\partial z}\frac{\partial\, {}_pf_o}{\partial z} - \frac{\partial \Psi}{\partial y} = 0 \quad ; \quad \text{at } y = f_o(x,z) \quad . \tag{2.20}$$

Similarly, at the free surface the pressure vanishes, $\partial\psi/\partial t = 0$, and $(2.16)_2$ must hold. The former, namely

$$\frac{\partial \Psi}{\partial t} = 0 \quad ; \quad \text{at} \quad y = 0 \qquad . \tag{2.21}$$

serves as boundary condition on ψ, while the latter, namely $(2.16)_2$ is an equation determining the deformed surface

$$\rho \frac{\partial\, {}_ff_1}{\partial t} = \frac{\partial \Psi}{\partial y} \quad . \tag{2.22}$$

The union of the equations (2.18), (2.19), (2.20), (2.21) and (2.22) suffice to describe the interaction of floating ice with water, once ${}_pf_o(\cdot)$ and ${}_pf_1(\cdot)$ are known.

We would like to emphasize that these equations are only adequate for extremely small wave lengths or <u>high frequencies</u> because the boundary condition at the free surface does not include its deformation. This condition obviously is satisfied for artificially induced seismic waves and for small ice plates where only high frequencies can develop. For all other interactions the boundary condition of the free surface must involve the function describing it. This is particularly the case when the depth of the water is comparable to the wave lengths of the surface waves. It clearly includes all tidal waves, natural earthquakes and interaction phenomena with coastal ice.

2.2) Incompressible Inviscid Fluid - Small Amplitude Approximation

In this section we list the basic equations and boundary conditions for an incompressible inviscid fluid. Except where large horizontal drift velocities do occur they apply in the low frequency range, thus in all cases mentioned above with the exception of rivers and tidal waves. Compressibility is not important, because the propagation of disturbances is not comparable to the one of sound waves.

To begin with, note that the dynamic equations now assume the form

$$\nabla \cdot \underset{\sim}{v} = 0$$

and

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p + \rho \underset{\sim}{g} = -\nabla [p - g \underset{\sim}{p} \cdot \underset{\sim}{e}] \quad .$$

$\underset{\sim}{p}$ is the position vector and $\underset{\sim}{e}$, $\|\underset{\sim}{e}\| = 1$, points in the direction of gravity. Accordingly, the velocity field is lamellar and thus we may write $\underset{\sim}{v} = \nabla \Phi$, implying that Φ is potential function:

$$\nabla^2 \Phi = 0 \quad . \tag{2.23}$$

The momentum equation can now be integrated in the usual way to yield the Bernoulli equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + \frac{p}{\rho} - g \, \underset{\sim}{p} \cdot \underset{\sim}{e} = 0 \quad , \tag{2.24}$$

which must be considered to be an equation for p, once Φ is known.

The boundary conditions are easily derived from (2.7), (2.8) and (2.24), respectively. On the ground

$$\frac{\partial\Phi}{\partial n} = 0 \ . \tag{2.25}$$

At the plate interface

$$\frac{\partial\, {}_p f}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\, {}_p f}{\partial x} + \frac{\partial\Phi}{\partial z}\frac{\partial\, {}_p f}{\partial z} - \frac{\partial\Phi}{\partial y} = 0 \ . \tag{2.26}$$

However, on the free surface

$$\frac{\partial\, {}_f f}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\, {}_f f}{\partial x} + \frac{\partial\Phi}{\partial z}\frac{\partial\, {}_f f}{\partial z} - \frac{\partial\Phi}{\partial y} = 0$$

and (2.27)

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - g\, \underset{\sim}{p}\cdot\underset{\sim}{e} = 0 \ ,$$

where we have set the pressure equal to zero.

Within the context of incompressible materials the equations (2.23) - (2.27) are exact. They are nonlinear through the boundary conditions, which makes their integration difficult.

Several approximations are possible. Here we only treat the theory of waves of small amplitude. Our derivation follows Stoker, [6].

The basic assumption thereby is that the velocity for the water particles, the free surface elevation $y = {}_f f(x,z,t)$, the interface displacement of water and ice plate $y = {}_p f(x,z,t)$ and their derivatives be small quantities. In fact we make the perturbation expansions

$$\Phi = \sum_{n=1}^{\infty} \epsilon^n \Phi_n \quad ;$$

$$_f f(\cdot) = \sum_{n=1}^{\infty} \epsilon^n {}_f f_n(\cdot) \quad ; \tag{2.28}$$

$$_p f(\cdot) = \sum_{n=o}^{\infty} \epsilon^n {}_p f_n(\cdot) \quad .$$

Substituting these expansions into the equations (2.23) - (2.28), respectively, it follows first of all that each of the functions $\Phi_n(x,y,z,t)$ is harmonic

$$\nabla^2 \Phi_n = 0 \quad . \tag{2.29}$$

Moreover, at the bottom wall

$$\frac{\partial \Phi_k}{\partial n} = 0 \quad ; \quad \text{at} \quad y = {}_g f(x,z) \quad . \tag{2.30}$$

At the plate-water-interface

$$\frac{\partial {}_p f_1}{\partial t} + \frac{\partial \Phi_1}{\partial x} \frac{\partial {}_p f_o}{\partial x} + \frac{\partial \Phi_1}{\partial z} \frac{\partial {}_p f_o}{\partial z} - \frac{\partial \Phi_1}{\partial y} = 0 \quad ;$$

$$\frac{\partial {}_p f_2}{\partial t} + \frac{\partial \Phi_2}{\partial x} \frac{\partial {}_p f_o}{\partial x} + \frac{\partial \Phi_2}{\partial z} \frac{\partial {}_p f_o}{\partial z} - \frac{\partial \Phi_2}{\partial y} + \frac{\partial \Phi_1}{\partial x} \frac{\partial {}_p f_1}{\partial x} + \frac{\partial \Phi_1}{\partial z} \frac{\partial {}_p f_1}{\partial z} - \frac{\partial^2 \Phi_1}{\partial_y^2} {}_p f_1 = 0$$

$$\vdots \qquad \text{at} \quad y = {}_p f_o(x,z) \tag{2.31}$$

On the free surface, however,

$$g \cdot {}_f f_1 + \frac{\partial \Phi_1}{\partial t} = 0 \;;$$
$$g \cdot {}_f f_2 + \frac{\partial \Phi_2}{\partial t} = -\frac{1}{2}(\nabla \Phi_1)^2 - {}_f f_1 \frac{\partial^2 \Phi_1}{\partial y \partial t} \tag{2.32a}$$ *

and

$$\frac{\partial {}_f f_1}{\partial t} = \frac{\partial \Phi_1}{\partial y} \;;$$
$$\frac{\partial {}_f f_2}{\partial t} = \frac{\partial \Phi_2}{\partial y} - \frac{\partial \Phi_1}{\partial x}\frac{\partial {}_f f_1}{\partial x} - \frac{\partial \Phi_1}{\partial z}\frac{\partial {}_f f_1}{\partial z} + \frac{\partial^2 \Phi_1}{\partial y^2} {}_f f_1 \;. \tag{2.32b}$$

The equations (2.32) all have to be satisfied at $y = 0$. This theory therefore is a development in the neighborhood of the rest position of equilibrium of water.

The relations (2.29) - (2.32) thus, in principle, furnish a means of calculating successively the coefficients of the series (2.28), clearly assuming that such series exist and do converge.

To lowest order, the interaction of ice plates with the underlying water is described by the Laplace equation

$$\nabla^2 \Phi_1 = 0 \;, \tag{2.33}$$

which satisfies the boundary conditions

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* We have chosen $\underset{\sim}{p} \cdot \underset{\sim}{e} = -y$

$$\frac{\partial \Phi_1}{\partial n} = 0 \; ; \qquad \text{on the ground}$$

$$\frac{\partial \, {}_p f_1}{\partial t} + \frac{\partial \Phi_1}{\partial x}\frac{\partial \, {}_p f_o}{\partial x} + \frac{\partial \Phi_1}{\partial z}\frac{\partial \, {}_p f_o}{\partial z} - \frac{\partial \Phi_1}{\partial y} = 0 \; ; \qquad \text{at the plate} \qquad (2.34)$$

$$\frac{\partial^2 \Phi_1}{\partial t^2} + g\frac{\partial \Phi_1}{\partial y} = 0 \; ; \; \text{at} \;\; y = 0 \; ,$$

where $(2.34)_2$ has been obtained from $(2.32a)_1$ and $(2.32b)_1$ by means of eliminating ${}_f f_1$. Once Φ_1 is known, the equation of the free surface can be obtained from equation $(2.32a)_1$.

We can now see the great simplifications which result through the small amplitude approximation. Not only has the problem become linear, but also the domain in which its solution is to be determined becomes fixed a priori. Surface wave problems in this formulation therefore belong from the mathematical point of view, to the classical problems of potential theory.

2.3) Shallow Water Approximations:

A different kind of approximation from the foregoing linear theories of waves of small amplitudes, is obtained when it is assumed that the depth of the water be sufficiently small as compared to some other significant length, such as for example the horizontal wave length or the curvature of the free surface and the plate interface, respectively. In this theory it is not assumed that the amplitudes of the disturbances is small nor is it assumed that the displacement and slope of the water surface are small. The emerging theory, the shallow water theory not only governs tidal waves but equally describes wave propagation in channels, [6]. As is well known, in both applications one assumes that the fluid be inviscid. As was already mentioned before, however, this assumption cannot be kept, when rivers are covered by ice, because boundary layer effects change the flow pattern completely. It is for this reason that we use the viscous fluid representation.

To begin with, the pertinent equations for coastal sea and lake ice and for the tidal waves will be derived from a set of equations emerging from the inviscid equations of motion. On the other hand, the governing equations for rivers emanate from the corresponding viscous equations. The scalings in the two cases, however, are different.

The above statements will be substantiated in the following subsections. Here we simply state the scaling procedure common to all cases. To this end, consider the dynamical equations which with the notation $\underset{\sim}{v} = (u,v,w)$ assume the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad ;$$

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \eta\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad ; \\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \eta\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + g \quad ; \\
\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\,\frac{\partial p}{\partial z} + \eta\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \quad .
\end{aligned} \tag{2.35}$$

Provided η is set equal to zero, the velocity field is lamellar

$$\begin{aligned}
\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} &= 0 \quad ; \\
\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} &= 0 \quad ; \\
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \quad .
\end{aligned} \tag{2.36}$$

On the other hand the boundary conditions must be satisfied. In view of the presence of viscosity effects, they must be changed, however, because there cannot be a slip between the liquid and its bounding surfaces. On the free surface we still have

$$\left.\begin{aligned}
\frac{\partial_f f}{\partial t} + u\,\frac{\partial_f f}{\partial x} + w\,\frac{\partial_f f}{\partial z} &= v \quad ; \\
p &= 0 \quad ,
\end{aligned}\right\} \tag{2.37a}$$

while on the ground

$$\underset{\sim}{v} = \underset{\sim}{0} \ . \tag{2.37b}$$

On the plate-water interface we must differentiate between the material points of the plate and those of the neighboring liquid.* That, there is no slip may be expressed by the statement that the material points of the plate at the interface possess the same velocity as those of the liquid. Let $\underset{\sim}{w}$ denote the velocity of the particles of the plate at the interface and $\underset{\sim}{v}$ that of the fluid particles at the immediate neighboring position.

Then the boundary condition is

$$\underset{\sim}{w} = \underset{\sim}{v} \ . \tag{2.37c}$$

Note that in case viscosity is discarded, (2.37b) and (2.37c) change and are replaced by

$$\begin{aligned} u \frac{\partial_g f}{\partial x} + w \frac{\partial_g f}{\partial z} &= v \qquad \text{on the ground} \\ \frac{\partial_p f}{\partial t} + u \frac{\partial_p f}{\partial x} + w \frac{\partial_p f}{\partial z} &= v \quad \text{at the plate} \ . \end{aligned} \tag{2.37d}$$

The above equations are now subjected to the following scalings:

$$\begin{aligned} x &= k\bar{x} \ ; \ z = k\bar{z} \ ; \ y = d\bar{y} \ ; \ \tau = \omega t \ ; \\ u &= V_o\bar{u} \ ; \ w = V_o\bar{w} \ ; \ v = V_1\bar{v} \ ; \ P = p_o\bar{p} \end{aligned} \tag{2.38}$$

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* Note that thus far we always have interpreted $y = {}_pf(\cdot)$ to be the representation of the material points of the fluid in their present configuration.

In the above, k and d are characteristic lengths, ω is a characteristic frequency, V_o and V_1 are characteristic velocities and p_o is a characteristic pressure. Substituting (2.38) into the equations (2.35) results in

$$\frac{d}{k}\frac{V_o}{V_1}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{w}}{\partial\bar{z}}\right) + \frac{\partial\bar{v}}{\partial\bar{y}} = 0 \quad ;$$

$$\omega V_o \frac{\partial\bar{u}}{\partial\tau} + \frac{V_o^2}{k}\left[\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}}\right] + \frac{V_1V_0}{d}\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} = -\frac{p_o}{\rho k}\frac{\partial\bar{p}}{\partial\bar{x}} + \frac{\eta V_o}{k^2}\left[\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{z}^2}\right]$$

$$+ \frac{\eta V_o}{d^2}\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \quad ;$$

$$\omega V_1 \frac{\partial\bar{v}}{\partial\tau} + \frac{V_oV_1}{k}\left[\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{v}}{\partial\bar{z}}\right] + \frac{V_1^2}{d}\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} = -\frac{p_o}{\rho d}\frac{\partial\bar{p}}{\partial\bar{y}} - g + \frac{\eta V_1}{k^2}\left[\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{\partial^2\bar{v}}{\partial\bar{z}^2}\right] \qquad (2.39)$$

$$+ \frac{\eta V_1}{d^2}\frac{\partial^2\bar{v}}{\partial\bar{y}^2} \quad ;$$

$$\omega V_o \frac{\partial\bar{w}}{\partial\tau} + \frac{V_o^2}{k}\left[\bar{u}\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}}\right] + \frac{V_oV_1}{d}\bar{v}\frac{\partial\bar{w}}{\partial\bar{y}} = -\frac{p_o}{\rho k}\frac{\partial\bar{p}}{\partial\bar{z}} + \frac{\eta V_o}{k^2}\left[\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \frac{\partial^2\bar{w}}{\partial\bar{z}^2}\right]$$

$$+ \frac{\eta V_o}{d^2}\frac{\partial^2\bar{w}}{\partial\bar{y}^2} \quad .$$

Moreover, for inviscid fluids, (2.36) becomes

$$\frac{\partial\bar{w}}{\partial\bar{y}} - \frac{V_1}{V_o}\frac{d}{k}\frac{\partial\bar{v}}{\partial\bar{z}} = 0 \;;$$

$$\frac{\partial\bar{u}}{\partial\bar{z}} - \frac{\partial\bar{w}}{\partial\bar{x}} = 0 \;; \tag{2.40}$$

$$\frac{\partial\bar{u}}{\partial\bar{y}} - \frac{V_1}{V_o}\frac{d}{k}\frac{\partial\bar{v}}{\partial\bar{x}} = 0 \;.$$

Similarly, we also scale the equations $y = {}_af(x,z,t)$, $a = p, g, f$, such that

$${}_af(x,z,t) = d_a\hat{f}(kx,kz,\frac{1}{\omega}\tau) = d_a\tilde{f}(\bar{x},\bar{z},\tau) \;.$$

With this representation, and with $\underset{\sim}{w} = (V_o\, w_x, V_1\, w_y, V_o\, w_z)$ the boundary conditions (2.37) assume the form

$$\underset{\sim}{v} = \underset{\sim}{0} \;; \quad \text{at the ground}$$

$$\underset{\sim}{w} = \underset{\sim}{v} \;; \quad \text{at the plate interface} \tag{2.41}$$

$$(\frac{\omega d}{V_1})\frac{\partial_f\tilde{f}}{\partial\tau} + \frac{V_o d}{V_1 k}\bar{u}\frac{\partial_f\tilde{f}}{\partial\bar{x}} + \frac{V_o d}{V_1 k}\bar{w}\frac{\partial_f\tilde{f}}{\partial\bar{z}} = \bar{v} \;; \quad \bar{p} = 0 \;; \quad \text{at the free surface}$$

when viscosity is accounted for, and

$$\frac{V_o d}{V_1 k}\bar{u}\frac{\partial_g\tilde{f}}{\partial\bar{x}} + \frac{V_o d}{V_1 k}\bar{w}\frac{\partial_g\tilde{f}}{\partial\bar{z}} = \bar{v} \;; \quad \text{on the ground}$$

$$(\frac{\omega d}{V_1})\frac{\partial_p\tilde{f}}{\partial\tau} + \frac{V_o d}{V_1 k}\bar{u}\frac{\partial_p\tilde{f}}{\partial\bar{x}} + \frac{V_o d}{V_1 k}\bar{w}\frac{\partial_p\tilde{f}}{\partial\bar{z}} = \bar{v} \;; \quad \text{at the plate} \tag{2.41a}$$

$$(\frac{\omega d}{V_1})\frac{\partial_f\tilde{f}}{\partial\tau} = \frac{V_o d}{V_1 k}\bar{u}\frac{\partial_f\tilde{f}}{\partial\bar{x}} + \frac{V_o d}{V_1 k}\bar{w}\frac{\partial_f\tilde{f}}{\partial\bar{z}} = \bar{v} \;; \quad p = 0 \;; \quad \text{at the free surface}$$

in the inviscid case.

In the following section we shall derive the approximate equations which hold for the shallow water approximations.

2.4) The Shallow Water Theory - Tidal Waves

In the preceding section we have scaled the Navier-Stokes or Euler equations together with their boundary conditions in such a way that the stretching in the horizontal direction was different from that in the vertical direction. Moreover, we have not specified the characteristic quantities as yet. These will be chosen in such a way that the well known shallow water theory results (see [6]).

First we neglect viscosity. But we do not assume that disturbances are small.

In the classical shallow water theory the scalings are chosen as follows:

$$p_o = \rho d g \ ;$$

$$V_1/V_o = k/d \ ; \ V_o V_1 = kg \ ; \tag{2.42}$$

and

$$\omega V_1 = g \ .$$

Accordingly, the reference pressure p_o is given by the hydrostatic pressure at some characteristic depth. The ratio of the reference velocities in the vertical and horizontal direction equals the inverse ratio of the scaling lengths. Finally, the frequency is scaled with g and V_1. It follows from $(2.42)_2$ that

$$V_o = \sqrt{gd} \ ; \ V_1 = (k/d)V_o$$

which means that horizontal velocities are scaled with the propagation speed as known from elementary hydrodynamics. For $O(k/d) \gg 1$, as we shall assume we have $\|V_1/V_o\| \gg O(1)$ and therefore $\|\bar{u}\| = O(1)$ and $\|\bar{w}\| = O(1)$, but $\|\bar{v}\| = o(1)$. Thus, the scaling (2.42) expresses the fact that horizontal velocities are more important than vertical velocities. Finally ω is scaled according to $(2.42)_3$ or $\omega = (d/k)\sqrt{g/d}$ which means that time processes are extremely small. Another choice could be

$$\Omega V_o = g \quad ; \quad \Omega = \sqrt{g/d} \quad ,$$

which means that the time evolution of the processes involved would be faster. With the scaling (2.42), the <u>inviscid</u> equations (2.39) and (2.40) assume the form

$$\epsilon\left[\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{w}}{\partial\bar{z}}\right] + \frac{\partial\bar{v}}{\partial\bar{y}} = 0 \quad ;$$

$$\epsilon\left[\frac{\partial u}{\partial\tau} + u\,\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{x}}\right] + v\,\frac{\partial\bar{u}}{\partial\bar{y}} = 0 \quad ;$$

$$\epsilon\left[\frac{\partial\bar{v}}{\partial\tau} + u\,\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\bar{v}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{y}} + 1\right] + \bar{v}\,\frac{\partial\bar{v}}{\partial\bar{y}} = 0 \quad ; \qquad (2.43)$$

$$\epsilon\left[\frac{\partial\bar{w}}{\partial\tau} + \bar{u}\,\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\bar{w}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{z}}\right] + \bar{v}\,\frac{\partial\bar{w}}{\partial\bar{y}} = 0 \quad ;$$

$$\frac{\partial\bar{w}}{\partial\bar{y}} = \frac{\partial\bar{v}}{\partial\bar{z}} \quad ; \quad \frac{\partial\bar{u}}{\partial\bar{z}} = \frac{\partial\bar{w}}{\partial\bar{x}} \quad ; \quad \frac{\partial\bar{v}}{\partial\bar{x}} = \frac{\partial\bar{u}}{\partial\bar{y}} \quad ,$$

where we have set

$$\epsilon = (d/k)^2 \ll 1 \quad . \qquad (2.44)$$

Similarly, the boundary conditions (2.41a) may be written as

$$\epsilon[\bar{u}\,\frac{\partial\,{}_g\tilde{f}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\,{}_g\tilde{f}}{\partial\bar{z}}] \;=\; \bar{v} \quad ; \quad \text{on the ground}$$

$$\epsilon[\frac{\partial\,{}_p\tilde{f}}{\partial\tau} + \bar{u}\,\frac{\partial\,{}_p\tilde{f}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\,{}_p\tilde{f}}{\partial\bar{z}}] \;=\; \bar{v} \quad ; \quad \text{at the plate} \tag{2.45}$$

$$\left.\begin{array}{l} \epsilon[\frac{\partial\,{}_f\tilde{f}}{\partial\tau} + \bar{u}\,\frac{\partial\,{}_f\tilde{f}}{\partial\bar{x}} + \bar{w}\,\frac{\partial\,{}_f\tilde{f}}{\partial\bar{z}}] \;=\; \bar{v} \quad ; \\ \\ \bar{p} \;=\; 0 \quad ; \end{array}\right\} \text{at the free surface} \;.$$

Hence, all equations have been reduced to a form depending on a single parameter ϵ. Because of the assumption (2.44) we construct solutions to (2.43) and (2.45) by a perturbation expansion. To this end we assume the power series developments for u, v, w, p, ${}_f f$ and ${}_p f$:

$$\begin{aligned} \bar{u} &= \sum_{n=o}^{\infty} \epsilon^n u_n \quad ; \quad \frac{\|u_{n+1}\|}{\|u_n\|} \to 0 \text{ as } \epsilon \to 0 \\ \bar{v} &= \sum_{n=o}^{\infty} \epsilon^n v_n \quad ; \quad \frac{\|v_{n+1}\|}{\|v_n\|} \to 0 \text{ as } \epsilon \to 0 \\ \bar{w} &= \sum_{n=o}^{\infty} \epsilon^n w_n \quad ; \quad \frac{\|w_{n+1}\|}{\|w_n\|} \to 0 \text{ as } \epsilon \to 0 \\ \bar{p} &= \sum_{n=o}^{\infty} \epsilon^n p_n \quad ; \quad \frac{\|p_{n+1}\|}{\|p_n\|} \to 0 \text{ as } \epsilon \to 0 \end{aligned} \tag{2.46}$$

$$ {}_f\tilde{f} = \sum_{n=o}^{\infty} \epsilon^n \, {}_f\tilde{f}_n \quad ; \quad \frac{\| {}_f\tilde{f}_{n+1} \|}{\| {}_f\tilde{f}_n \|} \to 0 \text{ as } \epsilon \to 0 $$

$$ {}_p\tilde{f} = \sum_{n=o}^{\infty} \epsilon^n \, {}_p\tilde{f}_n \quad ; \quad \frac{\| {}_p\tilde{f}_{n+1} \|}{\| {}_p\tilde{f}_n \|} \to 0 \text{ as } \epsilon \to 0 $$

and insert them in the equations (2.43) and (2.45). Upon comparing coefficients of like powers in ϵ, equations for the successive coefficients in the series, which are, of course functions of x, y, z and t are obtained. The terms of zeroth order yield the equations

$$ \frac{\partial v_o}{\partial \bar{y}} = 0 \ ; $$

$$ v_o \frac{\partial u_o}{\partial \bar{y}} = 0 \ ; \quad v_o \frac{\partial v_o}{\partial \bar{y}} = 0 \ ; \quad v_o \frac{\partial w_o}{\partial \bar{y}} = 0 \ ; $$

$$ \frac{\partial w_o}{\partial \bar{y}} - \frac{\partial \bar{v}_o}{\partial \bar{z}} = 0 \ ; \quad \frac{\partial u_o}{\partial \bar{z}} - \frac{\partial w_o}{\partial \bar{x}} = 0 \ ; \quad \frac{\partial v_o}{\partial \bar{x}} - \frac{\partial u_o}{\partial \bar{y}} = 0 \ ; $$

and

$$ \begin{aligned} v_o &= 0 \ ; \ p_o = 0 \ ; && \text{at } y = {}_f\tilde{f}_o(x,z) \\ v_o &= 0 \ ; && \text{at } y = {}_g\tilde{f}(x,z) \qquad (2.47) \\ v_o &= 0 \ ; && \text{at } y = {}_p\tilde{f}(x,z) \ . \end{aligned} $$

It follows that

$$v_o = 0 \ ;$$

$$w_o = w_o(x,z,t) \ ;$$

$$u_o = u_o(x,z,t) \ ; \qquad (2.48)$$

$$p_o(x, {}_f f_o, z, t) = 0 \ ,$$

which contains the important result that the vertical velocity component is zero and the horizontal components are , to the lowest order independent of the vertical coordinate y . Note that at this level of determination no variable is fixed except v_o.

The first order terms in turn yield the system of equations

$$\frac{\partial u_o}{\partial \bar{x}} + \frac{\partial w_o}{\partial \bar{z}} = - \frac{\partial v_1}{\partial \bar{y}} \ ;$$

$$\frac{\partial u_o}{\partial \tau} + u_o \frac{\partial u_o}{\partial \bar{x}} + w_o \frac{\partial u_o}{\partial \bar{z}} + \frac{\partial p_o}{\partial \bar{z}} = 0 \ ; \qquad (2.49)$$

$$\frac{\partial p_o}{\partial \bar{y}} + 1 = 0 \ ;$$

$$\frac{\partial w_o}{\partial \tau} + u_o \frac{\partial w_o}{\partial \bar{x}} + w_o \frac{\partial w_o}{\partial \bar{z}} + \frac{\partial p_o}{\partial \bar{z}} = 0$$

and

$$\left.\begin{aligned} \frac{\partial_f f_o}{\partial \tau} + u_o \frac{{}_f\tilde{f}_o}{\partial \bar{x}} + w_o \frac{{}_f\tilde{f}_o}{\partial \bar{z}} &= v_1 \ ; \\ p_1 &= 0 \ ; \end{aligned}\right\} \text{ at } y = {}_f f_o = 0$$

$$\frac{\partial\, {}_p f_o}{\partial \tau} + u_o \frac{\partial\, {}_p f_o}{\partial \bar{x}} + w_o \frac{\partial\, {}_p f_o}{\partial \bar{z}} = v_1 \quad ; \quad \text{at} \quad y = {}_p f_o = 0$$

$$u_o \frac{\partial\, {}_g f}{\partial \bar{x}} + w_o \frac{\partial\, {}_g f}{\partial \bar{z}} + v_1 = 0 \quad ; \quad \text{at} \quad y = {}_g f \ .$$

In view of (2.48), equations $(2.49)_1$ can be integrated directly to give

$$v_1 = -[\frac{\partial u_o}{\partial \bar{x}} + \frac{\partial w_o}{\partial \bar{z}}]y + F(\bar{x},\bar{z},\tau) \ . \tag{2.50}$$

Using the boundary condition at the ground, $(2.49)_8$, $F(\cdot)$ may be determined. Inserting the result into (2.50), we obtain

$$v_1 = -[\frac{\partial u_o}{\partial \bar{x}} + \frac{\partial w_o}{\partial \bar{z}}]\bar{y} - [\frac{\partial}{\partial \bar{x}}(u_o\ {}_g f(\cdot)) + \frac{\partial}{\partial \bar{z}}(w_o\ {}_f f(\cdot))]\Big|_{y={}_g\tilde{f}} \ . \tag{2.51}$$

In a similar fashion the third equation (2.49) can be integrated to give

$$p_o(\bar{x},\bar{y},\bar{z},t) = -\bar{y} + \pi(\bar{x},\bar{z},t) \tag{2.52a}$$

where $\pi(\cdot)$ must be determined from the upper boundary conditions. This yields

$$p_o(\bar{x},\bar{y},\bar{z},t) = -\bar{y} + {}_f f(\bar{x},\bar{z},t) \tag{2.52b}$$

on the free surface, while $\pi(\cdot)$ must remain undetermined below the plate. Substituting (2.51) and (2.52) into $(2.49)_{2,4,5}$ we obtain

$$\frac{\partial u_o}{\partial t} + u_o \frac{\partial u_o}{\partial \bar{x}} + w_o \frac{\partial u_o}{\partial \bar{z}} + \frac{\partial\, {}_f f_o}{\partial \bar{z}} = 0 \;;$$

$$\frac{\partial w_o}{\partial t} + u_o \frac{\partial w_o}{\partial \bar{x}} + w_o \frac{\partial w_o}{\partial \bar{z}} + \frac{\partial\, {}_f f_o}{\partial \bar{z}} = 0 \;; \qquad (2.53)$$

$$\frac{\partial\, {}_f f_o}{\partial t} + \frac{\partial}{\partial \bar{x}}[u_o({}_f f_o + {}_g f)] + \frac{\partial}{\partial \bar{z}}[u_o({}_f f_o + {}_g f)] = 0 \;,$$

the governing system of equations valid for that domain $(\bar{x}, \bar{z})$, where the surface is free of ice.

On the other hand, below the ice plate we have

$$\frac{\partial u_o}{\partial t} + u_o \frac{\partial u_o}{\partial \bar{x}} + w_o \frac{\partial u_o}{\partial \bar{z}} + \frac{\partial \pi}{\partial \bar{x}} = 0 \;;$$

$$\frac{\partial w_o}{\partial t} + u_o \frac{\partial w_o}{\partial \bar{x}} + w_o \frac{\partial w_o}{\partial \bar{z}} + \frac{\partial \pi}{\partial \bar{z}} = 0 \;; \qquad (2.54)$$

$$\frac{\partial\, {}_p f_o}{\partial t} + \frac{\partial}{\partial \bar{x}}[u_o({}_p f_o + {}_g f)] + \frac{\partial}{\partial \bar{z}}[u_o({}_p f_o + {}_g f)] = 0 \;,$$

where these equations still need to be coupled with the governing equations for the plate.

The system (2.53) and (2.54), complemented by an equation for the plate must be connected by continuity conditions. Along the ice-water separation curve $\mathcal{C}$, u_o and w_o should be continuous. Moreover, there should not be a jump in pressure at that position. With reference to Fig. (2.2) and the formulas (2.52) this condition may be expressed as

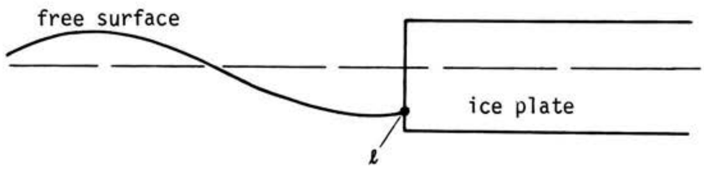


Fig 2.2

$$\pi(x,z,t)\Big|_{\ell} = {}_{f}f(x,z,t)\Big|_{\ell} \quad .$$

Further conditions are to be satisfied on the plate, (moment shear force).

In concluding this subsection we summarize: By a straightforward stretching of the coordinates and a regular perturbation approach to the stretched inviscid hydrodynamic equations we have been able to establish a set of nonlinear partial differential equations (2.53) and (2.54) governing the motion of the water having free surface or being covered by a flexible plate. The sets of equations for positions where the water surface is free or covered by the ice plate are different, but they can be connected by a set of continuity conditions, which are the following: the velocity and the pressure must not jump at the separating curve ℓ of ice and water.

2.5) Shallow Water Viscous Fluid, Rivers

In this section we collect the equations which govern the fluid motion in rivers when they are covered by ice sheets. That the flow patterns underneath an ice plate and in a river with free surface are different has already been mentioned earlier.

A reasonable description of the situation dealt with here, however, cannot be achieved on the basis of the shallow water approximation. Because the motion induced by the deformation of the plate is small as compared to the drift velocities when a rigid immobile plate is assumed it seems reasonable to perturb the full Navier-Stokes equations about the undeformed state. With the choice k = d in the scalings (2.42) the continuity and momentum equations assume the form

$$\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}} + \frac{\partial\bar{w}}{\partial\bar{z}} = 0 \;;$$

$$\frac{\partial\bar{u}}{\partial\tau} + \bar{u}\,\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\,\frac{\partial\bar{u}}{\partial\bar{y}} + \bar{w}\,\frac{\partial\bar{u}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{x}} + \sin\alpha = \frac{1}{R}(\nabla^2\bar{u}) \;;$$

$$\frac{\partial\bar{v}}{\partial\tau} + \bar{u}\,\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\,\frac{\partial\bar{v}}{\partial\bar{y}} + \bar{w}\,\frac{\partial\bar{v}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{y}} + \cos\alpha = \frac{1}{R}(\nabla^2\bar{v}) \;; \qquad (2.55)$$

$$\frac{\partial\bar{w}}{\partial\tau} + \bar{u}\,\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{v}\,\frac{\partial\bar{w}}{\partial\bar{y}} + \bar{w}\,\frac{\partial\bar{w}}{\partial\bar{z}} + \frac{\partial\bar{p}}{\partial\bar{z}} = \frac{1}{R}(\nabla^2\bar{w}) \;.$$

In the above equations, R is the Reynolds number defined by

$$R = \frac{\sqrt{gd}\;d}{\eta} \;, \qquad (2.56)$$

where d is the scaling length which may conveniently be chosen to be an average water depth. Furthermore, the coordinate system has been chosen as indicated in Fig. 2.2. The angle α is thus a measure for

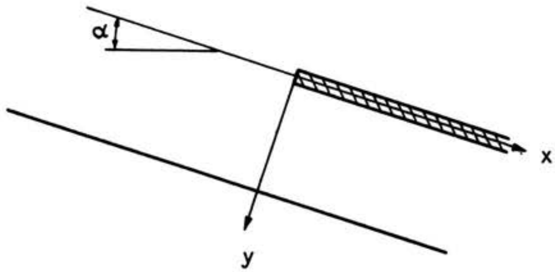


Fig 2.2

the average inclination of the water surface.

We now assume the following decomposition of the field variables

$$\bar{\underset{\sim}{v}} = \bar{\underset{\sim}{v}}^{(0)} + \bar{\underset{\sim}{v}}^{(1)} \quad ; \quad \bar{p} = \bar{p}^{(0)} + \bar{p}^{(1)} \quad . \tag{2.57}$$

Here, $\underset{\sim}{v}^{(0)}$ is the velocity field due to gravitational forces but in presence of an ice cover, which is assumed to be rigid. $\underset{\sim}{v}^{(1)}$, on the other hand, is the correction due to the deformation of the ice sheet. We assume that $v^{(1)} \ll v^{(0)}$ and may therefore neglect all products of $v^{(1)}$ with itself; similarly for the pressure, but no smallness of $p^{(1)}$ must be assumed. Substituting (2.57) into (2.55), assuming no deformation of the ice sheet yields again a system of the form (2.55) where the velocity $\underset{\sim}{v} = (u,v,w)$ is replaced by $v^{(0)} = (u^{(0)},v^{(0)}, w^{(0)})$. This system is

$$\frac{\partial\bar{u}^{(0)}}{\partial\bar{x}} + \frac{\partial\bar{v}^{(1)}}{\partial\bar{y}} + \frac{\partial\bar{w}^{(0)}}{\partial\bar{z}} = 0 \quad :$$

$$\frac{\partial\bar{u}^{(0)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{y}} + \bar{w}^{(0)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(0)}}{\partial\bar{x}} + \sin\alpha = \frac{1}{R}\nabla^2\bar{u}^{(0)} \quad ;$$

$$\frac{\partial\bar{v}^{(0)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{v}^{(0)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{v}^{(0)}}{\partial\bar{y}} + \bar{w}^{(0)}\frac{\partial\bar{v}^{(0)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(0)}}{\partial\bar{y}} + \cos\alpha = \frac{1}{R}\nabla^2\bar{v}^{(0)} \quad ;$$

$$\frac{\partial\bar{w}^{(0)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{y}} + \bar{w}^{(0)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(0)}}{\partial\bar{z}} = \frac{1}{R}\nabla^2\bar{w}^{(0)} \quad ;$$

(2.58)

with the corresponding boundary conditions

$$v^{(0)} = 0 \quad ; \quad \text{on the ground and at the ice-water interface}$$

$$\left.\begin{array}{l}\text{pressure}\\ \text{shear stress}\end{array}\right\} = 0 \quad ; \quad \text{at the free surface} \quad .$$

(2.59)

The equations for the perturbed quantities on the other hand are

$$\frac{\partial\bar{u}^{(1)}}{\partial\bar{x}} + \frac{\partial\bar{v}^{(1)}}{\partial\bar{y}} + \frac{\partial\bar{w}^{(1)}}{\partial\bar{z}} = 0$$

$$\frac{\partial\bar{u}^{(1)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{u}^{(1)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{u}^{(1)}}{\partial\bar{x}} + \bar{w}^{(0)}\frac{\partial\bar{u}^{(1)}}{\partial\bar{z}} + \bar{u}^{(1)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{x}} + \bar{v}^{(1)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{y}}$$

$$+ \bar{w}^{(1)}\frac{\partial\bar{u}^{(0)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(1)}}{\partial\bar{x}} = \frac{1}{R}(\nabla^2\bar{u}^{(1)})$$

(2.60)

$$\frac{\partial\bar{v}^{(1)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{v}^{(1)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{v}^{(1)}}{\partial\bar{y}} + \bar{w}^{(0)}\frac{\partial\bar{v}^{(1)}}{\partial\bar{z}} + \bar{u}^{(1)}\frac{\partial\bar{v}^{(0)}}{\partial\bar{x}} + \bar{v}^{(1)}\frac{\partial\bar{v}^{(1)}}{\partial\bar{y}} + \bar{w}^{(1)}\frac{\partial\bar{v}^{(1)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(1)}}{\partial\bar{y}} = \frac{1}{R}(\nabla^2\bar{v}^{(1)})$$

$$\frac{\partial\bar{w}^{(1)}}{\partial\tau} + \bar{u}^{(0)}\frac{\partial\bar{w}^{(1)}}{\partial\bar{x}} + \bar{v}^{(0)}\frac{\partial\bar{w}^{(1)}}{\partial\bar{y}} + \bar{w}^{(0)}\frac{\partial\bar{w}^{(1)}}{\partial\bar{z}} + \bar{u}^{(1)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{x}} + \bar{v}^{(1)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{y}} + \bar{w}^{(1)}\frac{\partial\bar{w}^{(0)}}{\partial\bar{z}} + \frac{\partial\bar{p}^{(1)}}{\partial\bar{z}} = \frac{1}{R}(\nabla^2\bar{w}^{(1)}) \quad .$$

Further simplification is achieved if the viscosity term in (2.60) is neglected. This is justified, because the zeroth order solution chiefly bears boundary layer character and this latter is contained in (2.60) through the terms $\bar{u}^{(0)}$, $\bar{v}^{(0)}$ and $\bar{w}^{(0)}$.

The boundary conditions for this first order inviscid approximation therefore are obtained as follows:

$$\bar{u}^{(1)}\frac{\partial_g\tilde{f}}{\partial\bar{x}} + \bar{w}^{(1)}\frac{\partial_g\tilde{f}}{\partial\bar{z}} = \bar{v}^{(1)} \qquad \text{on the ground}$$

$$\frac{\partial_p\tilde{f}}{\partial\tau} + \bar{u}^{(1)}\frac{\partial_p\tilde{f}}{\partial\bar{x}} + \bar{w}^{(1)}\frac{\partial_p\tilde{f}}{\partial\bar{z}} = \bar{v}^{(1)} \qquad \text{at the plate-water interface} \quad (2.61)$$

$$\left.\begin{aligned} &\frac{\partial_f\tilde{f}}{\partial\tau} + \bar{u}^{(1)}\frac{\partial_f\tilde{f}}{\partial\bar{x}} + \bar{w}^{(1)}\frac{\partial_f\tilde{f}}{\partial\bar{z}} = \bar{v}^{(1)} \; ; \\ &\bar{p}^{(1)} = 0 \; ; \end{aligned}\right\} \qquad \text{at the free surface} \quad .$$

These boundary conditions express that there is at most a slip of the velocity tangential to the boundary ${}_g\tilde{f}$, ${}_p\tilde{f}$ and ${}_f\tilde{f}$ are the equations $\bar{y} = \tilde{f}(\bar{x},\bar{y},\bar{z})$ of the ground, plate-water interface and the free surface, respectively.

Chapter 3.) The Plate Equations:

3.1) Introduction

In the last Chapter we studied the hydrodynamic equations which for various situations describe the interaction of ice with water. Here in this Section we list the equations for the ice plate. Without being exhaustive we nevertheless attempt to be general enough to cover most situations occuring in this field.

With regard to various physical aspects the description of the ice cover is much more complex than the one of its underlying water. First of all, we no longer assume that the ice plate is in an isothermal state as it was done for the water. This is so, because the water-ice interface is at the freezing temperature while its upper surface depends on the air temperature and may vary quite considerably. The time scale of these thermal processes is much larger than the one of the wave motion so that it seems justified to neglect true thermoelastic effects and to simply assume that the plate material constants are inhomogeneous. All material coefficients depend upon a parameter - the temperature - which in turn is a function of position.

Apart from this complication in the physical description of the ice plate other effects should also be included. Due to the thermal stress effects prestress either as tension or pressure, may occur and a proper description should also take these effects into account. Moreover, the freezing process is such that various degrees of anisotropy are induced in the actual ice plate, which in general thus should not be treated as an isotropic material. Furthermore, the thickness of ice plates is rarely constant and this should imply that only a full three dimensional theory could describe the dynamic response of ice plates. Clearly such a description would be very complex.

The treatments in the existing literature are generally based upon the theory of thin elastic plates, governed by the equation

$$D\nabla^4 W + p + m\frac{\partial^2 W}{\partial t^2} = q \tag{3.1}$$

Here, D is an effective plate rigidity, m the mass per unit area, p the pressure which the liquid base exerts upon the bottom surface of the plate and q represents the static intensity as well as the inertia of the external load. In accord with the basic small deflection assumption we have discarded the convective acceleration in (3.1).

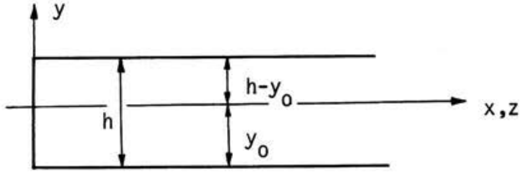
The effective plate rigidity D in general is not the classical plate rigidity as we allow the modulus of elasticity to be a function of position. In case its variation is only in the y direction it has been shown [2] that

$$D = \frac{1}{1-\nu^2}\int_{-y_o}^{h-y_o} y^2 E(y)\,dy \tag{3.2}$$

where y_o is determined from (see Fig. 3.1)

$$\int_{-y_o}^{h-y_o} yE(y)\,dy = 0$$

Fig 3.1



Historically, the equation (3.1) was introduced by Hertz in 1888 to describe the behavior of floating ice sheets. Within this context it has not been modified ever since, except in an article by Nevel [1], who applies (3.1) to a Maxwell type visco-elastic material. He shows by comparison with experiments that for external loads, slowly varying with time the visco-elastic treatment is essential.

The technique to derive the governing equations of two dimensional elasticity, which we shall use is one which has become increasingly fashionable in recent years. The plate equations shall be derived by a "smearing procedure" of the governing equations of three dimensional elasticity. The method goes back to Cauchy [23], but has increasingly been applied in the recent past. Mindlin was the first to derive the two dimensional plate equations by a Cauchy series expansion from the equations of three dimensional elasticity. Mindlin and Medick [11] derive a linear theory of plates using expansion procedures. Gol'denveizer, [15], discusses the possible aspects of such asymptotic or iterative methods. Similar methods are used by Green, Laws and Naghdi [24] and by Dökmeci [19], [20] and Dökmeci and Hutter [21] in other contexts of general plate and shell theories.

The major objective in this present Chapter is a consistent derivation of equations similar to those of von Kàrmàn's plate equations. The conventional Kirchhoff-Love hypothesis, that is the assumption that directors perpendicular to the middle surface remain perpendicular under deformation, is abrogated. The theory is developed by means of the Hamiltonian principle and a separation of variables technique is used by which the three dimensional field equations are converted into two dimensional ones. A sequence of approximate equations which include the effects of transvers shear and normal strains, acceleration and rotatory inertia is thus consistently constructed. The governing equations consist of the macroscopic equations of motion together with the relevant boundary conditions, the constitutive equations, the strain displacement relations and the strain energy expression.

3.2 Kinematic Variables:

Consider an open regular region $\mathfrak{D}$ with boundary $\partial\mathfrak{D}$ in three dimensional Euclidean space $\mathcal{E}^3$. Let $\partial\mathfrak{D} = \partial\mathfrak{D}_U \cup \partial\mathfrak{D}_L \cup \partial\mathfrak{D}_\epsilon$ where $\partial\mathfrak{D}_U$, $\partial\mathfrak{D}_L$ and $\partial\mathfrak{D}_\epsilon$ are, respectively, the upper and lower faces and the edge boundary face. Clearly,

$$\mathfrak{D} \cup \partial\mathfrak{D} = \overline{\mathfrak{D}} \;;$$
$$\partial\mathfrak{D}_U \cap \partial\mathfrak{D}_L = \emptyset \;;$$
$$\partial\mathfrak{D}_L \cap \partial\mathfrak{D}_\epsilon = \emptyset \;;$$
$$\partial\mathfrak{D}_U \cap \partial\mathfrak{D}_\epsilon = \emptyset \;,$$

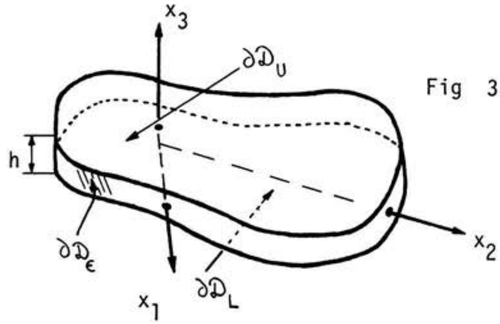


Fig 3.2

where $\overline{\mathfrak{D}}$ is the closure of $\mathfrak{D}$. The edge boundary surface is taken as a cylindrical surface perpendicular to the flat middle surface of the undeformed plate. The plate in its reference configuration is referred to a right handed coordinate system of which the first two axes lie in the undeformed reference surface. The positive direction of the third coordinate, x_3 is taken to be upward and $x_3 = 0$ coincides with a plane between the upper and lower faces, the position of which will be determined in the course of calculations.

We assume the thickness of the plate to be much smaller than any of its dimensions in the (x_1,x_2)-plane. Denoting the smallest value of the length dimension in the horizontal reference plane by L we thus have

$$d/L \ll 1 \;. \tag{3.3}$$

This assumption allows us to treat the plate as a two dimensional model of a three dimensional deformable body. Moreover, assumption (3.3) implies that stress and displacement fields do not vary violently across the thickness of the plate.

In the subsequent calculations we shall choose the Lagrangian description for the motion of the plate. All coordinates are consequently taken in the reference frame.

On the above basis the displacement components of a generic point in $\bar{\mathfrak{D}}$ can be represented as

$$u_i(x_k,t) = \sum_{m=o}^{\infty} P_m(x_3)u_i^{(m)}(x_\alpha,t) \quad ; \quad i,\ k = 1,\ 2,\ 3 \quad ; \quad \alpha = 1,2 \tag{3.4}$$

with

$$P_o(x_3) = 1 \quad ; \quad P_1(x_3) = x_3 \quad . \tag{3.5}$$

Here, from the mathematical point of view, a separation of variable solution is sought for the nonlinear field equations, which are presented in the following Sections. Therefore the vector functions $u_i^{(m)}$ are unknown a priori and independent functions defined on $\bar{\mathfrak{D}}$. Moreover, it is assumed that the functions $u_i^{(m)}$ exist, are single valued and of class C^2 at least.

In the final analysis only the two functions P_o and P_1 will be used. They will be considered to be taken from the set $P_m(x_3) = x_3^m$. If the displacement vector $\underset{\sim}{u}$ is analytic in $\bar{\mathfrak{D}}$ with respect to the coordinate x_3 then (3.2) can be interpreted as a Taylor series expansion of $\underset{\sim}{u}$ about $x_3 = 0$ which is uniformly convergent in $\bar{\mathfrak{D}}$. However, in our case $\underset{\sim}{u}^{(m)}$ are independent. In fact, in (3.4), P_m

could be any other convenient functions such as Legendre or Jacobi polynomials.

By virtue of the representation (3.4) the Kirchhoff-Love hypothesis is eliminated i.e. directors perpendicular to the middle surface need not remain perpendicular to the deformed middle surface in the course of the motion.

3.3 Strain Displacement Relations:

The right Cauchy-Green deformation tensor, denoted by C_{ij} is expressed in terms of the displacement components [9] by

$$C_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \delta_{ij} \tag{3.6}$$

and

$$C_{ij} = 2\tilde{E}_{ij} + [\tilde{E}_{ki}\tilde{R}_{kj}+\tilde{E}_{kj}\tilde{R}_{ki}+\tilde{E}_{ik}\tilde{R}_{jk}+\tilde{R}_{ki}\tilde{R}_{kj}] + \delta_{ij} \tag{3.7}$$

with

$$\begin{aligned} \tilde{E}_{ij} &= \tfrac{1}{2}(u_{i,j}+u_{j,i}) = u_{(i,j)} \quad ; \\ \tilde{R}_{ij} &= \tfrac{1}{2}(u_{i,j}-u_{j,i}) = u_{[i,j]} \quad . \end{aligned} \tag{3.8}$$

Following Truesdell and Toupin [9] we do not neglect in (3.7) the first two terms in the bracket as was done by Novoshilov [25].

The series expansions in all displacement components as assumed in (3.4) imply that the Cauchy-Green deformation tensor be of the following form:

$$C_{ij} = \delta_{ij} + \sum_{m=o}^{\infty} x_3^m C_{ij}^{(m)}(x_\alpha, t) \quad ; \tag{3.9}$$

where

$$\begin{aligned} C_{ij}^{(m)} = {} & u_{i,\alpha}^{(m)}\delta_{j\alpha} + u_{j,\alpha}^{(m)}\delta_{i\alpha} + (u_i^{(m+1)}\delta_{3j} + u_j^{(m+1)}\delta_{3i})(m+1) \\ & + \sum_{p=0}^{m}[u_{k,\alpha}^{(m-p)}u_{k,\beta}^{(p)}\delta_{i\alpha}\delta_{j\beta}+(m-p+1)u_k^{(m-p+1)}u_{k,\beta}^{(p)}\delta_{i3}\delta_{j\beta} \\ & +(p+1)u_{k,\alpha}^{(m-p)}u_k^{(p+1)}\delta_{i\alpha}\delta_{j3}+(p+1)(m-p+1)u_k^{(m-p+1)}u_k^{(p+1)}\delta_{i3}\delta_{j3}] \end{aligned} \tag{3.10a}$$

is a measure of the strain of order (m). Equivalently $u_i^{(m)}$ is the displacement field of order (m).

The representation (3.6) is exact. In the event where shear deformation and extension are small, however, $\tilde{E}_{ij}$ is small as compared to $\tilde{R}_{ij}$. Then the terms $\tilde{E}_{ij}\tilde{E}_{kj}$, $\tilde{E}_{kj}\tilde{R}_{ki}$ and $\tilde{E}_{ki}\tilde{R}_{kj}$ may be neglected as compared to $\tilde{R}_{ki}\tilde{R}_{kj}$. In this case the Cauchy-Green deformation tensor still assumes the form (3.9) but (3.10a) is now approximated by

$$C_{ij}^{(m)} = u_{i,\alpha}^{(m)}\delta_{j\alpha} + u_{j,\alpha}^{(m)}\delta_{i\alpha} + (m+1)[u_i^{(m+1)}\delta_{3j}+u_j^{(m+1)}\delta_{3j}] + \frac{1}{4}[\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}] \tag{3.10b}$$

where

$$\begin{aligned}
\mathrm{I} = \sum_{p=o}^{m} [&u_{k,\alpha}^{(m-p)}u_{k,\beta}^{(p)}\delta_{i\alpha}\delta_{j\beta}+(m-p+1)u_k^{(m-p+1)}u_{k,\beta}^{(p)}\delta_{i3}\delta_{j\beta} \\
&+(p+1)u_{k,\alpha}^{(m-p)}u_k^{(p+1)}\delta_{i\alpha}\delta_{j3}+(p+1)(m-p+1)u_k^{(m-p+1)}u_k^{(p+1)}\delta_{i3}\delta_{j3}] \\
\mathrm{II} = \sum_{p=o}^{m} [&u_{i,\beta}^{(m-p)}u_{\beta,\gamma}^{(p)}\delta_{j\gamma}+(m-p+1)u_{3;\gamma}^{(p)}u_i^{(m-p+1)}\delta_{j\gamma} \\
&+(m-p+1)(p+1)u_i^{(m-p+1)}u_3^{(p+1)}\delta_{j3}] \qquad (3.10c) \\
\mathrm{III} = \sum_{p=o}^{m} [&u_{\gamma,\beta}^{(m-p)}u_{j,\gamma}^{(p)}\delta_{i\beta}+(m-p+1)u_\beta^{(m-p+1)}u_{j,\beta}^{(p)}\delta_{i3}+(p+1)u_{3,\beta}^{(m-p)}u_j^{(p+1)}\delta_{i\beta}\delta_{k3} \\
&+(p+1)(m-p+1)u_3^{(m-p+1)}u_j^{(p+1)}\delta_{i3}] \\
\mathrm{IV} = \sum_{p=o}^{m} [&u_{i,\beta}^{(m-p)}u_{j,\beta}^{(p)}+(m-p+1)(m-p+1)u_i^{(m-p+1)}u_j^{(p+1)}\delta_{33}] \quad .
\end{aligned}$$

In this form the strain displacement relations for the approximate theory look even more complicated than in the exact theory. In simplifying situations however it is more convenient.

An even simpler approximate theory emerges when the Cauchy-Green deformation tensor is approximated by

$$C_{ij} \cong \delta_{ij} + u_{i,j} + u_{j,i} + u_{3,\alpha} u_{3,\beta} \delta_{i\alpha} \delta_{j\beta} \quad . \tag{3.9a}$$

The representation (3.9) still remains correct, but now

$$C_{ij}^{(m)} = u_{i,\alpha}^{(m)} \delta_{j\alpha} + u_{j,\alpha}^{(m)} \delta_{i\alpha} + (m+1) u_i^{(m+1)} \delta_{3j} + (m+1) u_j^{(m+1)} \delta_{i3}$$
$$+ \sum_{p=o}^{m} u_{3,\alpha}^{(m-p)} u_{3,\beta}^{(p)} \delta_{i\alpha} \delta_{j\beta} \quad . \tag{3.10d}$$

In thin plates with moderately large displacements it is appropriate to retain only first order terms and of the nonlinearities of (3.10d) only the zeroth order terms; then

$$\frac{1}{2} C_{\alpha\beta}^{(o)} = E_{\alpha\beta}^{(o)} = u_{(\alpha,\beta)}^{(o)} + \frac{1}{2} \frac{\partial\eta}{\partial x_\alpha} \frac{\partial\eta}{\partial x_\beta}$$
$$\frac{1}{2} C_{13}^{(o)} = E_{13}^{(o)} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x_1} - \varphi\right) \quad ; \tag{3.10e}$$
$$\frac{1}{2} C_{23}^{(o)} = E_{23}^{(o)} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - \psi\right) \quad ;$$

and

$$\tfrac{1}{2}C_{11}^{(1)} = E_{11}^{(1)} = -\frac{\partial\varphi}{\partial x} ;$$

$$\tfrac{1}{2}C_{22}^{(1)} = E_{22}^{(1)} = -\frac{\partial\psi}{\partial y} ; \tag{3.10e}$$

$$\tfrac{1}{2}C_{12}^{(1)} = E_{12}^{(1)} = \tfrac{1}{2}\left(\frac{\partial\psi}{\partial x}+\frac{\partial\varphi}{\partial y}\right) ,$$

with

$$\varphi = -u_1^{(1)} ; \quad \psi = -u_2^{(1)} ; \quad \eta = u_3^{(0)} , \tag{3.10f}$$

while

$$C_{33}^{(0)} = u_3^{(1)} .$$

We shall call this approximation the von Kàrmàn approximation, because it corresponds to the one occuring in his plate theory, [26].

The structure of the formulas (3.9) and (3.10) is so complex that an interpretation in physical terms is not easily available. Some limiting situations can, however, be interpreted. The most important feature of the representation (3.4) emanates from (3.7). Its linear version gives

$$C_{ij} = 2\tilde{E}_{ij} + \delta_{ij} ; \quad C_{ij}^{(m)} = 2\tilde{E}_{ij}^{(m)}$$

with

$$2\tilde{E}_{ij}^{(m)} = u_{i,j}^{(m)} + u_{j,i}^{(m)} .$$

3.4) The Equations of Balance:

The dynamic equations of balance of mass, linear momentum, moment of momentum and energy may essentially be stated in two different forms, dependent upon, whether they are referred to the reference or present configuration. Here, where we have chosen the reference configuration they assume the following form:

Balance of mass:

$$\rho_o = \rho J \quad ; \qquad (3.11)$$

Balance of linear momentum:

$$\rho_o \ddot{u}_i = T_{ik,k} + \rho_o F_i \quad ; \qquad (3.12)$$

Balance of moment of momentum:

$$F_{ik} T_{jk} = F_{jk} T_{ik} \quad ; \qquad (3.13)$$

Balance of energy:

$$\rho_o \dot{\epsilon} = T_{ik} \dot{F}_{i,k} - Q_{k,k} + \rho_o r \quad . \qquad (3.14)$$

ρ_o denotes the density in the reference configuration, while ρ is the density in the present configuration. F_i and T_{ij} are the body force and Piola-Kirchhoff stress tensor, respectively, Q_k is the heat flux vector, r the energy supply and ϵ the internal energy density. Moreover,

$$F_{ik} = u_{i,k} + \delta_{ik} \quad ; \quad J = \mathrm{Det}(F_{ik}) \qquad (3.15)$$

and

$$T_{ik} = F_{i\ell}\Sigma_{k\ell} \tag{3.16}$$

where Σ_{ik} is the symmetric second Piola-Kirchhoff stress tensor adopting the definition of Truesdell and Noll, [10].

We shall henceforth neglect heat sources and restrict our considerations to processes for which $Q_{k,k} = 0$. The balance laws of moment of momentum and energy are then satisfied identically once the constitutive equations are formulated. The only remaining field equation is then (3.12), because the balance law of mass can be considered to be an equation for ρ.

Let $\underset{\sim}{t}^*$ and $\underset{\sim}{u}^*$ be the prescribed values of the stress and displacement vectors on the boundary surface. More precisely, let $\partial\mathfrak{D}_u$ and $\partial\mathfrak{D}_\sigma$ be disjoint sets of the boundary surface such that $\partial\mathfrak{D} = \partial\mathfrak{D}_u \cup \partial\mathfrak{D}_\sigma$; then the boundary conditions can be written in the form

$$u_k^* = u_k = 0 \ ; \qquad (x_1,x_2)\epsilon\partial\mathfrak{D}_u \tag{3.17}$$

and

$$t_k^* - t_k = 0 \ ; \qquad (x_1,x_2)\epsilon\partial\mathfrak{D}_\sigma \tag{3.18}$$

with

$$t_k = T_{k\ell}N_\ell \tag{3.19}$$

where N_ℓ is the normal vector in the reference configuration.

We proceed to formulate the variational principle. To this end, let t_1 and t_2 be two arbitrary but fixed times such that $t_2 > t_1$ and let δ indicate variation. Then it follows from (3.12), (3.17) and

(3.18) that

$$\delta\mathfrak{J} = \delta\mathfrak{J}_1 + \delta\mathfrak{J}_2 + \delta\mathfrak{J}_3$$

$$\delta\mathfrak{J} = \int_{t_1}^{t_2} dt \left\{ \int_{\mathfrak{D}} [T_{ik,k} - \rho_o(\ddot{u}_i - F_i)]\delta u_i dv + \int_{\partial\mathfrak{D}_u} (u_k^* - u_k)\delta t_k ds + \int_{\partial\mathfrak{D}_\sigma} (t_k^* - t_k)\delta u_k ds \right\} \tag{3.20}$$

is equivalent to the local equations (3.12), (3.17) and (3.18). In fact, because the variations δt_k and δu_i are arbitrary they can be varied independently, implying that the coefficients of δu_i and δt_k must vanish separately over the body $\mathfrak{D}$ and the boundary $\partial\mathfrak{D}_\sigma$ and $\partial\mathfrak{D}_u$, respectively. In fact, choosing $\delta t_k = 0$ and δu_k with compact support in $\bar{\mathfrak{D}}$ * implies

$$\delta\mathfrak{J}_1 = 0 \; . \tag{3.20a}$$

Similarly, one can show

$$\delta\mathfrak{J}_2 = 0 \tag{3.20b}$$

and

$$\delta\mathfrak{J}_3 = 0 \; . \tag{3.20c}$$

The variational integrals will be used in the following sections to derive the macroscopic equations. We emphasize that its applicability is limited to the case when thermal effects are known a priori or when their change under the processes under investigation is insignificant. In particular it cannot be used for the derivations of plate equations in thermoelasticity and other complex theories.

* A set of functions {u} is said to have compact support in $\bar{\mathfrak{D}}$ if $u = 0$ on $\partial\mathfrak{D}$.

3.5) Load and Stress Resultants:

In order to facilitate notation in the subsequent analysis it is advantageous to introduce the following shorthand notation.

We define a body force resultant of order m by

$$F_i^{(m)} = \int_h \rho_o F_i x_3^m ds \quad , \tag{3.21}$$

a stress resultant

$$T_{ij}^{(m)} = \int_h x_3^m \Sigma_{ij} ds \tag{3.22}$$

surface loads of order m

$$S_k = [x_3^m \Sigma_{3k} N_3] \qquad ; \quad x_3 \text{ on the} \begin{Bmatrix} \text{upper} \\ \text{or} \\ \text{lower} \end{Bmatrix} \text{surface} \tag{3.23}$$

In the above relations ds is the line element and integration is over the height of the undeformed plate. N_i is the unit exterior normal vector.

Note that the definitions (3.21) - (3.23) are more or less arbitrary. In particular one could also define $F_i^{(m)}$, etc. by integrating over the height of the deformed body or one could define $T_{ij}^{(m)}$ in terms of T_{ij} rather than Σ_{ij}. This arbitrariness, however does not correspond to a nonuniqueness in the emerging theory. But it means that the averaged quantities $T_{ij}^{(m)}$, $S_i^{(m)}$ should by no means be given explicit physical meaning. Uniqueness is not required except for physically measurable quantities such as displacements, velocity and the like. A formulation must be unique only with respect to these definitely observable quantities. On the other hand, that (3.22) and (3.23) are the most convenient choices will be seen in Section 3.6. Similarly to (3.21) - (3.23) we define an <u>acceleration resultant</u> by

$$\ddot{U}_i^{(m)} = \sum_{p=o}^{\infty} I^{(m+p)} \ddot{u}_i^{(p)} \tag{3.24}$$

where the m-th moment of inertia is given by

$$I^{(m)} = \int_h x_3^m ds \quad . \tag{3.25}$$

Moreover, the prescribed <u>stress resultant</u> of order m is given by

$$t_k^{*(m)} = \int_h x_3^m t_k^* ds \quad . \tag{3.26}$$

Later, we shall also need the temperature resultant of order m which is given by $\theta^{(m)}$ or $\Theta^{(m)}$:

$$\theta^{(m)} = \int_h x_3^m \vartheta \; ds \quad ; \quad \vartheta = \sum_{n=o}^{\infty} x_3^n \Theta^{(n)} \quad . \tag{3.27}$$

It seems appropriate to recall here that we do not assume that

$$I^{(1)} = 0 \ ,$$

which would imply that the plane $x_3 = 0$ would lie midway between the upper and lower surfaces.

3.6) Constitutive Equations - Elastic Materials:

Following the usual lines of arguments, introducing the Clausius-Duhem inequality

$$\rho_o \dot{\eta} - \left(\frac{Q_k}{\vartheta}\right)_{,k} \geq \rho_o \frac{r}{\vartheta} \tag{3.28}$$

where ϑ denotes the temperature and η the entropy, equation (3.14) and inequality (3.28) imply that

$$-\rho_o \dot{\psi} - \rho_o \eta \dot{\vartheta} + T_{ik} \dot{F}_{ik} - \frac{Q_k \vartheta_{,k}}{\vartheta} \geq 0 \quad , \tag{3.29}$$

where also the Helmholtz free energy

$$\psi = \epsilon - \eta \vartheta \tag{3.30}$$

has been introduced.

We now define an <u>elastic material with parameter ϑ</u> to be of the form

$$\begin{aligned} T_{ij} &= \mathfrak{T}_{ij}(F_{\ell m}, \vartheta) \; ; \\ \epsilon &= \mathfrak{E}(F_{\ell m}, \vartheta) \; ; \\ \psi &= \mathfrak{P}(F_{\ell m}, \vartheta) \; ; \\ \eta &= \mathfrak{S}(F_{\ell m}, \vartheta) \; ; \\ Q_i &= \mathfrak{Q}_i(F_{\ell m}, \vartheta) \; . \end{aligned} \tag{3.31}$$

Then

$$\dot{\psi} = \frac{\partial \mathfrak{P}}{\partial F_{\ell m}} \dot{F}_{\ell m} + \frac{\partial \mathfrak{P}}{\partial \vartheta} \dot{\vartheta}$$

and thus (3.29) becomes

$$-\rho_o\left[\frac{\partial \mathfrak{P}}{\partial \vartheta} + \eta\right]\dot{\vartheta} + \left[-\rho_o \frac{\partial \mathfrak{P}}{\partial F_{\ell m}} + T_{\ell m}\right]\dot{F}_{\ell m} - \frac{Q_k \vartheta_{,k}}{\vartheta} \geq 0 \quad , \tag{3.32}$$

implying, since this inequality is linear in $\dot{\vartheta}$, $\dot{F}_{\ell m}$ and $\vartheta_{,k}$ that

$$\eta = -\frac{\partial \mathfrak{P}}{\partial \vartheta} \quad ; \quad T_{ik} = \rho_o \frac{\partial \mathfrak{P}}{\partial F_{ik}} \quad ; \quad Q_k = 0 \quad . \qquad (3.33)^*$$

We now require that the constitutive equations (3.31) be materially objective. Since ϵ, ψ and η are scalar valued functions it is easily shown that for the constitutive equations to be invariant under rigid body motion, one must have

$$\psi = \tilde{\mathfrak{P}}(C_{ij}, \vartheta) \quad ,$$

where C_{ij} is given by (3.7) and similar expressions hold also for $\mathfrak{E}(\cdot)$ and $\mathfrak{G}(\cdot)$. Thus, by chain rule

$$T_{ik} = \rho_o \left[\frac{\partial \tilde{\mathfrak{P}}}{\partial C_{\ell m}} \frac{\partial C_{\ell m}}{\partial F_{ik}}\right] = \rho_o 2 \frac{\partial \tilde{\mathfrak{P}}}{\partial C_{km}} F_{im} \quad . \tag{3.34}$$

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* We have assumed that the constitutive variables are functions of the deformation gradient and the temperature only. Had we included also the temperature gradient, then Q_k would not vanish and Fourier's heat conduction equation could be derived. This would then enter the energy equation from which the change in temperature due to deformation could be determined. We are aware that this is the more physical situation than the one described above. If one keeps temperature changes aside, then both treatments lead to exactly the same plate equations.

Using (3.16) and (3.11), the relation (3.34) is equivalent to

$$\Sigma_{ij} = 2\rho_o \frac{\partial \tilde{\mathfrak{P}}}{\partial C_{ij}} \quad . \tag{3.35}$$

It is advantageous to introduce the elongation tensor

$$2E_{ij} = C_{ij} - \delta_{ij} \tag{3.36}$$

and the function $\mathring{\mathfrak{P}}(E_{ij}, \vartheta)$ such that

$$\Sigma_{ij} = 2\rho_o \frac{\partial \tilde{\mathfrak{P}}}{\partial E_{ij}} = \frac{\partial \mathring{\mathfrak{P}}}{\partial E_{ij}} \quad . \tag{3.37}$$

Note that this is particularly convenient because according to (3.22) we have

$$T_{ij}^{(m)} = \int_h x_3^{(m)} \frac{\partial \mathring{\mathfrak{P}}}{\partial E_{ij}} ds \quad . \tag{3.38}$$

The representation

$$\mathring{\mathfrak{P}} = \frac{1}{2}\, \mathcal{C}_{ijk\ell}(E_{ij} - \vartheta\omega_{ij})(E_{k\ell} - \vartheta\omega_{k\ell})$$

produces the constitutive law

$$\Sigma_{ij} = \mathcal{C}_{ijk\ell}(E_{k\ell} - \vartheta\omega_{ij}) \quad , \tag{3.39}$$

which is linear in the elongation tensor E_{ij}. Note, however, that E_{ij} is still the Lagrangian strain of finite elasticity. In the above equations $\mathcal{C}_{ijk\ell}$ and ω_{ij} are the isothermal first order elasticity coefficient and thermal expansion coefficient, respectively. They satisfy the symmetry relations

$$\mathcal{C}_{ijk\ell} = \mathcal{C}_{jik\ell} = \mathcal{C}_{k\ell ij} \; ; \; \omega_{ij} = \omega_{ji} \tag{3.40}$$

of hyperelasticity. In case of isotropic material they reduce to

$$\begin{aligned} \mathcal{C}_{ijk\ell} &= \lambda\delta_{ij}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}) \; ; \\ \omega_{ij} &= \omega\delta_{ji} \; , \end{aligned} \tag{3.41}$$

where ω is the coefficient of linear thermal expansion and λ and μ are Lamé's constants. One usually assumes

$$\mu > 0 \; ; \; 3\lambda + 2\mu > 0 \; . \tag{3.42}$$

With $\mathring{\mathfrak{P}}(E_{ij}, \vartheta)$ being an unspecified nonlinear function of the elongation tensor and the temperature not much simplification can be achieved when calculating the stress resultants. When $\mathring{\mathfrak{P}}(\cdot)$ is a polynomial then a reduction is possible. In particular the macroscopic constitutive equations in the linear form are obtained by substituting (3.39) into (3.22):

$$T_{ij}^{(m)} = \int_h x_3^m \, \mathcal{C}_{ijk\ell}(\vartheta)(E_{k\ell}-\omega_{k\ell}\vartheta)ds \quad , \tag{3.43}$$

In case $\mathcal{C}_{ijk\ell}$ is not temperature dependent this gives

$$T_{ij}^{(m)} = \mathcal{C}_{ijk\ell} \sum_{p=o}^{\infty} I^{(m+p)}(E_{k\ell}^{(p)}-\omega_{k\ell}\Theta^{(p)}) \tag{3.44}$$

or for the isotropic case

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} I^{(m+p)}[\lambda(E_{kk}^{(p)}-3\omega\Theta^{(p)})\delta_{ij}+2\mu(E_{ij}^{(p)}-\omega\Theta^{(p)}\delta_{ij}) \; , \tag{3.45}$$

where use has been made of (3.27) and $(3.9)_2$ with $E_{ij}^{(m)} = \frac{1}{2}C_{ij}^{(m)}$.

In floating ice plates, however, it is not justified to assume that the elastic constants be temperature independent. When thermal expansion is neglected ($\omega_{ij} = 0$), then (3.43) can be put into the form

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} \mathfrak{C}_{ijk\ell}^{(m+p)} E_{k\ell}^{(p)} \tag{3.46a}$$

with

$$\mathfrak{C}_{ijk\ell}^{(m)} = \int_h x_3^m \, \mathcal{C}_{ijk\ell}(\vartheta)ds \quad . \tag{3.46b}$$

In the more general situation of (3.43), however we obtain

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} \mathfrak{C}_{ijk\ell}^{(m+p)} [E_{k\ell}^{(p)} - \omega_{k\ell}\Theta^{(p)}],$$

or (3.47a)

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} \left\{ \Lambda^{(m+p)}(E_{kk}^{(p)} - 3\omega\Theta^{(p)})\delta_{ij} + 2M^{(m+p)}(E_{ij}^{(p)} - \omega\Theta^{(p)}\delta_{ij}) \right\}$$

with

$$\Lambda^{(m)} = \int_h x_3^m \lambda(\vartheta)ds \quad ; \quad M^{(m)} = \int_h x_3^m \mu(\vartheta)ds \quad . \tag{3.47b}$$

It is this law, we shall use in the linearized theory of deflection of ice plates. Finally, from (3.47) it follows that

$$T_{ij}^{(m)} = \frac{1}{2}\left[\frac{\partial \bar{\mathfrak{A}}}{\partial E_{ij}^{(m)}} + \frac{\partial \bar{\mathfrak{A}}}{\partial E_{ji}^{(m)}}\right] ,$$

where

$$\bar{\mathfrak{P}} = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} \mathfrak{C}_{ijk\ell}^{(p+1)} [E_{ij}^{(p)} - \omega_{ij}\theta^{(p)}][E_{k\ell}^{(p)} - \omega_{k\ell}\theta^{(p)}] \quad . \tag{3.48}$$

The kinetic energy density (per unit area) on the other hand is

$$\mathfrak{K} = \frac{1}{2}\int_h \rho \dot{u}_i \dot{u}_i ds = \frac{1}{2}\sum_{p=o}^{\infty} \sum_{q=o}^{\infty} I^{(p+q)} \dot{u}_k^{(p)} \dot{u}_k^{(q)}$$

With these formulas the apparatus for the derivation of the plate equations has been obtained.

Added in proof:*

A different representation from (3.44) and (3.47) is obtained when $(3.27)_1$ is used instead of $(3.27)_2$. (3.43) then becomes

$$T_{ij}^{(m)} = \ell_{ijkl}\left\{ \sum_{p=o}^{\infty} I^{(m+p)} E_{kl}^{(p)} - \omega_{kl}\theta^{(m)} \right\} \tag{3.49}$$

while (3.47a) is replaced by

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} \mathfrak{C}_{ijkl}^{(m+p)} E_{kl}^{(p)} - \alpha_{ij}^{(m)}$$

where

$$\alpha_{ij}^{(m)} = \int_h x_3^m \ell_{ijkl}(\vartheta)\, \omega_{kl}(\vartheta)\, \vartheta \, ds \quad .$$

This representation lacks the symmetry properties enjoyed by (3.44) and (3.47) but may be advantageous in special approximations.

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* The author is grateful to Prof.A.D.Kerr, Princeton University, for the sug-- stion that a formulation without a Taylor series expansion of the temperatu distribution should be possible. (private communication)

3.7) The Plate Equations of Motion:

We now proceed to develop the nonlinear field equations of plates in terms of the displacement components. In this Section our starting equation is the variational principle (3.20).

To begin with, consider the contribution from the volume integral

$$\delta \mathfrak{J}_1 = \int_{t_1}^{t_2} dt \left[\int_{\mathcal{D}} [T_{ij,j} - \rho_o(\ddot{u}_i - F_i)]\delta u_i \right] dv$$

$$= \int_{t_1}^{t_2} dt \left[\int_{\mathcal{A}} dA \int_h [T_{ij,j} - \rho_o(\ddot{u}_i - F_i)]\delta u_i \right] ds \quad . \tag{3.50}$$

Substituting for u_i and correspondingly for δu_i the representation (3.4), we obtain

$$\delta \mathfrak{J}_1 = \int_{t_1}^{t_2} dt \int_{\mathcal{A}} dA \left[\sum_{m=o}^{\infty} [-\rho_o \ddot{U}_i^{(m)} + F_i^{(m)} + P_i^{(m)} + N_i^{(m)}] \right] \delta u_i^{(m)} \quad , \tag{3.51}$$

where

$$N_i^{(m)} = T^{(m)}_{\beta\alpha,\beta}\delta_{i\alpha} + T^{(m)}_{\beta 3,\beta}\delta_{i3} - mT^{(m-1)}_{3\alpha}\delta_{i\alpha} - mT^{(m-1)}_{33}\delta_{i3}$$

$$+ \sum_{p=o}^{\infty} \Big\{ T^{(m+p)}_{\beta\alpha,\beta} u^{(m)}_{i,\alpha} - (m+p) u^{(m)}_{i,\alpha} T^{(m+p-1)}_{3\alpha} + T^{(m+p-1)}_{\beta 3,\beta}\, p\, u_i^{(p)}$$

$$- p(m+p-1) T^{(m+p-2)}_{33} u_i^{(p)} + T^{(m+p)}_{\beta\alpha} u^{(p)}_{i,\alpha\beta} + p\, T^{(m+p-1)}_{\beta 3} u^{(p)}_{i,\beta}$$

$$+ p\, T^{(m+p-1)}_{3\alpha} u^{(p)}_{i,\alpha} + p(p-1) T^{(m+p-2)}_{33} u_i^{(p)} \Big\}$$

and

$$P_i^{(m)} = \left[\sum_{p=o}^{\infty}(S_\alpha^{(m+p)}u_{i,\alpha}^{(m)}+p\ S_3^{(m+p-1)}u_i^{(p)})+S_i^{(m)}\right]_L^U . \quad (3.52)$$

Here $[(\cdot)]_L^U = (\cdot)_U + (\cdot)_L$ is the sum of the quantity $(\cdot)$ evaluated at the upper and lower boundary face, respectively.

Further approximations are achieved by neglecting special terms. In the von Kàrmàn approximation we apparently must set

$$\sum_{m=o}^{1}\int_{t_1}^{t_2} dt \int_{A} da \int_{h} x_3^m(\Sigma_{j\ell}(u_{i,\ell}+\delta_{i\ell}))_{,j} ds\ \delta u_i^{(m)}$$

$$= \sum_{m=o}^{1}\int_{t_1}^{t_2} dt \int_{A} da \int_{h} x_3^m(\Sigma_{j\ell}(u_{3,\alpha}^{(0)}\delta_{i3}\delta_{\alpha\ell}+\delta_{i\ell}))_{,j} ds\ \delta u_i^{(m)}$$

$$= \sum_{m=o}^{1}\int_{t_1}^{t_2} dt \int_{A} da[N_i^{(m)}+p_i^{(m)}] ,$$

with

$$N_i^{(m)} = T_{\beta i,\beta}^{(m)} - mT_{3i}^{(m-1)} + \sum_{p=o}^{1}\delta_{3i}\Big\{T_{\beta\alpha}^{(m+p)}u_3^{(m)}-(m+p)u_{3,\alpha}^{(m)}T_{3\alpha}^{(m+p-1)}$$

$$+pT_{\beta3,\beta}^{(m+p-1)}u_3^{(p)}-p(m+p-1)T_{33}^{(m+p-2)}u_3^{(p)}+T^{(m+p)}u_{3,\alpha}^{(p)}$$

$$+pT_{\beta3}^{(m+p-1)}u_{3,\beta}^{(p)}+pT_{3\alpha}^{(m+p-1)}u_{3,\alpha}^{(p)}+p(p+1)T_{33}^{(m+p-2)}u_3^{(p)}\Big\} \quad (3.52a)$$

$$P_i^{(m)} = [S_i^{(m)}]_L^U + \delta_{i3}\left[\sum_{p=o}^{1}(S^{(m+p)}u_{i,\alpha}^{(m)}+pS_3^{(m+p-1)}u_i^{(p+1)})\right]_L^U .$$

The surface integrals in (3.20) are

$$\delta \mathfrak{J}_2 = \int_{t_1}^{t_2} dt \int_{\partial \mathfrak{D}_d} (u_k^* - u_k) \delta t_k dA \tag{3.53}$$

and

$$\delta \mathfrak{J}_3 = \int_{t_1}^{t_2} dt \left\{ \int_{\partial \mathfrak{D}_\sigma} (t_k^* - t_k) \delta u_k ds + \int_{\partial \mathfrak{D}_t} (t_k^* - t_k^*) \delta u_k ds \right\} . \tag{3.54}$$

Here $\partial \mathfrak{D}_d$ is that part of the surface where the displacement is prescribed. $\partial \mathfrak{D}_\sigma$ is the surface portion on $\partial \mathfrak{D}_L$ or $\partial \mathfrak{D}_U$, respectively, where the traction is prescribed, while $\partial \mathfrak{D}_t$ is that portion of $\partial \mathfrak{D}_\epsilon$ where the stresses are prescribed.

Performing the integrations in (3.53) and (3.54) we obtain

$$\delta \mathfrak{J}_2 = \int_{t_1}^{t_2} dt \int_{\partial \mathfrak{D}_d} \sum_{m=o}^{\infty} x_3^m (u_k^{*(m)} - u_k^{(m)}) \delta t_k ds \tag{3.55}$$

$$\delta \mathfrak{J}_3^{(1)} = \int_{t_1}^{t_2} dt \int_{\partial \mathfrak{D}_\sigma} [t_k^{*(m)} - S_k^{(m)} - \sum_{p=o}^{\infty} S_l^{(m+p)} u_{k,l}^{(m)}]_{-L}^{U} \delta u_k^{(m)} dA$$

$$\delta \mathfrak{J}_3^{(2)} = \int_{t_1}^{t_2} dt \oint_{\mathcal{C}_t} \left\{ t_k^{*(m)} - T_{\alpha k}^{(m)} N_\alpha - \sum_{p=o}^{\infty} [T_{\alpha\beta}^{(m+p)} N_\alpha u_{k,\beta}^{(p)} + p\; T_{\alpha 3}^{(m+p-1)} N_\alpha u_k^{(p)}] \right\} \delta u_k^{(m)} ds . \tag{3.56}$$

Here the integrand in (3.56), $[(\cdot)]_{-L}^{U}$ is the difference, $[(\cdot)]_{-L}^{U} = (\cdot)_U - (\cdot)_L$ of $(\cdot)$ evaluated at the upper and lower face, respectively.

With (3.50), (3.55) and (3.56) we are now in position to apply the Hamiltonian principle. On setting each variation of the functionals

$\mathfrak{I}_k$ separately to zero, viz.,

$$\delta\mathfrak{I}_1 = \delta\mathfrak{I}_2 = \delta\mathfrak{I}_3^{(1)} = \delta\mathfrak{I}_3^{(2)} = 0$$

for arbitrary variations of the displacement and traction vector δu_i and δt_k, the following hierarchy of boundary value problems is obtained:

$$\rho_o \ddot{U}_i^{(m)} = F_i^{(m)} + N_i^{(m)} + P_i^{(m)} \quad ; \qquad (x_1,x_2)\in\mathfrak{D}$$

$$u_k^{*(m)} - u_k^{(m)} = 0 \quad ; \qquad (x_1,x_2)\in\partial\mathfrak{D}_d$$

$$[t_k^{*(m)} - S_k^{(m)} - S_\ell^{(m+n)} u_{k,\ell}^{(m)}]_{-L}^{U} = 0 \quad ; \qquad (x_1,x_2)\in\partial\mathfrak{D}_\sigma \quad (3.57)$$

$$t_k^{*(m)} - T_{\alpha k}^{(m)} N_\alpha - \sum_{n=o}^{\infty} [T_{\alpha\beta}^{(m+n)} N_\alpha u_{k,\beta}^{(n)} + n\, T_{\alpha 3}^{(m+n-1)} N_\alpha u_k^{(n)}] = 0 \quad ;$$

$$(x_1,x_2)\in\mathfrak{D}_t .$$

These equations will henceforth be called the macroscopic equations of motion and boundary conditions of order m.

Thus far, a fully nonlinear plate theory has been established. It consists of a one parameter family of differential equations and boundary conditions (3.57). The stress resultants $T_{ij}^{(m)}$ are given in terms of the elongation tensor E_{ij} in (3.45) or (3.48), which in turn are defined in Section 3.3 in terms of the strains. Note that for a well set initial value problem the above equations must be complemented by initial conditions. Let u_o and $\dot{u}_o$ be the displacement and velocity field prescribed at time t_o. Then, by a Taylor series expansion

$$u_o = \sum_{m=o}^{\infty} x_3^m \left. \frac{\partial^m u_o}{m!\partial x_3^m} \right|_{x_3=0} = \sum_{m=o}^{\infty} x_3^m u_o^{(m)}$$

$$\dot{u}_o = \sum_{m=o}^{\infty} x_3^m \left. \frac{\partial^m \dot{u}_o}{m!\partial x_3^m} \right|_{x_3=0} = \sum_{m=o}^{\infty} x_3^m \dot{u}_o^{(m)} \tag{3.58}$$

one obtains the initial values for the $u_o^{(m)}$ and $\dot{u}_o^{(m)}$.

Clearly, the set of equations introduced above is complex. In fact it forms an infinity of equations and in this form cannot be used for practical analysis. We must search for a consistent reduction of the equations by truncating the series. Thus the plate theory of order (M) is defined by

$$u_i(x_k,t) = \sum_{m=o}^{M} P_m(x_3) u_i^{(m)}(x_\alpha,t) \tag{3.59}$$

together with the condition

$$u_k^{(m)} = 0 \quad \text{for all } m > M. \tag{3.60}$$

Accordingly, only those quantities in (3.59) are considered of which the order is not greater than M. This results in a finite set of nonlinear partial differential equations and corresponding boundary conditions.

3.8) Further Simplifications:

The general theory which characterizes the nonlinear behavior of plates has been formulated in the preceding Sections. This theory is still very complex and calls for further simplifications. Various such simplifications are possible dependent upon the degree of further neglections.

a. Linear Theory:

Dropping all nonlinear terms in the equations of Section (3.7) one obtains a fully linear theory of plates which is governed by the set of equations

$$\rho_o \ddot{U}_i^{(m)} = F_i^{(m)} + T_{\beta\alpha,\beta}^{(m)}\delta_{i\alpha} + T_{\beta 3,\beta}^{(m)}\delta_{i3} - mT_{3\alpha}^{(m-1)}\delta_{i\alpha} + P_i^{(m)} \quad ; \quad (x_1,x_2)\epsilon\mathfrak{D}$$

$$u_k^{(m)} - u_k^{*(m)} = 0 \quad ; \qquad (x_1,x_2)\epsilon\partial\mathfrak{D}_d \quad ;$$

$$[t_k^{*(m)} - T_{\alpha k}^{(m)}N_\alpha] = 0 \quad ; \qquad (x_1,x_2)\epsilon\mathfrak{D}_t \quad ; \tag{3.61}$$

$$[t_k^{*(m)} - S_k^{(m)}]_{-L}^{U} = 0 \quad ; \qquad (x_1,x_2)\epsilon\partial\mathfrak{D}_\sigma \quad .$$

Here,

$$P_i^{(m)} = [S_i^{(m)}]_L^U$$

and (3.62)

$$T_{ij}^{(m)} = \sum_{p=o}^{\infty} \mathfrak{C}_{ijk\ell}^{(m+p)}(\tilde{E}_{k\ell}^{(p)} - \omega_{k\ell}\Theta^{(p)})$$

and where we have set

$$E_{ij} = \tilde{E}_{ij} \quad .$$

The above equations constitute the linear plate theory. It is geometrically and physically linear, both simplifications emerging from the fact that the pertinent equations have been consistently linearized. This theory includes effects such as shear and rotatory inertia and it may be considered to be a generalized version of the Reissner plate theory. Apart from the slightly more general constitutive treatment (see (3.62)) it coincides in this form with the Mindlin plate theory provided the appropriate simplifications are made.

Special situations are readily available as demonstrated below.

b. Extensional motion only:

We now investigate the linear theory presented above for the special case of extensional motion only.

Setting all derivatives with respect to x_3 equal to zero and requiring that $u_3 \equiv 0$ one obtains from (3.61)

$$F_3^{(m)} + P_3^{(m)} \equiv 0 \tag{3.63}$$

as a necessary condition for consistency of the stated assumption and the equations

$$\rho_o \ddot{U}_\alpha^{(m)} = F_\alpha^{(m)} + P_\alpha^{(m)} + T_{\beta\alpha,\beta}^{(m)} - mT_{3\alpha}^{(m-1)} \quad . \tag{3.64}$$

Restriction to terms of order m = 0 implies that

$$m\ddot{u}_\alpha = T_{\beta\alpha,\beta} + F_\alpha + P_\alpha \quad , \tag{3.65}$$

where we have omitted the indices (0). For the remainder of this section this will be done consistently. The stress resultants are

$$T_{\alpha\beta} = \mathfrak{C}_{\alpha\beta\gamma\delta}(\tilde{E}_{\gamma\delta} - \omega_{\gamma\delta}\Theta) \quad ,$$

or

$$T_{\alpha\beta} = \Lambda(\tilde{E}_{\gamma\gamma} - 3\omega\Theta)\delta_{\alpha\beta} + 2M(\tilde{E}_{\alpha\beta} - \omega\Theta\delta_{\alpha\beta}) \quad .$$

Assuming that Λ, M and Θ are not functions of x_1 and x_2 (this assumption is a valid one for ice plates) we may write (3.65) in the form

$$m\ddot{u}_{\alpha} = (\Lambda+M)u_{\beta,\beta\alpha} + Mu_{\alpha,\beta\beta} + F_{\alpha} + P_{\alpha} \quad , \tag{3.66}$$

which formally agrees with the two dimensional equations of elasticity. Neglecting F_α and P_α and considering the static situation only, we obtain

$$T_{\beta\alpha,\beta} = 0 \quad ,$$

which is identically satisfied when the stress function Φ is introduced such that

$$T_{\beta\alpha} = (-1)^{\delta_{\alpha\beta}} \frac{\partial^2\Phi}{\partial x_\beta \partial x_\alpha} \quad ; \quad \text{(no summation)} \quad . \tag{3.67}$$

The governing equation for Φ is now obtained from the compatibility conditions which must be satisfied by the displacement field. To derive them, note that in this linear version

$$\tilde{E}_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{3.68}$$

and that $\tilde{E}_{\alpha\beta}$ satisfy the identity

$$\tilde{E}_{\alpha\beta,\gamma\delta} + \tilde{E}_{\gamma\delta,\alpha\beta} - \tilde{E}_{\beta\delta,\alpha\gamma} - \tilde{E}_{\alpha\gamma,\beta\delta} \equiv 0 \quad , \tag{3.69}$$

as can easily be seen by substituting (3.68) into (3.69). We invert the constitutive equation for stress:

$$\tilde{E}_{\mu\nu} = \mathfrak{C}^{-1}_{\alpha\beta\mu\nu} T_{\alpha\beta} + \omega_{\mu\nu}\Theta \quad , \tag{3.70}$$

where $\mathfrak{C}^{-1}_{\alpha\beta\mu\nu}$ may be denoted as compliance tensor, which in the isotropic case assumes the form

$$\mathfrak{C}^{-1}_{\alpha\beta\mu\nu} = \Lambda^{-1}\delta_{\alpha\beta}\delta_{\mu\nu} - M^{-1}(\delta_{\alpha\mu}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\mu}) \tag{3.71}$$

with

$$\Lambda^{-1} = -\frac{\Lambda}{2M(3\Lambda+2M)} \quad ; \quad M^{-1} = 1/(2M) \quad .$$

Substituting (3.71) into (3.69), setting $\gamma = \delta$ results in

$$\Lambda^{-1}\delta_{\alpha\beta}T_{\mu\mu,\gamma\gamma} + T_{\gamma\gamma,\alpha\beta}(\Lambda^{-1}+2M^{-1}) + 2M^{-1}T_{\alpha\beta,\gamma\gamma} = 0 \quad , \tag{3.72}$$

which by setting $\alpha = \beta$ implies

$$T_{\mu\mu,\beta\beta} = 0 \quad , \tag{3.73}$$

so that $\mathrm{tr}(\underset{\sim}{T})$ is harmonic. This implies that the first term of (3.72) vanishes by itself. Taking the Laplacian of (3.72) shows that $T_{\alpha\beta}$ is biharmonic. On the other hand, differentiating (3.72) with respect to x_β and observing that $T_{\alpha\beta,\beta} = 0$ leads to

$$\frac{\partial}{\partial x_\alpha}(\nabla^4\Phi) = 0 \quad ; \quad \nabla^4\Phi = \text{constant.}$$

The constant is fixed by satisfying (3.73). Thus

$$\nabla^4 \Phi = 0 \; , \tag{3.74}$$

which is the familiar bipotential equation in plain strain elasticity.

For the solution of the dynamic problem one uses the Helmholtz theorem and decomposes the displacement vector into a solenoidal and lamellar vector as follows:

$$u_\beta = \Psi,_\beta + \epsilon_{\beta j \gamma} A_{j,\gamma} \; , \tag{3.75}$$

where ψ and A_j are unique to within a gradient of a scalar. Substituting (3.75) into (3.66) gives

$$[m\ddot{\Psi} + (\Lambda+2M)\Psi,_{\beta\beta}],_\alpha + \epsilon_{\alpha j \gamma}[m\ddot{A}_{j,\gamma} - MA_{j,\gamma\beta\beta}] = 0 \; , \tag{3.76}$$

which is satisfied if

$$\begin{aligned} \ddot{\Psi} &= c_p^2 \Psi,_{\beta\beta} \; ; \quad c_p^2 = (\Lambda+2M)/m \; ; \\ \ddot{A}_3 &= c_s^2 A_{3,\beta\beta} \; ; \quad c_s^2 = M/m \; , \end{aligned} \tag{3.77}$$

where c_p and c_s are the primary and secondary wave speeds, respectively.

A question arises as to whether every solution of (3.77) is also solution of (3.66) and opposite. This question is answered by the completeness theorem, [27], stating that every solution of (3.66) admits a decomposition (3.75) with Ψ and A_3 satisfying the equations (3.77).

c) Plate theory accounting for shear effects (Generalized Reissner Theory):

We now use the set of equations (3.61) to formulate a theory of plates for flexural motion which accounts for shear deformation and rotatory inertia. We neglect thermal effects (homothermal conditions) and assume that

$$T_{11}^{(o)} = T_{22}^{(o)} = T_{33}^{(o)} = T_{12}^{(o)} = 0 \ , \tag{3.78}$$

but*

$$\begin{aligned} T_{13}^{(o)} &= Q_x = 2M^{(o)}\tilde{E}_{13}^{(o)} \ ; \\ T_{23}^{(o)} &= Q_y = 2M^{(o)}\tilde{E}_{23}^{(o)} \ . \end{aligned} \tag{3.79}$$

It follows from (3.78) with the use of (3.47b) that

$$\tilde{E}_{11}^{(o)} = \tilde{E}_{22}^{(o)} = \tilde{E}_{33}^{(o)} = \tilde{E}_{12}^{(o)} = 0 \ . \tag{3.80}$$

On the other hand, from (3.47a)

$$\begin{aligned} T_{11}^{(1)} &= M_x = \Lambda^{(2)}\tilde{E}_{kk}^{(1)} + 2M^{(2)}\tilde{E}_{11}^{(1)} \ ; \\ T_{22}^{(1)} &= M_y = \Lambda^{(2)}\tilde{E}_{kk}^{(1)} + 2M^{(2)}\tilde{E}_{22}^{(1)} \ ; \\ T_{12}^{(1)} &= -M_{xy} = 2M^{(2)}\tilde{E}_{12}^{(1)} \ , \end{aligned} \tag{3.81}$$

while we assume that

---

* Although the coordinates x, y and z have been chosen differently in the last Chapter, we choose this order here, namely $x_1 = x, x_2 = y$ and $x_3 = z$, because the emerging equations then can easily be compared with those in the existing literature.

$$T_{33}^{(1)} = 0 = \Lambda^{(2)}\tilde{E}_{kk}^{(1)} + 2M^{(2)}\tilde{E}_{33}^{(1)} \quad ;$$

$$T_{13}^{(1)} = 0 = 2M^{(2)}\tilde{E}_{13}^{(1)} \tag{3.82}$$

$$T_{23}^{(1)} = 0 = 2M^{(2)}\tilde{E}_{23}^{(1)} \quad .$$

Thus:

$$\tilde{E}_{23}^{(1)} = \tilde{E}_{13}^{(1)} = 0 \quad . \tag{3.83}$$

But

$$\tilde{E}_{33}^{(1)} = -\frac{\Lambda^{(2)}}{\Lambda^{(2)}+2M^{(2)}}\tilde{E}_{\alpha\alpha}^{(1)} \tag{3.84}$$

and

$$\tilde{E}_{kk}^{(1)} = \frac{2M^{(2)}}{\Lambda^{(2)}+2M^{(2)}}\tilde{E}_{\alpha\alpha}^{(1)} \quad . \tag{3.85}$$

Using this result in (3.81) gives

$$M_x = D(\tilde{E}_{11}^{(1)}+N\tilde{E}_{22}^{(1)}) \quad ;$$

$$M_y = D(\tilde{E}_{22}^{(1)}+N\tilde{E}_{11}^{(1)}) \quad ; \tag{3.86}$$

$$M_{xy} = -2M^{(2)}\tilde{E}_{12}^{(1)} \quad ,$$

with

$$D = \frac{2\Lambda^{(2)}M^{(2)}}{\Lambda^{(2)}+2M^{(2)}} + 2M^{(2)} \quad ; \quad N = \frac{\Lambda^{(2)}}{2(\Lambda^{(2)}+M^{(2)})} \quad . \tag{3.87}$$

Defining

$$\eta \equiv u_3^{(o)} \quad ; \quad \varphi \equiv -u_1^{(1)} \quad ; \quad \psi \equiv -u_2^{(1)} \quad , \tag{3.88}$$

we obtain from the linearized version of (3.10)

$$\tilde{E}^{(o)}_{13} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x} - \varphi\right) \quad ; \quad \tilde{E}^{(o)}_{23} = \frac{1}{2}\left(\frac{\partial\eta}{\partial y} - \psi\right) \tag{3.89}$$

and

$$\tilde{E}^{(1)}_{11} = -\frac{\partial\varphi}{\partial x} \quad ; \quad \tilde{E}^{(1)}_{22} = -\frac{\partial\psi}{\partial y} \quad ; \quad \tilde{E}^{(1)}_{12} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial y} + \frac{\partial\psi}{\partial x}\right) \ . \tag{3.90}$$

Substituting these expressions into (3.79) and (3.86), respectively, yields

$$\begin{aligned} Q_x &= M^{(o)}\left(\frac{\partial\eta}{\partial x} - \varphi\right) \quad ; \\ Q_y &= M^{(o)}\left(\frac{\partial\eta}{\partial y} - \psi\right) \end{aligned} \tag{3.91}$$

and

$$\begin{aligned} M_x &= -D\left(\frac{\partial\varphi}{\partial x} + N\frac{\partial\psi}{\partial y}\right) \quad ; \\ M_y &= -D\left(\frac{\partial\psi}{\partial y} + N\frac{\partial\varphi}{\partial x}\right) \quad ; \\ M_{xy} &= M^{(2)}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\right) \quad . \end{aligned} \tag{3.92}$$

The equations of motion can be derived from (3.61) and are given by

$$\begin{aligned} \rho_o A\,\frac{\partial^2\eta}{\partial t^2} &= \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q \quad ; \\ -\rho_o I^{(2)}\,\frac{\partial^2\varphi}{\partial t^2} &= \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x + T_x \quad ; \\ -\rho_o I^{(2)}\,\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y + T_y \quad , \end{aligned} \tag{3.93}$$

where we have set

$$q = F_3^{(o)} + P_3^{(o)} \quad ; \quad F_1^{(1)} + P_1^{(1)} = T_x \quad , \quad F_2^{(1)} + P_2^{(1)} = T_y \quad .$$

Furthermore $u_1^{(o)} = u_2^{(o)} = 0$. The equations (3.91) - (3.93) form a determinate set of 8 equations for 8 unknowns $Q_x, Q_y, M_x, M_y, M_{xy}, \eta, \varphi$ and ψ. Similar, but not entirely the same equations have been derived by Reissner, [28], and by a different method of derivation also by Green, [29] (see also [30]).

Eliminating Q and M from (3.93) gives

$$\rho_o A \frac{\partial^2 \eta}{\partial t^2} = M^{(o)}[\nabla^2 \eta - \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}] - q \quad ;$$

$$\rho_o I^{(2)} \frac{\partial^2 \varphi}{\partial t^2} = D \frac{\partial^2 \varphi}{\partial x^2} + M^{(1)} \frac{\partial^2 \varphi}{\partial y^2} + (DN + M^{(1)}) \frac{\partial^2 \psi}{\partial x \partial y} + M^{(o)}(\frac{\partial \eta}{\partial x} - \varphi) - T_x \quad ;$$

$$\rho_o I^{(2)} \frac{\partial^2 \psi}{\partial t^2} = D \frac{\partial^2 \psi}{\partial y^2} + M^{(1)} \frac{\partial^2 \psi}{\partial x^2} + (DN + M^{(1)}) \frac{\partial^2 \varphi}{\partial x \partial y} + M^{(o)}(\frac{\partial \eta}{\partial y} - \varphi) - T_y \quad , \tag{3.94}$$

a system of three equations in three unknowns. Once η, φ and ψ are determined moments and shear forces can be obtained from (3.91) and (3.92).

As for the boundary conditions, these are obtained from (3.61). On a $\partial\mathfrak{D}_d$ boundary one prescribes η, φ and ψ by η^*, φ^* and ψ^*, while on the edge $\mathcal{C}_t$ the resultant traction vector is prescribed: or equivalently Q_n, M_n and M_{nt} which are, respectively, shear force, bending moment and twisting moment on the boundary edge $\mathcal{C}_t$ with normal direction $\underset{\sim}{N}$ and tangential direction $\underset{\sim}{t}$. Applications of this theory are postponed to later investigations.

d) A Generalized von Kàrmàn Plate Theory

The foregoing calculations have shown that the nonlinear plate theory developed in this Chapter leads to useful subtheories in simplified situations. The two situations are, however, still too simple to describe some effects on floating ice plates. In both theories thermal stresses were neglected. It therefore seems to be natural to consider a slightly more general case, which contains a coupling between flexural and extensional vibrations and also accounts for temperature effects and especially does not assume that the material constants be constant across the plate thickness.

To begin with recall that the macroscopic constants $\Lambda^{(i)}$ and $M^{(i)}$ for plates isotropic in the (x_1,x_2) plane as given in (3.47b) are known, once the temperature variation across the plate thickness is prescribed. Note further that we have not fixed the (x_1,x_2)-plane yet. This will be done now with the condition

$$\Lambda^{(1)} = \int_h x_3 \lambda(\vartheta) ds = 0 . \tag{3.95}$$

We assume henceforth that the coordinates (x_1,x_2,x_3) are chosen accordingly. To be precise, however, one must mention that the normalization (3.95) is only useful provided the temperature distribution is such that the surface $x_3=0$ is flat. This implies that λ does not vary within the (x_1,x_2)-plane.

Satisfying (3.95) does not, in general, imply

$$M^{(1)} = \int_h x_3 \mu(\vartheta) ds = 0 , \tag{3.96}$$

but if it does, then $\mu(\vartheta) = K\lambda(\vartheta)$ and this in turn implies that Poisson's ratio ν must be independent of the temperature. We shall not assume it in the sequel, because we are interested in its influence and can always neglect corresponding terms a posteriori.

The theory presented here can be considered to be an order "1" theory. Accordingly we shall keep O(1)-terms, while neglecting all higher order terms. With regard to the strain displacement relations we shall keep the nonlinear terms which correspond to the von Kàrmàn assumption. The theory presented here may therefore be called a physically linear but geometrically partially nonlinear theory. We also assume isotropy*, so that the stress strain relations $(3.47a)_2$ can be used. These formulas are used to calculate the zeroth order stress resultants:

$$
\begin{aligned}
T_{13}^{(o)} &= Q_x = 2M^{(o)}E_{13}^{(o)} + 2M^{(1)}E_{13}^{(1)} \quad ; \\
T_{23}^{(o)} &= Q_y = 2M^{(o)}E_{23}^{(o)} + 2M^{(1)}E_{23}^{(1)} \quad ; \\
T_{11}^{(o)} &= N_x = \Lambda^{(o)}E_{kk}^{(o)} + 2M^{(o)}E_{11}^{(o)} - \omega(3\Lambda^{(o)}+2M^{(o)})\Theta^{(o)} + 2M^{(1)}(E_{11}^{(1)}-\omega\Theta^{(1)}) ; \\
T_{22}^{(o)} &= N_y = \Lambda^{(o)}E_{kk}^{(o)} + 2M^{(o)}E_{11}^{(o)} - \omega(3\Lambda^{(o)}+2M^{(o)})\Theta^{(o)} + 2M^{(1)}(E_{22}^{(1)}-\omega\Theta^{(1)}) ; \\
T_{12}^{(o)} &= N_{xy} = 2M^{(o)}E_{12}^{(o)} + 2M^{(1)}E_{12}^{(1)} \quad ,
\end{aligned}
\tag{3.97}
$$

which we shall assume to be nonzero. The resultant $T_{33}^{(o)}$, however is assumed to be zero, (at least far from applied loads). Thus

---

* More precisely we assume that the macroscopic equations are isotropic. This only requires the equations of three dimensional elasticity to be orthotropic.

$$T_{33}^{(o)} = \Lambda^{(o)}E_{kk}^{(o)} + 2M^{(o)}E_{33}^{(o)} - (3\omega\Lambda^{(o)}+2M^{(o)})\Theta^{(o)} + 2M^{(1)}(E_{33}^{(1)}-\omega\Theta^{(1)}) = 0. \tag{3.98}$$

As far as first order stress resultants are concerned it seems appropriate to assume that $T_{11}^{(1)}$, $T_{22}^{(1)}$ and $T_{12}^{(1)}$ are nonzero, while all other first order stress resultants are set equal to zero, viz:

$$T_{11}^{(1)} = M_x = 2M^{(1)}E_{11}^{(o)} + \Lambda^{(2)}E_{kk}^{(1)} + 2M^{(2)}E_{11}^{(1)}$$

$$- \{2M^{(1)}\omega\Theta^{(o)} + 3\Lambda^{(2)}\omega\Theta^{(1)} + 2M^{(2)}\omega\Theta^{(1)}\} \quad ;$$

$$T_{22}^{(1)} = M_y = 2M^{(1)}E_{22}^{(o)} + \Lambda^{(2)}E_{kk}^{(1)} + 2M^{(2)}E_{22}^{(1)} \tag{3.99}$$

$$- \{2M^{(1)}\omega\Theta^{(o)} + 3\Lambda^{(2)}\omega\Theta^{(1)} + 2M^{(2)}\omega\Theta^{(1)}\} \quad ;$$

$$T_{12}^{(1)} = -M_{xy} = 2M^{(1)}E_{12}^{(o)} + 2M^{(2)}E_{12}^{(1)}$$

and

$$T_{13}^{(1)} = 2M^{(1)}E_{13}^{(o)} + 2M^{(2)}E_{13}^{(1)} = 0 \quad ;$$

$$T_{23}^{(1)} = 2M^{(1)}E_{23}^{(o)} + 2M^{(2)}E_{23}^{(1)} = 0 \quad ; \tag{3.100}$$

$$T_{33}^{(1)} = \Lambda^{(2)}(E_{kk}^{(1)}-3\omega\Theta^{(1)}) + 2M^{(2)}(E_{33}^{(1)}-\omega\Theta^{(1)}) + 2M^{(1)}(E_{33}-\omega\Theta^{(o)}) = 0 .$$

It follows from (3.100) that some zeroth and first order strains are not independent. In particular $(3.100)_{1,2}$ imply that

$$E_{13}^{(1)} = \frac{M^{(1)}}{M^{(2)}}E_{13}^{(o)} \quad ; \quad E_{23}^{(1)} = \frac{M^{(1)}}{M^{(2)}}E_{23}^{(o)} \quad , \tag{3.101}$$

while $(3.100)_3$ is an equation relating first and zeroth order strain $E_{33}^{(o)}$ and $E_{33}^{(1)}$. A second equation for these quantities is (3.98). Thus we may solve (3.98) and $(3.101)_3$ for $E_{33}^{(o)}$ and $E_{33}^{(1)}$. The result of this calculation is

$$
\begin{aligned}
E_{33}^{(o)} &= A^{(o)}E_{\alpha\alpha}^{(o)} + B^{(o)}E_{\alpha\alpha}^{(1)} + C^{(o)} \quad ; \\
E_{33}^{(1)} &= A^{(1)}E_{\alpha\alpha}^{(o)} + B^{(1)}E_{\alpha\alpha}^{(1)} + C^{(1)}
\end{aligned}
\qquad (3.102)
$$

with

$$
\begin{aligned}
A^{(o)} &= -\Lambda^{(o)}(\Lambda^{(2)}+2M^{(2)})/\Delta \quad ; \\
B^{(o)} &= 2\Lambda^{(2)}M^{(1)}/\Delta \quad ; \\
A^{(1)} &= 2\Lambda^{(o)}M^{(1)}/\Delta \quad ; \\
B^{(1)} &= -\Lambda^{(2)}(\Lambda^{(o)}+2M^{(o)})/\Delta \quad ;
\end{aligned}
$$

$$
\begin{aligned}
C^{(o)} = \omega[&(\Lambda^{(2)}+2M^{(2)})(3\Lambda^{(o)}\Theta^{(o)}+2M^{(o)}\Theta^{(o)}+2M^{(1)}\Theta^{(1)}) \\
&-2M^{(1)}(3\Lambda^{(2)}\Theta^{(1)}+2M^{(2)}\Theta^{(1)}+2M^{(1)}\Theta^{(o)})]/\Delta \quad ; \\
C^{(1)} = \omega[&(\Lambda^{(o)}+2M^{(o)})(3\Lambda^{(2)}\Theta^{(1)}+2M^{(2)}\Theta^{(1)}+2M^{(1)}\Theta^{(o)}) \\
&-2M^{(1)}(3\Lambda^{(o)}\Theta^{(o)}+2M^{(o)}\Theta^{(o)}+2M^{(1)}\Theta^{(1)}]/\Delta \quad ; \\
\Delta = [&\Lambda^{(o)}+2M^{(o)}][\Lambda^{(2)}+2M^{(2)}]-4(M^{(1)})^2 \quad .
\end{aligned}
\qquad (3.103)
$$

Note that $B^{(o)}$ and $A^{(1)}$ vanish when $M^{(1)} = 0$. Substituting (3.101) and (3.102) into (3.97) we obtain

$$Q_x = \Gamma_x E_{13}^{(o)} \quad ;$$
$$Q_y = \Gamma_y E_{23}^{(o)} \quad , \tag{3.104}$$

with

$$\Gamma_x = \Gamma_y = (2M^{(o)} + \frac{2M^{(1)^2}}{M^{(2)}}) \tag{3.105}$$

and

$$N_x = \mathfrak{D}^{(o)}(E_{11}^{(o)} + \mathcal{N}^{(o)}E_{22}^{(o)}) + \mathfrak{D}^{(1)}(E_{11}^{(1)} + \mathcal{N}^{(1)}E_{22}^{(1)}) + \mathcal{T}_x$$

$$N_y = \mathfrak{D}^{(o)}(\mathcal{N}^{(o)}E_{11}^{(o)} + E_{22}^{(o)}) + \mathfrak{D}^{(1)}(\mathcal{N}^{(1)}E_{11}^{(1)} + E_{22}^{(1)}) + \mathcal{T}_y \tag{3.106}$$

$$N_{xy} = 2M^{(o)}E_{12}^{(o)} + 2M^{(1)}E_{12}^{(1)} \quad ,$$

where

$$\mathfrak{D}^{(o)} = [\Lambda^{(o)}(A^{(o)}+1)+2M^{(o)}] \quad ; \quad \mathcal{N}^{(o)} = \frac{\Lambda^{(o)}}{\mathfrak{D}^{(o)}}(A^{(o)}+1) \quad ;$$

$$\mathfrak{D}^{(1)} = [\Lambda^{(o)}B^{(o)}+2M^{(1)}] \quad ; \quad \mathcal{N}^{(1)} = \frac{1}{\mathfrak{D}^{(1)}}(\Lambda^{(o)}B^{(o)}) \quad ; \tag{3.107}$$

$$\mathcal{T}_x = \mathcal{T}_y = [\Lambda^{(o)}C^{(o)} - \omega(3\Lambda^{(o)}+2M^{(o)})\Theta^{(o)} - 2\omega M^{(1)}\Theta^{(1)}] \quad .$$

$M^{(1)} = 0$ implies $\mathfrak{D}^{(1)} = 0$ and $\mathfrak{D}^{(1)}\mathcal{N}^{(1)} = 0$, in which case the relations (3.106) simplify considerably.

Similarly, one obtains from (3.99) the following relations

$$M_x = \mathfrak{D}^{(o)}(E_{11}^{(o)} + \mathfrak{n}^{(o)}E_{22}^{(o)}) + \mathfrak{D}^{(1)}(E_{11}^{(1)} + \mathfrak{n}^{(1)}E_{22}^{(1)}) + \mathfrak{T}_x$$

$$M_y = \mathfrak{D}^{(o)}(\mathfrak{n}^{(o)}E_{11}^{(o)} + E_{22}^{(o)}) + \mathfrak{D}^{(1)}(\mathfrak{n}^{(1)}E_{11}^{(1)} + E_{22}^{(1)}) + \mathfrak{T}_y \qquad (3.108)$$

$$M_{xy} = -[2M^{(1)}E_{12}^{(o)} + 2M^{(2)}E_{12}^{(1)}]$$

where

$$\mathfrak{D}^{(o)} = (\Lambda^{(2)}A^{(1)} + 2M^{(1)}) \quad ; \quad \mathfrak{n}^{(o)} = \frac{1}{\mathfrak{D}^{(o)}}(\Lambda^{(2)}A^{(1)}) \quad ;$$

$$\mathfrak{D}^{(1)} = (\Lambda^{(2)}(B^{(1)}+1) + 2M^{(2)}) \quad ; \quad \mathfrak{n}^{(1)} = \frac{1}{\mathfrak{D}^{(1)}}(\Lambda^{(2)}(B^{(1)}+1)) \quad ; \qquad (3.109)$$

$$\mathfrak{T}_x = \mathfrak{T}_y = -\left\{\omega[2M^{(1)}\Theta^{(o)} + 3\Lambda^{(2)}\Theta^{(1)} + 2M^{(2)}\Theta^{(1)}] - \Lambda^{(2)}C^{(1)}\right\} .$$

Observe again that in case $M^{(1)} = 0$, then $\mathfrak{D}^{(o)} = 0$ and $\mathfrak{n}^{(o)} = 0$. In this case there is a true separation of (3.108) from (3.106).

Thus far the generalized stress strain relations (3.106) and (3.108) have been derived. They are now complemented by the equations of motion and the strain displacement relations.

As for the strain displacement relations we choose the von Kàrmàn approximations (3.10c). On substituting these equations into (3.106) and (3.108), respectively, one would obtain the stress resultants in terms of the displacements.

Before proceeding, however, with further reductions we list the equations of motion which, with the definitions

$$\rho_o I^{(o)} = \mathfrak{M} \quad ; \quad \mathfrak{S} = \rho_o I^{(1)} \quad ; \quad \mathfrak{J} = \rho_o I^{(2)} \qquad (3.110)$$

can be obtained in the form

$$\mathfrak{M}\ddot{u} - \mathfrak{S}\ddot{\varphi} = \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + T_x^{(o)} \quad ;$$

$$\mathfrak{M}\ddot{v} - \mathfrak{S}\ddot{\psi} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + T_y^{(o)} \quad ;$$

$$\mathfrak{M}\ddot{\eta} + \mathfrak{S}\ddot{u}_3^{(1)} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + N_x\frac{\partial^2\eta}{\partial x^2} + 2N_{xy}\frac{\partial^2\eta}{\partial x\partial y} + N_y\frac{\partial^2\eta}{\partial y^2} \qquad (3.111)$$

$$+ \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\frac{\partial\eta}{\partial x} + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\frac{\partial\eta}{\partial y}$$

$$+ \left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right)\frac{\partial\eta}{\partial x} - \left(\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y}\right)\frac{\partial\eta}{\partial y} \quad ,$$

where we have defined

$$T_x^{(i)} = F_x^{(i)} + P_x^{(i)} \quad ; \quad T_y^{(i)} = F_y^{(i)} + P_x^{(i)} \quad ; \quad q = F_z^{(i)} + P_z^{(i)}$$

and

$$u = u_1 \quad , \quad v = u_2 \quad .$$

Note that the unknown $u_3^{(1)}$ in the equations (3.111) can be expressed in terms of u, v, φ, ψ and η. To this end, observe that (see (3.10d))

$$C_{33}^{(o)} = 2E_{33}^{(o)} = u_3^{(1)} \quad .$$

On the other hand, equation $(3.102)_1$ must hold. Thus

$$\ddot{u}_3^{(1)} = \left\{A^{(o)}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2\right] - B^{(o)}\left[\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right] + C^{(o)} + C^{(1)}\right\} . \quad (3.112)$$

Furthermore the first order equations are

$$\mathfrak{S}\ddot{u} - \mathfrak{J}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x + T_x^{(1)} ;$$

$$\mathfrak{S}\ddot{v} - \mathfrak{J}\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + T_y^{(1)} . \tag{3.113}$$

With this set of equations a definite set has been obtained describing the dynamic response of ice plates to instationary motions. The equations of motion are (3.111) and (3.113); they form a set of 5 equations for the unknowns u, v, φ, ψ, and η. The stress-strain relations for the macroscopic equations are (3.104), (3.106) and (3.108), while the strains are given in (3.10c).

The theory presented here is a thermal stress theory valid for plates for which the temperature may be nonuniform with depth. This implies that several otherwise vanishing terms must be included in the theory. Among these terms is $\mathfrak{S}$ which is the static moment of the cross section of unit width. Only when $x_3 = 0$ coincides with the middle plane, that is the plane midway between the bottom and top surface, is $\mathfrak{S} = 0$. On the other hand it was already mentioned before that some simplifications may be achieved by assuming that Poisson's ratio be temperature independent. In this event $\mathfrak{D}^{(1)} = 0$, $\mathfrak{N}^{(1)} = 0$ and $\mathfrak{D}^{(o)} = 0$ and $\mathfrak{R}^{(o)} = 0$, which simplifies the strain displacement relations (3.106) and (3.108). Note further that extensional and flexural motion, u, v, and φ and η, respectively, are coupled, even when nonlinear terms in (3.111) are neglected.

Because $\mathfrak{S} = O(h^2)$, and we have kept only terms of order h in the expansion procedure of this theory, we may justly neglect the terms involving Ξ. Hence the equations of motion are reduced considerably. Moreover, when it is assumed that $\lambda(\vartheta)$ and $\mu(\vartheta)$ are affine to each other, one also has $M^{(1)} = 0$. For this simplified situation, which corresponds to the consistent approximation for uniformly distributed temperature, the governing equations are only a slight generalization of the von Kàrmàn equations. The difference lies in the following facts, which make the theory presented here applicable to a wider class of problems than the von Kàrmàn theory: First we deal with the dynamic situation, second, our sèt of equations includes shear and rotatory inertia and avoids, as does the previously presented generalized Reissner theory too, the controversial edge singularities. The present theory thus provides enough generality, that all boundary conditions can be satisfied. Thirdly we have included thermal stress effects. Moreover, of the nonlinear terms in (3.111), the von Kàrmàn theory only keeps the first terms.

For the remainder of this section we list the equations for the theory where the following simplifying assumptions are made:

(i) $\Xi = 0$;

(ii) $M^{(1)} = 0$;

(iii) $T_x^{(o)} = T_y^{(o)} = T_x^{(1)} = T_y^{(1)} = 0$.

This theory could justly be called <u>generalized Reissner-von Kàrmàn</u> theory. Its governing equations are

Equations of Motion:

$$\mathfrak{M}\ddot{u} = \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \quad ;$$

$$\mathfrak{M}\ddot{v} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \quad ;$$

$$\begin{aligned} \mathfrak{M}\ddot{\eta} = {} & \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + N_x \frac{\partial^2 \eta}{\partial x^2} + 2N_{xy} \frac{\partial^2 \eta}{\partial x \partial y} + N_y \frac{\partial^2 \eta}{\partial y^2} + \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\frac{\partial \eta}{\partial x} \\ & + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\frac{\partial \eta}{\partial y} + \left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right)\frac{\partial \eta}{\partial x} + \left(\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}\right)\frac{\partial \eta}{\partial y} \quad ; \end{aligned} \tag{3.114}$$

$$-\mathfrak{I}\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x \quad ;$$

$$-\mathfrak{I}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y \quad ;$$

The stress-strain relation

$$N_x = \mathfrak{D}^{(o)}(E_{11}^{(o)} + \mathcal{N}^{(o)}E_{22}^{(o)}) + \mathfrak{T}_x \quad ;$$

$$N_y = \mathfrak{D}^{(o)}(E_{22}^{(o)} + \mathcal{N}^{(o)}E_{11}^{(o)} + \mathfrak{T}_y \; ; \tag{3.115}$$

$$N_{xy} = 2M^{(o)}E_{12}^{(o)} \quad ;$$

$$M_x = \mathfrak{D}^{(1)}(E_{11}^{(1)} + \mathfrak{n}^{(1)}E_{22}^{(1)}) + \mathfrak{T}_x \; ; \qquad Q_x = \Gamma_x E_{13}^{(o)} \; ;$$

$$M_y = \mathfrak{D}^{(1)}(E_{22}^{(1)} + \mathfrak{n}^{(1)}E_{11}^{(1)}) + \mathfrak{T}_y \; ; \qquad Q_y = \Gamma_y E_{23}^{(o)} \; . \tag{3.116}$$

$$M_{xy} = -2M^{(2)}E_{12}^{(1)} \; ;$$

The strain-displacement relations:

$$E_{\alpha\beta}^{(o)} = \frac{1}{2}\left[u_{\alpha,\beta}^{(o)} + u_{\beta,\alpha}^{(o)} + \frac{\partial\eta}{\partial x_\alpha}\frac{\partial\eta}{\partial x_\beta}\right] \; ; \quad E_{11}^{(1)} = -\frac{\partial\varphi}{\partial x} \; ;$$

$$E_{13}^{(o)} = \frac{1}{2}\left(\frac{\partial\eta}{\partial x} - \varphi\right) \; ; \quad E_{22}^{(1)} = -\frac{\partial\psi}{\partial y} \; ; \tag{3.117}$$

$$E_{23}^{(o)} = \frac{1}{2}\left(\frac{\partial\eta}{\partial y} - \psi\right) \; ; \quad E_{12}^{(1)} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right) \; .$$

For flexural vibrations these equations may be reduced further by disregarding horizontal accelerations $\ddot{u}$ and $\ddot{v}$. In this event the equilibrium equations $(3.114)_{1,2}$ for the membrane forces are satisfied identically by introducing the stress function

$$N_x = \frac{\partial^2 F}{\partial y^2} \; ; \quad N_y = \frac{\partial^2 F}{\partial x^2} \; ; \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \; .$$

If these expressions are substituted into $(3.115)_{1,2,3}$ and the resulting equations are inverted the zeroth order strain components become

$$E_{11}^{(o)} = \frac{1}{\mathfrak{D}^{(o)}(1-\mathfrak{N}^{(o)2})}\left\{\frac{\partial^2 F}{\partial y^2} - \mathfrak{N}^{(o)}\frac{\partial^2 F}{\partial x^2} - (\mathfrak{T}_x - \mathfrak{N}^{(o)}\mathfrak{T}_y)\right\}$$

$$E_{22}^{(o)} = \frac{1}{\mathfrak{D}^{(o)}(1-\mathfrak{N}^{(o)2})}\left\{\frac{\partial^2 F}{\partial x^2} - \mathfrak{N}^{(o)}\frac{\partial^2 F}{\partial y^2} - (\mathfrak{T}_y - \mathfrak{N}^{(o)}\mathfrak{T}_x)\right\} \qquad (3.118)$$

$$E_{12}^{(o)} = \frac{-1}{2M^{(o)}}\frac{\partial^2 F}{\partial x \partial y} \ .$$

On the other hand, (3.117) imply

$$\frac{\partial^2}{\partial y^2}(E_{11}^{(o)}) + \frac{\partial^2}{\partial x^2}(E_{22}^{(o)}) - 2\frac{\partial^2}{\partial x\partial y}(E_{12}^{(o)}) = \left(\frac{\partial^2\eta}{\partial x\partial y}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2}, \qquad (3.119)$$

which when (3.115) is used becomes

$$\frac{\partial^4 F}{\partial x^4} + \mathfrak{H}\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = \mathfrak{E}\left[\left(\frac{\partial^2\eta}{\partial x\partial y}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2}\right] \qquad (3.120)$$

with

$$\mathfrak{H} = \frac{\mathfrak{D}^{(o)}(1-\mathfrak{N}^{(o)^2})}{2M^{(o)}} - 2\,\mathfrak{N}^{(o)} \ ;$$

$$\qquad (3.121)$$

$$\mathfrak{E} = \mathfrak{D}^{(o)}(1-\mathfrak{N}^{(o)^2}) \ ,$$

where use has been made of the fact that temperature varies only in the direction perpendicular to the plate reference plane. Note that an equation similar to (3.117) also occurs in the von Kàrmàn plate theory with $\mathfrak{H} = 2$. Here, we note that this assumption cannot be made.

In view of the particular form of equation (3.117) it seems to be appropriate to pause and to compare the linear version of (3.117) with

(3.74) which was derived by neglecting all nonlinear terms and restricting to zeroth order terms only. The difference in these two limiting cases lies only in the coefficient $\mathfrak{H}$, which differs here from 2. The reason for this fictitious discrepancy is the following: Equation (3.78) is based upon two dimensional deformation and thus valid for plain strain situation. On the other hand in equation (3.117) or its linear counterpart, the coefficient $\mathfrak{H}$ is determined by a condition, which could be called plane stress situation. The zeroth and first order stress resultants $T_{33}^{(0)}$ and $T_{33}^{(1)}$ have been set to zero, which determined the corresponding strain components in terms of others. Clearly, setting $T_{33}^{(0)}$ and $T_{33}^{(1)}$ to zero does not imply a true plane stress situation, but an approximate one. Based on this condition the coefficient $\mathfrak{H}$ is obtained. Hence, the deviation of $\mathfrak{H}$ from 2 is based upon the difference of plane strain-plane stress situation and does not have any connection with the nonlinearities involved in the theory.

Finally, substituting (3.117) into the third equation of (3.111) and using (3.113) in (3.112) the second set of equations for the dynamic response is obtained:

$$\mathfrak{M}\ddot{\eta} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 \eta}{\partial x^2} + 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 \eta}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 \eta}{\partial y^2}$$

$$+ \left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right)\frac{\partial \eta}{\partial x} + \left(\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x}\right)\frac{\partial \eta}{\partial y} \quad ;$$

$$- \mathfrak{J}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x \quad ;$$

$$- \mathfrak{J}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y \quad ;$$

$$M_x = - \mathfrak{D}^{(1)}[\frac{\partial\varphi}{\partial x} + \mathfrak{n}\frac{\partial\psi}{\partial y}] + \mathfrak{T}_x \quad ;$$

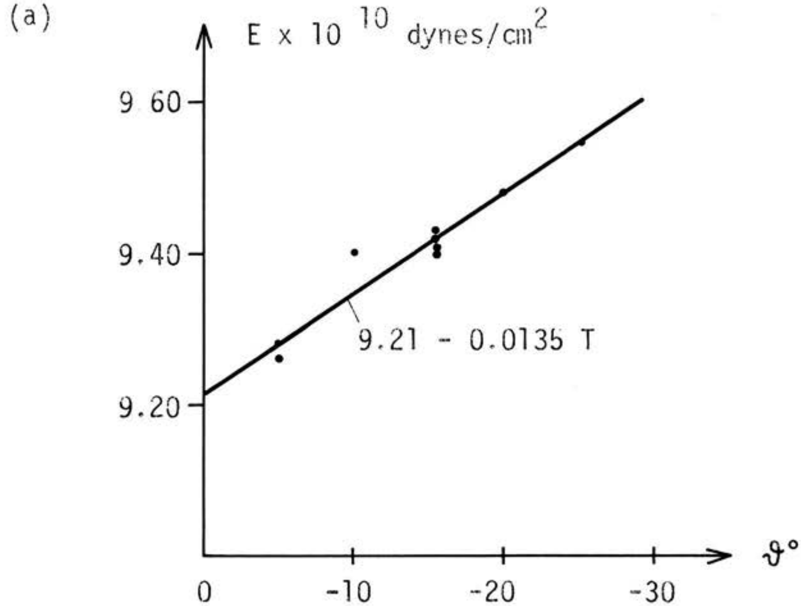
$$M_y = - \mathfrak{D}^{(1)}[\frac{\partial\varphi}{\partial y} + \mathfrak{n}\frac{\partial\psi}{\partial x}] + \mathfrak{T}_y \quad ; \qquad (3.122)$$

$$M_{xy} = -M^{(2)}(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}) \quad ;$$

$$Q_x = \frac{1}{2}\Gamma_x(\frac{\partial\eta}{\partial x} - \varphi) \quad ;$$

$$Q_y = \frac{1}{2}\Gamma_y(\frac{\partial\eta}{\partial y} - \psi) \quad .$$

Moreover, neglecting all nonlinear terms we obtain the bending theory (3.91), (3.92) and (3.93).



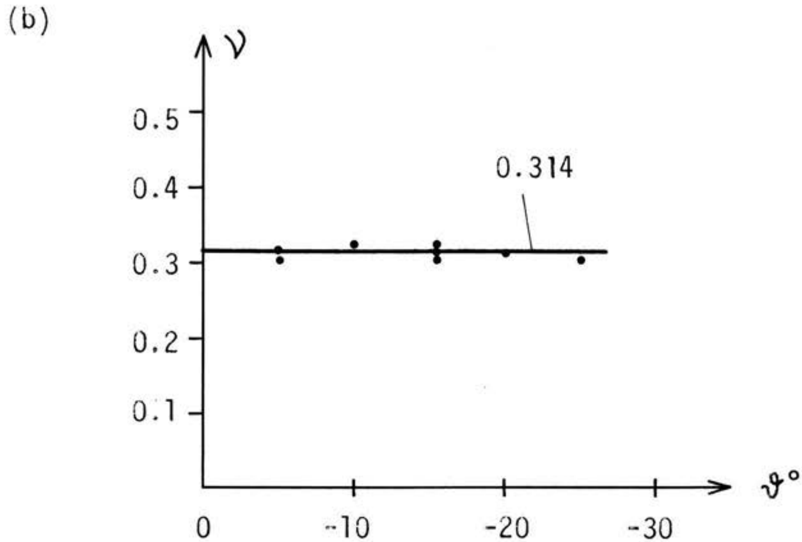
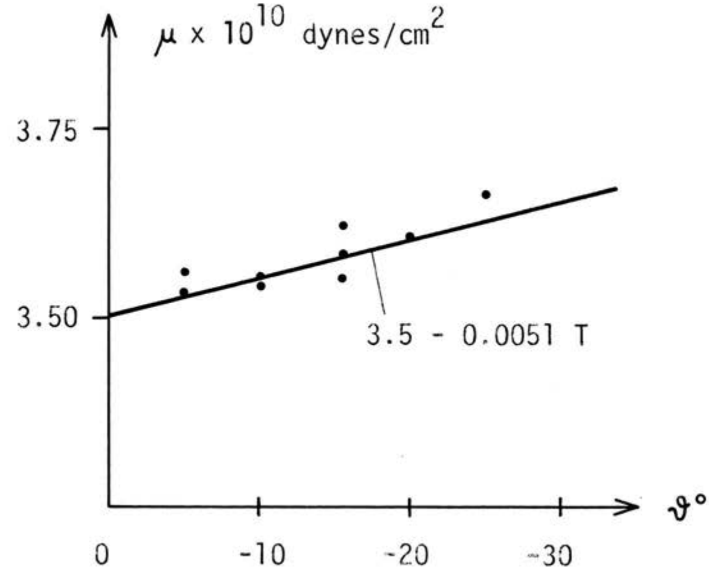


Fig.3.3: Temperature variation of the elastic constants

(c)



(d)

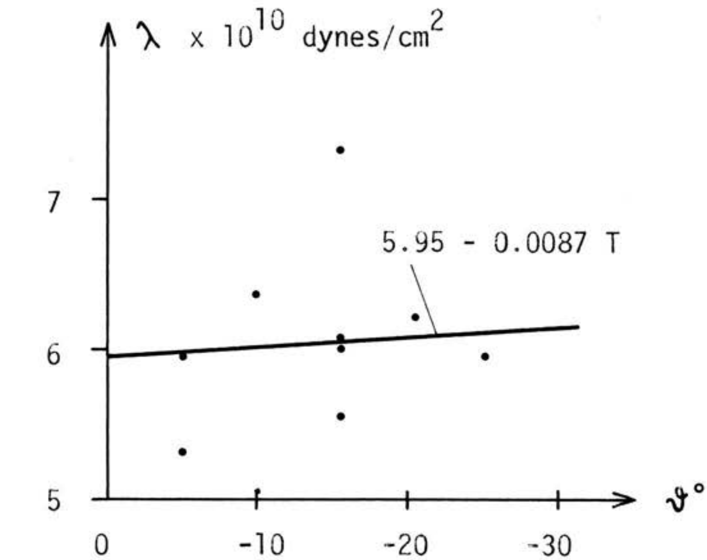


Fig.3.3: Temperature variation of the elastic constants

3.9) Numerical Values for the Plate Constants.

The preceding calculations make use of various constants which are different from the familiar ones in ordinary plate theory. To make this theory amenable to explicit calculations we shall list the pertinent constants in this section. Because temperature variation is essential in this theory, we first need some information with regard to the temperature variation of the elastic constants, such as for Young's modulus, the shear modulus, Poisson's ratio and Lamé's constants, respectively. For polycrystalline ice with randomly oriented crystals these constants have been calculated by various authors from the corresponding single crystal properties. Röthlisberger [31] summarizes work of Jona and Scherrer[32], Bass [33] et al, Brockamp and Querfurth [34] and Bennett [35]. Figs. 3.3a, b and c are direct reinterpretations of Table Va of Röthlisberger and Fig. 3.3d is calculated from known relations of Lamé's constants to Young's modulus and Poisson's ratio. It is seen that to a good order of approximation Poisson's ratio may be assumed to be independent of the temperature with value*

$$\nu = 0.314 \quad ; \quad \text{(polycrystalline ice)} \quad .$$

Furthermore, while E, μ and ν do not show considerable scatter, this is not so for λ. The reason might be a higher sensitivity to errors in the experimental values.

Two constants suffice to determine any other two. The averages, shown as straight lines in Figure 3.3c and d are thus calculated from

* Note that for sea ice ν cannot be assumed to be a constant.

those given for E and ν. A linear approximation suffices. It is for this reason that the average value as based upon the linear temperature dependence may, despite the considerable scatter, nevertheless be considered to be an appropriate value. Thus we obtain:

$$\frac{E(\vartheta)}{E(0)} = 1 - 0.00146\vartheta \quad ; \quad E(0) = 9.21\,[\text{dynes cm}^{-2}] \quad ;$$

$$\nu(\vartheta) = 0.314 \quad ;$$

$$\frac{\mu(\vartheta)}{\mu(0)} = 1 - 0.00146\vartheta \quad ; \quad \mu(0) = 3.5 \quad ; \quad [\text{dynes cm}^{-2}] \quad ;$$

$$\frac{\lambda(\vartheta)}{\lambda(0)} = 1 - 0.00146\vartheta \quad ; \quad \lambda(0) = 5.95 \quad ; \quad [\text{dynes cm}^{-2}] \quad .$$

In these formulas the temperature is given in centigrades.

In order to obtain numerical values for $\Lambda^{(m)}$ and $M^{(m)}$ as given by (3.47b) we assume for the sake of simplicity a linear distribution of the temperature across the thickness of the ice with temperatures ϑ^u and ϑ^l at the upper and lower face, respectively. Without loss of generality we may assume $\vartheta^l = 0$, because the water temperature at the lower face is at the freezing point. Then (see Fig. 3.4)

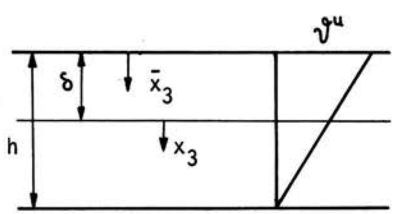


Fig 3.4

$$\Lambda^{(1)} = \int_h x_3 \lambda(\vartheta)\,ds = 0$$

implies

$$\int_h (\bar{x}_3 - \delta)\lambda(\vartheta)\,ds = 0 \ ,$$

or

$$\delta = \frac{\int_h \bar{x}_3 \lambda(\vartheta)\,ds}{\int_h \lambda(\vartheta)\,ds} \ . \tag{3.123}$$

Because

$$\vartheta(\bar{x}_3) = (h-\bar{x}_3)(\vartheta^u/h) \ ,$$

one has

$$\lambda(\vartheta) = \lambda(0)[1-0.00146\frac{h-\bar{x}_3}{h}\vartheta^u] \ , \tag{3.124}$$

which when used in the integrals (3.123) gives

$$\frac{\delta}{h} = \frac{(0.5-0.000244\ \vartheta^u)}{(1\ -0.000732\ \vartheta^u)} \ .$$

Its asymptotic value as $\vartheta^u \to -\infty$ is $(\delta/h)_\infty = 0.334$, but this value is never significant. For the range $0° \leq \vartheta \leq -100°$ namely, a Taylor series representation of (3.124) is accurate enough. Thus

$$\frac{\delta}{h} \cong 0.5 + 0.000122 \cdot \vartheta^u \ , \tag{3.125}$$

where ϑ^u is to be substituted in centigrades.

Once the position of the reference plane (x_1, x_2) has been fixed, we are also in position to determine $\Lambda^{(m)}$ and $M^{(m)}$ for $m \neq 1$. We obtain from (3.47b) and (3.121)

$$\Lambda^{(o)} = \lambda(0)h[1-0.000732 \; \vartheta^u] \; ;$$

$$\Lambda^{(2)} = \frac{\lambda(0)h^3}{12}[1-0.000293 \; \vartheta^u] \; ;$$

$$M^{(o)} = \mu(0)h[1-0.000732 \; \vartheta^u] \; ; \tag{3.126}$$

$$M^{(2)} = \frac{\mu(0)h^3}{12}[1-0.000293 \; \vartheta^u] \; .$$

With the above numerical values the plate rigidities D, N, (3.87), $\mathfrak{D}^{(o)}$, $\mathfrak{N}^{(o)}$, $\mathfrak{D}^{(1)}$, $\mathfrak{N}^{(1)}$, (3.107), and $\mathfrak{D}^{(o)}$ $\mathfrak{n}^{(o)}$ and $\mathfrak{D}^{(1)}$ $\mathfrak{n}^{(1)}$ and Γ_x and Γ_y can be evaluated. After some manipulations we obtain

$$D = \frac{h^3}{12}\frac{2\mu(0)}{1-\nu}(1-0.000586 \; \vartheta^u) \; ;$$

$$N = \nu \; ;$$

$$\mathfrak{D}^{(o)} = \frac{2\mu(0)}{1-\nu}h(1-0.000732 \, \vartheta^u) \; ;$$

$$\mathfrak{N}^{(o)} = \nu \; ; \tag{3.127}$$

$$\mathfrak{D}^{(1)} = \frac{h^3}{12}\frac{2\mu(0)}{1-\nu}(1-0.000293 \; \vartheta^u) \; ;$$

$$\mathfrak{n}^{(1)} = \nu \; ;$$

$$\Gamma_x = \Gamma_y = 2\mu(0)h(1-0.000732 \; \vartheta^u) \; .$$

With these quantities, all plate constants used in the governing equations are known, once the temperature at the upper face is known. For the evaluation of the thermal effects, however, we also need $\mathfrak{T}_x$, $\mathfrak{T}_y$, $\mathfrak{I}_x$ and $\mathfrak{I}_y$, as given in (3.107) and (3.109). For their evaluation the zeroth and first temperature resultants are needed. According to (3.27) these are obtained by a Taylor series expansion in the variable x_3 of the temperature function. Clearly, for a linear distribution only $\Theta^{(o)}$ and $\Theta^{(1)}$ survive

$$\vartheta(x_3) = (1-\frac{\delta}{h})\,\vartheta^u - \frac{x_3}{h}\,\vartheta^u = \Theta^{(o)} + x_3\Theta^{(1)} \tag{3.128}$$

Thus

$$\begin{aligned} \Theta^{(o)} &= (1-\frac{\delta}{h})\,\vartheta^u \cong 0.5\,\vartheta^u \quad ; \\ \Theta^{(1)} &= -\,\vartheta^u/h \quad . \end{aligned} \tag{3.129}$$

Substituting the expressions (3.126) into (3.103) and the results into the last equations (3.107) and (3.109), respectively, we obtain

$$\begin{aligned} \mathfrak{T}_x &= \mathfrak{T}_y = -3\alpha h\,\frac{K}{1-\nu}(1-0.000732\,\vartheta^u) \quad ; \\ \mathfrak{I}_x &= \mathfrak{I}_y = -\alpha h^3\frac{K}{4(1-\nu)}(1-0.000732\,\vartheta^u) \quad , \end{aligned} \tag{3.130}$$

where K is the bulk modulus, given by

$$K = (3\lambda(0)+2\mu(0))/3 \quad . \tag{3.131}$$

Before concluding this section, we give some numerical values for the constants $\mathfrak{S}$, $\mathfrak{H}$ and $\mathfrak{K}$. It is easily seen that in view of the definition of $\mathfrak{S}$ and because of (3.121) one has

$\mathfrak{S} = h^3 \times 0.000122\ \vartheta^u$, $\mathfrak{J} \cong 2$ and $\mathfrak{E} = 2\mu(0)(1+\nu)(1-0.000732\ \vartheta^u)$.

This shows that in case ν is not a function of the temperature, $\mathfrak{J} \cong 2$, which agrees with the plane strain situation. Furthermore, the extreme small value of $\mathfrak{S}$ justifies its neglection.

Chapter 4.) Plate as a Viscoelastic Material

4.1) Introduction

In the preceding Chapter the plate equations have been derived on the basis of a purely elastic behavior of the material. Clearly, such simplified behavior can only be assumed when the processes under investigation are fast. Slow processes should be based upon a model which takes dissipation into account. For floating ice this has been evidenced by Nevel [1], who on the basis of the classical plate theory, in which the elastic constants are replaced by their complex counterparts of viscoelasticity, shows that the stress relaxes with time. Consequently, also the displacement field must show an analogous property.

Nevel's result does not take into account a temperature variation of the ice with position; his relaxation and creep functions are temperature independent. Hence, for the present purpose, where a non-uniform temperature distribution is assumed, a more general approach must be considered. Moreover, Nevel uses a viscoelastic model consisting of a combination of springs and dashpots, an approach which generally overemphasizes the value of the model.

The above preliminary remarks seem to indicate that it suffices to develop a theory of viscoelasticity of which the relaxation and creep functions are temperature dependent. One could then average over the plate thickness as was done for the elastic case in Section 3.6. Careful investigation of such a procedure, however, shows that one does not obtain a term taking thermal expansion into account. Moreover, experimentally the relaxation and creep functions have not been systematically determined as functions of the temperature. The averaged quantities therefore could not be evaluated.

Considerable simplification is achieved, when the temperature dependence of the mechanical properties is based upon a postulate which is borrowed from thermodynamic theories of high polymeric solids. This special class of materials is referred to as being <u>thermorheologically simple</u>. The corresponding description of the temperature dependent properties was first proposed by Leaderman [36] and Ferry [37], [38]. It was further developed by Schwarzl and Staverman [34] who introduced the term "thermorheologically simple". Thermodynamic aspects are discussed by Staverman and Schwarzl [40] and by Hunter, [41]. The theory has been reinvestigated in two articles by Crochet and Naghdi [42], [43]. These authors extend the interpretation of the term "thermorheologically simple". Accounts of similar nature are by Morland and Lee [44], but their formulation is not general enough to account for finite elasticity. Further literature on thermorheologically simple material is to be found in the text books of Pipkin [45], Christensen [46] and Lockett [47].

The theory of Crochet and Naghdi is rather complex insofar as it requires the knowledge of some basic facts in functional analysis. However, it has various advantages as compared to the other more ad hoc treatments, because (i) it allows to formulate a <u>finite</u> linear theory and (ii) it contains the effect of thermal stresses. In a proper formulation both effects should be contained in the theory. Property (i) allows us to formulate the viscoelastic counterpart of the generalized Reissner - von Kàrmàn plate theory.

In concluding this section it seems appropriate to note that the constitutive assumptions (3.39) actually are the elastic analogue of the thermorheologically simple solid presented here.

4.2) Preliminaries and Functional Analytic Background

In Section 3.6 an elastic material with parameter ϑ (the temperature) was defined as a material the response functions of which were only dependent upon the deformation gradient F_{ij} and the parameter ϑ. In view of the hereditary effects which must be included now the response of a material also depends upon the values of the deformation gradient and temperature prior to the present time. In accord with common usage we define the history up to time t of the deformation gradient and of the temperature, at a fixed material point to be, respectively

$$F_{ij}^{t}(s) = F_{ij}(t-s) \quad ; \quad \vartheta^{t}(s) = \vartheta(t-s) \quad ; s\in[0,\infty). \qquad (4.1)$$

Moreover, the restriction of $F_{ij}^{t}(s)$ to the open interval $(0,\infty)$ is called the past history and will be denoted by $F_{ij_r}^{t}(s)$. A similar definition also holds for $\vartheta_r^t(s)$. Furthermore $F_{ij}^{t}(0) = F_{ij}^{(t)} = F_{ij}$ and $\vartheta^t(0) = \vartheta(t) = \vartheta$ are called the present values of the deformation gradient and temperature, respectively.

With these preliminary remarks we are now in a position to state the constitutive assumptions which replace the ones given in Section 3.6 (equation 3.31). To this end, let f be any one of the dependent variables. Then a simple material with parameter ϑ is defined as a material obeying the following constitutive assumption:

$$f = \mathop{\mathfrak{F}}_{s=0}^{\infty}(F_{ij_r}^{t}(s), \vartheta_r^t(s); F_{ij}, \vartheta) \quad . \qquad (4.2)$$

Here, $\overset{\infty}{\underset{s=o}{\mathfrak{F}}}(\cdot)$ is a functional over the history of the deformation gradient and the parameter ϑ. The notation $\overset{\infty}{\underset{s=o}{\mathfrak{F}}}$ indicates that s varies over the entire past history $(0,\infty)$. $\overset{\infty}{\underset{s=o}{\mathfrak{F}}}(\cdot)$ may be given for example by an integral over s.

The representation (4.2) is assumed to hold for the stress T_{ij}, free energy ψ, entropy η and heat flux Q_i, explicitly :

$$
\begin{aligned}
T_{ij} &= \overset{\infty}{\underset{s=o}{\mathfrak{T}}}{}_{ij}(F^t_{\ell m_r}(s),\ \vartheta^t_r(s);F_{\ell m},\vartheta) \quad ; \\
\psi &= \overset{\infty}{\underset{s=o}{\mathfrak{P}}}(F^t_{\ell m_r}(s),\ \vartheta^t_r(s);F_{\ell m},\vartheta) \quad ; \\
\eta &= \overset{\infty}{\underset{s=o}{\mathfrak{G}}}(F^t_{\ell m_r}(s),\ \vartheta^t_r(s);F_{\ell m},\vartheta) \quad ; \\
Q_i &= \overset{\infty}{\underset{s=o}{\mathfrak{Q}}}{}_i(F^t_{\ell m_r}(s),\ \vartheta^t_r(s);F_{\ell m},\vartheta)
\end{aligned}
\qquad (4.3)
$$

These are not materially objective yet, that is they are not invariant under time dependent orthogonal transformations of the form

$$x'_i = \mathcal{O}_{ij}(t)x_j + c_i(t) \quad . \qquad (4.4)$$

Under such transformations T_{ij}, ψ, η and Q_i transform as follows:

$$
\begin{aligned}
T'_{ij} &= \mathcal{O}_{ik}T_{kj} \quad ; \quad \psi' = \psi \quad ; \quad \eta' = \eta \quad ; \\
Q'_i &= \mathcal{O}_{ij}Q_j
\end{aligned}
\qquad (4.5a)
$$

while

$$F'_{ij} = \mathcal{O}_{ik}F_{kj} \quad ; \quad \vartheta' = \vartheta \quad . \qquad (4.5b)$$

Based upon these transformation properties, introducing the second Piola-Kirchhoff tensor

$$\Sigma_{ij} = (F^{-1})_{kj}T_{ki} \quad , \tag{4.6}$$

it can be shown that the constitutive equations must have the form

$$\begin{aligned}
\psi &= \mathop{\mathfrak{P}}_{s=o}^{\infty}(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ; \\
\eta &= \mathop{\mathfrak{S}}_{s=o}^{\infty}(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ; \\
\Sigma_{mn} &= \mathop{\mathfrak{T}}_{s=o}^{\infty}{}_{mn}(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ; \\
Q_m &= F_{mn}\mathop{\mathfrak{Q}}_{s=o}^{\infty}{}_n(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ,
\end{aligned} \tag{4.7}$$

where $E_{ij}(s)$ is the elongation tensor defined in (3.36). The domain of $E^t_{ij_r}(s)$ is the positive real line.

We assume that the response functionals (4.7) satisfy the principle of fading memory due to Coleman and Noll [48]. Accordingly let $h(s)$ be a po tive monotonically decreasing continuous function over $[0,\infty)$, which decays fast enough to be square integrable. Let $\mathcal{V}_{(7)}$ be the vector space of the ordered pairs $\Lambda_{ij} = (S_{\ell m}, \vartheta)(\ell,m=1,2,3;i,j=1,\ldots,4)$, where $S_{\ell m}$ are symmetric tensors. With the inner product

$$(\Lambda^1_{ij},\Lambda^2_{ij}) = S^1_{\ell m}S^2_{\ell m} + \vartheta^1 \vartheta^2 \quad , \tag{4.8}$$

$\mathcal{V}_{(7)}$ becomes a Hilbert space. Denoting the norm corresponding to (4.8) by $\|\cdot\|$, we introduce a Banach space $\mathcal{A}_{(7)}$ with the norm

$$\|\Lambda_{ij}\|_h = \left\{ \int_0^\infty h^2(s)\|\Lambda_{ij}(s)\|^2 ds \right\}^{1/2} . \qquad (4.9)$$

In what follows we restrict our attention to materials of which the response functionals depend in a continuously differentiable way upon the histories of E_{ij} and ϑ. Furthermore we presume that the free energy functional $\overset{\infty}{\underset{s=0}{\mathfrak{P}}}(\cdot)$ depends in a twice continuously differentiable way upon these histories. In other words we assume that the functionals be Fréchet differentiable over the Hilbert space $\mathcal{V}_{(7)}$, while the free energy functional possesses Fréchet derivatives up and including to the second order.

It has been shown by Coleman [49] that in a material satisfying the above smoothness properties and the thermodynamic principle as illustrated in Section 3.6, the response functionals are given as follows*:

$$\mathfrak{S} = -\frac{\partial}{\partial\vartheta}\overset{\infty}{\underset{s=0}{\mathfrak{P}}}(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ; \qquad (4.10)$$

$$\mathfrak{T}_{ij} = \frac{1}{\rho_o}\frac{\partial}{\partial E_{ij}}\overset{\infty}{\underset{s=0}{\mathfrak{P}}}(E^t_{ij_r}(s), \vartheta^t_r(s); E_{ij}, \vartheta) \quad ; \qquad (4.11)$$

$$\mathfrak{Q}_i = 0 . \qquad (4.12)$$

---
* Note that, because we do not include temperature gradients in our theory, the situation is slightly different from that dealt with by Coleman. However, an easy argument shows that the only difference is in (4.12).

Let ϑ_X be a fixed reference temperature at a particle X and consider processes such that

$$\vartheta^t(s) = \vartheta_X 1^+(s) \quad , \tag{4.13}$$

where $1^+(s)$ is the function which maps the positive real line into the real number 1, i.e. $1^+(s) = 1$; $s \in [0,\infty)$. Thus we define

$$\mathop{\mathfrak{P}}_{s=0}^{\infty}(E^t_{ij_r}(s), \vartheta_X 1^+(s), E, \vartheta_X) = \mathfrak{P}^*_X(E^t_{ij_r}(s), E_{ij}) \quad ;$$

$$-\frac{\partial}{\partial\vartheta}\mathop{\mathfrak{P}}_{s=0}^{\infty}(E^t_{ij_r}(s);\ \vartheta^t_r(s), E_{ij}, \vartheta)\Big|_{\vartheta^t_r(s) = \vartheta_X 1^+(s)} = \mathfrak{S}^*_X(E^t_{ij_r}(s), E_{ij});$$

$$\frac{\partial}{\partial E_{mn}}\mathop{\mathfrak{P}}_{s=0}^{\infty}(E^t_{ij_r}(s), \vartheta_X 1^+(s), E, \vartheta_X) = \frac{1}{\rho_o}\mathfrak{T}^*_{X_{mn}}(E^t_{ij_r}(s), E_{ij}) \quad . \tag{4.14}$$

Similarly, assuming the constitutive equation for stress $(4.7)_3$ to be invertible in the sense that

$$E_{mn} = \mathop{\mathfrak{E}}_{s=0}^{\infty}{}_{mn}(\Sigma^t_{ij_r}(s), \vartheta^t_r(s); \Sigma_{ij}, \vartheta) \quad , \tag{4.15}$$

we also may define the functional

$$\mathop{\mathfrak{E}}_{s=0}^{\infty}{}_{mn}(\Sigma^t_{ij_r}(s), \vartheta_X 1^+(s); \Sigma_{ij}, \vartheta_X) = \mathop{\mathfrak{E}}_{s=0}^{\infty}{}^*_{Xmn}(\Sigma^t_{ij_r}(s), \Sigma_{ij}) \quad . \tag{4.16}$$

The starred functionals will be called local <u>isothermal functionals</u>. On the other hand, let

$$0^{+}_{ij}(s) \;=\; 0 \qquad s\epsilon[0,\infty) \tag{4.17}$$

be the zero function and consider a history for which $E^{t}_{ij}(s) = 0^{+}_{ij}(s)$. Then we define

$$\mathop{\mathfrak{P}}_{s=0}^{\infty}(0^{+}_{ij}(s),\ \vartheta^{t}_{r}(s);0^{+}_{ij},\ \vartheta\,) \;=\; \bar{\mathfrak{B}}\,(\,\vartheta^{t}_{r}(s),\ \vartheta\,)\ . \tag{4.18}$$

4.3) Constitutive Equations for Thermorheologically Simple Materials

Suppose that the material is initially at rest with a temperature distribution ϑ_X (nonuniform). Let $\Sigma_{Y_{ij}}$ be the state of stress at zero strain and temperature ϑ_Y at a reference particle Y. Then, corresponding to the constant stress history the functional (4.15) assumes the form

$$\overset{\infty}{\underset{s=o}{\mathfrak{E}}}_{mn} (\Sigma_{Y_{ij}} 1^+(s),\ \vartheta_r^t(s); \Sigma_{Y_{ij}},\ \vartheta\ ) \ . \tag{4.19}$$

Its value for the temperature history $\vartheta(s) = \vartheta_Y 1^+(s)$ must vanish, by the very definition of $\Sigma_{Y_{ij}}$ and ϑ_Y. However, its value at any other particle with $\vartheta^t = \vartheta_X 1^+(s)$ need not vanish. Because $\Sigma_{Y_{ij}}$ is fixed, the functional in (4.19) can also be expressed as a new functional $\overset{\infty}{\underset{s=o}{\mathfrak{U}}}_{Y_{ij}}(\ \vartheta_r^t(s),\ \vartheta\ )$ its value will be called $\alpha_{ij}(t)$. Thus

$$\alpha_{ij}(t) \ = \ \overset{\infty}{\underset{s=o}{\mathfrak{U}}}_{Y_{ij}}(\ \vartheta_r^t(s),\ \vartheta\ )\ , \tag{4.20}$$

with

$$\overset{\infty}{\underset{s=o}{\mathfrak{U}}}_{Y_{ij}}(\ \vartheta_Y 1^+(s),\ \vartheta_Y) \ = \ 0\ . \tag{4.21}$$

We now decompose the strain functional in two parts, one that is given by (4.20) and is independent of the stress history and the other part given by

$$E_{ij}(t) - \alpha_{ij}(t) \ = \ \overset{\infty}{\underset{u=o}{\tilde{\mathfrak{E}}}}_{ij}(\Sigma^t_{k\ell_r}(u),\ \vartheta_r^t(u); \Sigma_{k\ell}, \vartheta). \tag{4.22}$$

As a direct consequence of this

$$E_{ij}^{t}(s) - \alpha_{ij}^{t}(s) = \hat{\mathfrak{E}}_{u=0}^{\infty}{}_{ij}(\Sigma_{k\ell}^{t}(s+u),\ \vartheta^{t}(s+u)) \quad . \tag{4.23}$$

Similarly (4.16) may be written as

$$E_{ij}^{t}(s)\Big|_{\vartheta^{t}(s) = \vartheta_{X}1^{+}(s)} = \overset{\infty}{\underset{u=0}{\mathfrak{E}}}{}^{*}_{X_{ij}}(\Sigma_{k\ell}^{t}(s+u)) \quad , \tag{4.24}$$

which according to $(4.14)_3$ and the assumed invertibility property of the stress functional has an inverse

$$\Sigma_{ij}^{t}(s) = \overset{\infty}{\underset{u=0}{\mathfrak{I}}}{}^{*}_{Y_{ij}}(E_{k\ell}^{t}(s+u)) \quad . \tag{4.25}$$

One more item is needed before the constitutive postulate can be established. We introduce a <u>modified time scale</u> which only depends upon the history of the temperature through the functional form

$$\xi^{t}(s) = \xi_{s} = \overset{\infty}{\underset{u=0}{\mathfrak{Z}}}(\ \vartheta_{r}^{t}(u), s) \quad , \tag{4.26}$$

where $\overset{\infty}{\underset{u=0}{\mathfrak{Z}}}(\cdot)$ has the properties such that

$$\begin{aligned} &\xi_{s}(s=0) = 0 \quad ; \\ &\partial\xi_{s}/\partial s > 0 \quad ; \\ &\xi_{s}\Big|_{\vartheta = \vartheta_{Y}} = s \quad , \end{aligned} \tag{4.27}$$

and where ϑ_{Y} designates the constant temperature at the fixed particle Y.

It is worthwhile to note again that for our situation the temperature distribution is known a priori so that at particle X, $\vartheta^t(s) = \vartheta_X 1^+(s)$. For this special case the functional (4.26) becomes a function of ϑ_X. A special form of (4.26) is then

$$\xi_s = (1/\chi(\vartheta_X))s \quad ,$$

or (4.28)

$$\xi_s \chi(\vartheta_X) = s$$

with the properties (see (4.27))

$$\chi(\vartheta_X) > 0 \quad ;$$

(4.29)

$$\chi(\vartheta_Y) = 1 \quad .$$

We shall restrict further considerations to the law (4.28) and (4.29).

We can now introduce the main hypothesis of thermorheologically simple solids (Crochet and Naghdi [42]): The functional relationship between $E^t_{ij}(s) - \alpha^t_{ij}(s)$ and Σ_{ij} is such that the constitutive functional $\hat{\mathfrak{E}}_{ij}(\cdot)$ in (4.22) is equivalent to the isothermal functional $\mathfrak{E}^*_{Y_{ij}}(\cdot)$ at the reference particle Y (given in (4.24)) in terms of the same history of stress, but with a modified time scale defined by (4.26) (or equivalently (4.28)). Hence we write*

---

* Note that E^t_{ij} and α^t_{ij} in (4.30) are evaluated at the particle X. Hence the notation

$$E^t_{X_{ij}}(\xi_s) - \alpha^t_{X_{ij}}(\xi_s) = \mathop{\hat{\mathfrak{E}}}_{u=0}^{\infty}{}^*_{Y_{ij}}(\Sigma^{\ t}_{X_{k\ell}}(\xi_{u+s}))$$

would be more precise. The same is true for the formulas (4.31)-(4.33).

$$E^t_{ij}(\xi_s) - \alpha^t_{ij}(\xi_s) = \mathop{\hat{\mathfrak{E}}}_{u=0}^{\infty}{}^{*}_{Y_{ij}}(\Sigma^t_{k\ell}(\xi_{u+s})) \quad . \tag{4.30}$$

Because $\hat{\mathfrak{E}}^{*}_{Y_{ij}}(\cdot)$ has the inverse $\mathfrak{I}^{*}_{Y_{ij}}(\cdot)$, (see (4.25)), it follows

$$\Sigma^t_{ij}(\xi_s) = \mathop{\mathfrak{I}}_{u=0}^{\infty}{}^{*}_{Y_{ij}}(E^t_{k\ell}(\xi_{u+s}) - \alpha^t_{k\ell}(\xi_{u+s})) \ . \tag{4.31}$$

In particular when $s = 0$, or equivalently $\xi_s = 0$, then

$$\Sigma_{ij}(t) = \mathop{\mathfrak{I}}_{u=0}^{\infty}{}^{*}_{Y_{ij}}({E_{k\ell}}^t(\xi_u)-{\alpha_{k\ell}}^t(\xi_u)) \ , \tag{4.32}$$

or when the past history and the present values are separated

$$\Sigma_{ij}(t) = \mathop{\mathfrak{I}}_{u=0}^{\infty}{}^{*}_{Y_{ij}}({E_{k\ell}}_r^t(\xi_u)-{\alpha_{k\ell}}_r^t(\xi_u);E_{k\ell}-\alpha_{k\ell}) \quad . \tag{4.33}$$

Thus the stress at the particle X is given by the isothermal functional of particle Y at the reference temperature ϑ_Y whereby the strain history is replaced by $(E^t_{ij}-\alpha^t_{ij})$. Since this functional is fixed, once the reference particle is chosen, one could refrain from listing the subscript Y. However, we shall not do it here, because the subscript is a reminder that the base temperature distribution is nonuniform. Note further that in view of (4.21) and $(4.29)_2$ the stress at the particle Y is given by the ordinary isothermal response (4.12).

It remains to establish explicit constitutive equations for the free energy ψ and for α_{ij}. Since we are restricting considerations to locally isothermal thermodynamic states, we do not present the complete theory which is given by Crochet and Naghdi [42], but only list the quantities which are of importance in our case. We choose the isothermal free energy as*

$$\mathfrak{F}^*_Y = \frac{1}{2}\rho_o \int_o^\infty \int_o^\infty [E_{ij_r}{}^t(s)-E_{ij}]M_{ijk\ell}(s,u)[E_{k\ell_r}{}^t(u)-E_{k\ell}]ds\, du \quad . \tag{4.34}$$

Here $M_{ijk\ell}$ is a fourth order tensor function of the variables s and u which we have assumed to be independent of the Lagrangian strain. Clearly,

$$M_{ijk\ell} = M_{jik\ell} = M_{ij\ell k} = M_{k\ell ij} \quad . \tag{4.35}$$

In (4.34) we have excluded a term linear in the strains, because it can be shown that thermodynamic restrictions require it to be zero. The stress at the reference particle Y is obtained from (4.34) by applying (4.14). Thus,

$$\Sigma_{Y_{ij}} = -\int_o^\infty (E_{k\ell_r}{}^t(u)-E_{k\ell}) \left\{\int_o^\infty M_{ijk\ell}(s,u)ds\right\} du \quad . \tag{4.36}$$

Defining

$$\tilde{G}_{ijk\ell}(s,u) = \int_s^\infty \int_u^\infty M_{ijk\ell}(\sigma,\rho)d\sigma\, d\rho \quad , \tag{4.37}$$

where

$$\tilde{G}_{ijk\ell} \to 0 \text{ as } u \to \infty \text{ or } s \to \infty \quad ,$$

we obtain by an integration by parts

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* In integrals like this the past history could be replaced by the history of E_{ij}, because only the equivalence class of functions which differ on a set of measure zero determines the integral. We shall restrict further considerations to smooth processes and thus replace $E_{ij_r}{}^t$ by E^t_{ij} .

$$\Sigma_{Y_{ij}} = -\int_0^\infty G_{ijk\ell}(u)\frac{d}{du}[E_{k\ell_r}{}^t(u)-E_{k\ell}]du$$
$$= -\int_0^\infty G_{ijk\ell}(u)\frac{d}{du}E_{k\ell_r}{}^t(u)du \quad , \tag{4.38}$$

where we have written $\tilde{G}_{ijk\ell}(0,u) = G_{ijk\ell}(u)$. Using the assumption of thermorheologically simple materials yields for the stress at a particle X, (see (4.33)), the expression

$$\Sigma_{ij} = -\int_0^\infty G_{ijk\ell}(u)\frac{d}{du}[E_{k\ell}(t-\xi_u)-\alpha_{k\ell}(t-\xi_u)]du \quad . \tag{4.39}$$

With the transformation

$$\tau = t - \xi_u \quad ; \quad \frac{d}{du} = \frac{d}{d\xi_u}\frac{d\xi_u}{du} = -\frac{d\xi_u}{du}\frac{d}{d\tau} \tag{4.40}$$

formula (4.39) becomes

$$\Sigma_{ij} = \int_{-\infty}^t G_{ijk\ell}(u)\frac{d}{d\tau}[E_{k\ell}(\tau)-\alpha_{k\ell}(\tau)]d\tau \quad , \tag{4.41}$$

where u still must be related to τ . From (4.40) and (4.28) it follows that

$$u = (t-\tau)\chi(\vartheta_X) \quad . \tag{4.42}$$

This brings us now into the position to define the stress resultants of order m. In fact, assuming a linear dependence upon the temperature for the function $\mathfrak{U}_{ij}(\cdot)$ we may write

$$\alpha_{ij} = \omega_{ij}(\vartheta_X - \vartheta_Y) \tag{4.43}$$

thereby defining the reference particle Y as that for which the thermal strains vanish. Thus defining

$$E_{k\ell}(\tau) = \sum_{m=0}^{\infty} x_3^m E_{k\ell}^{(m)}(\tau) \quad ; \quad \alpha_{k\ell}(\tau) = \sum_{m=0}^{\infty} x_3^m \omega_{ij} \Theta^{(m)}(\tau) \tag{4.44}$$

as in Chapter 3 and

$$\mathfrak{G}_{ijk\ell}^{(m)}(\tau) = \int_h x_3^{(m)} G_{ijk\ell}(\tau\chi(\vartheta(x_3)))ds \tag{4.45}$$

we obtain

$$T_{ij}^{(m)} = \sum_{p=0}^{\infty} \int_{-\infty}^{t} \mathfrak{G}_{ijk\ell}^{(m+p)}(t-\tau) \frac{d}{d\tau} \left\{ E_{k\ell}^{(p)}(\tau) - \omega_{k\ell} \Theta^{(p)}(\tau) \right\} d\tau \tag{4.46a}$$

where for isotropic material (temperature only a function of x_3)

$$\mathfrak{G}_{ijk\ell}^{(m)}(\tau) = \mathfrak{L}^{(m)}(\tau)\delta_{ij}\delta_{k\ell} + \mathfrak{M}^{(m)}(\tau)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \tag{4.46b}$$

4.4) Viscoelastic Plate Equations

With the formulas (4.46) we are now in a position to establish the governing equations for various approximate theories of viscoelastic plates.

We discuss here the situations of Section 3.8 only.

a) Extensional Motion only

The governing equation is again (3.65), but with the constitutive relation

$$T_{\alpha\beta} = \int_{-\infty}^{t} \mathfrak{G}_{\alpha\beta\gamma\delta}(t-\tau)\frac{d}{d\tau}[\tilde{E}_{\gamma\delta}(\tau) - \omega_{\gamma\delta}\theta(\tau)]d\tau \,. \tag{4.47}$$

In the above equation we have omitted a superscript (0) and have approximated the Lagrangian strain tensor $E_{k\ell}$ by $\tilde{E}_{k\ell}$, as is customary in the linear theories.

Note that (4.47) (and (4.46)) has convolution form. In case $\mathfrak{G}_{ijk\ell}$, E_{ij} and θ vanish for negative argument, this suggests the use of Laplace transforms in solving equations (3.65) and (4.47). To this end let f(t) be a function of time. Then

$$\bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t)dt \tag{4.48}$$

is its Laplace transform and

$$f(t) = \frac{1}{2\pi i}\oint e^{st}\,\bar{f}(s)ds \tag{4.49}$$

its inverse transform. The integration in (4.49) is along the Bromwich contour in the complex s-plane, that is along a vertical line to the right of all singularities of the integrand function $e^{st}f(s)$ in (4.49).

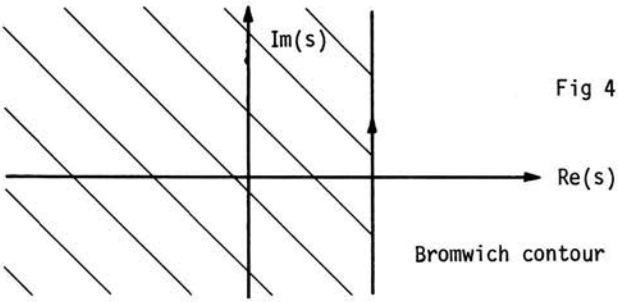


Fig 4.1

In order the function f(t) to possess a Laplace transform, it must, according to (4.48), asymtotically not grow faster than

$$f \sim e^{at} \quad \text{as} \quad t \to \infty \quad ; \quad \text{with } \mathrm{Re}(a) > \mathrm{Re}(s)$$

We shall henceforth tacitly assume that this condition is satisfied for all functions under investigation.

Performing a Laplace transform with equations (3.65) and (3.47) we obtain

$$sm\bar{u}_\alpha = \bar{T}_{\beta\alpha,\beta} + \bar{F}_\alpha + \bar{P}_\alpha + \dot{u}_o - su_o \tag{4.50}$$

and

$$\bar{T}_{\alpha\beta} = s\,\bar{\mathfrak{C}}_{\alpha\beta\gamma\delta}(s)[\bar{E}_{\gamma\delta}(s) - \omega_{\gamma\delta}\bar{\Theta}(s)] \tag{4.51}$$

or

$$-s^2 m\bar{u}_\alpha = \left\{s\,\bar{\mathfrak{C}}_{\alpha\beta\gamma\delta}(s)[\tfrac{1}{2}(\bar{u}_{\gamma,\delta} + \bar{u}_{\delta,\gamma}) - \omega_{\gamma\delta}\bar{\Theta}(s)]\right\} + \bar{F}_\alpha + \bar{P}_\alpha + \dot{u}_o + su_o \quad . \tag{4.52}$$

Here $\dot{u}_o = \dot{u}_o(x_1, x_2)$ and $u_o = u_o(x_1, x_2)$ are the initial distribution of the velocity and displacement fields. The equation (4.52) formally agrees with the corresponding Laplace transformed equation of the purely elastic theory (see equation (3.65) and the equation right thereafter) if the following correspondence is made:

$$s\,\bar{\mathfrak{G}}_{\alpha\beta\gamma\delta}(s) \longleftrightarrow \mathfrak{C}_{\alpha\beta\gamma\delta} \quad .$$

Hence, only the inverse transform of the solutions are different.

An even more useful correspondence exists in the quasistatic situation, when the acceleration term vanishes. In this case (4.52) reduces to

$$\{s\,\bar{\mathfrak{G}}_{\alpha\beta\gamma\delta}(s)[\tfrac{1}{2}(\bar{u}_{\gamma,\delta}+\bar{u}_{\delta,\gamma})-\omega_{\gamma\delta}\bar{\Theta}(s)]\}_{,\beta} + \bar{F}_{\alpha} + \bar{P}_{\alpha} = 0 \ , \tag{4.53}$$

while the elastic counterpart (equation (3.65)) becomes

$$\{\mathfrak{C}_{\alpha\beta\gamma\delta}[\tfrac{1}{2}(u_{\gamma,\delta}+u_{\delta,\gamma})-\omega_{\gamma\delta}\Theta]\}_{,\beta} + F_{\alpha} + P_{\alpha} = 0 \ . \tag{4.54}$$

Thus in the quasistatic approximation the correspondence exists between the Laplace transformed viscoelastic problem and the elastic problem (and not its Laplace transform). Given an elastic solution u_{α}, we replace in-there

$$\mathfrak{C}_{\alpha\beta\gamma\delta} \qquad \text{by} \qquad s\,\bar{\mathfrak{G}}_{\alpha\beta\gamma\delta}(s)$$

and determine an inverse Laplace transform of the emerging equations. This yields the solution to the corresponding viscoelastic problem.

b) Reissner Plate Theory for Viscoelastic Plates

The situation dealt with here is analogous to the one in subsection 3.8,c. Accordingly, we refer to that subsection and only list equations which differ from those given there. In particular, we assume homothermal conditions.

We neglect thermal expansion ($\omega_{k\ell}=0$) and again assume that the relations (3.78) and the first part of (3.82) hold. That is to say, we assume that all stress resultants vanish except Q_x, Q_y, M_x, M_y, and M_{xy}. For isotropic plates, using (4.46a, b), this again implies (3.80) and (3.83). On the other hand, the counterpart of equation (3.82) is now

$$\int_0^t \{ \mathcal{L}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{kk}^{(1)}(\tau))+2\,\mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{33}^{(1)}(\tau))\}d\tau = 0\,. \quad (4.55)$$

Taking a Laplace transform implies

$$\overline{\tilde{E}_{33}^{(1)}} = -\frac{\overline{\mathcal{L}^{(2)}}}{\overline{\mathcal{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}}}\overline{\tilde{E}_{\alpha\alpha}^{(1)}} \quad , \qquad (4.56)$$

and

$$\overline{\tilde{E}_{kk}^{(1)}} = \frac{\overline{\mathcal{L}^{(2)}}}{\overline{\mathcal{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}}}\overline{\tilde{E}_{\alpha\alpha}^{(1)}} \,. \qquad (4.57)$$

The shear forces and moments corresponding to (3.79) and (3.81) are

$$Q_x = 2\int_0^t \mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{13}^{(0)}(\tau))d\tau$$

$$Q_y = 2\int_0^t \mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{23}^{(0)}(\tau))d\tau \qquad (4.58)$$

and

$$M_x = \int_0^t \mathcal{L}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{kk}^{(1)}(\tau))d\tau + \int_0^t 2\,\mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{11}^{(1)}(\tau))d\tau; \qquad (4.59)_1$$

$$M_y = \int_0^t \mathfrak{L}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{kk}^{(1)}(\tau))d\tau + \int_0^t 2\,\mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{22}^{(1)}(\tau))d\tau \quad ; \tag{4.59$_2$}$$

$$-M_{xy} = \int_0^t 2\,\mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{12}^{(1)}(\tau))d\tau \quad .$$

Performing a Laplace transform with $(4.59)_{1,2}$ and using (4.56) gives

$$\bar{M}_x = s\overline{D(s)}\left\{\overline{\tilde{E}_{11}^{(1)}(s)}+\overline{N(s)\tilde{E}_{22}^{(1)}(s)}\right\} \quad ;$$

$$\bar{M}_y = s\overline{D(s)}\left\{\overline{\tilde{E}_{22}^{(1)}(s)}+\overline{N(s)\tilde{E}_{11}^{(1)}(s)}\right\} \quad ; \tag{4.60}$$

$$\bar{M}_{xy} = -s\,\overline{\mathfrak{M}^{(2)}(s)}\;\overline{E_{12}^{(1)}(s)} \quad ,$$

with

$$\overline{D(s)} = \frac{4\left\{\overline{\mathfrak{L}^{(2)}}\;\overline{\mathfrak{M}^{(2)}}+\left(\overline{\mathfrak{M}^{(2)}}\right)^2\right\}}{\overline{\mathfrak{L}^{(2)}}+2\,\overline{\mathfrak{M}^{(2)}}} \quad ;$$

$$\overline{N(s)} = \frac{\overline{\mathfrak{L}^{(2)}}}{2\left\{\overline{\mathfrak{L}^{(2)}}+\overline{\mathfrak{M}^{(2)}}\right\}} \quad . \tag{4.61}$$

Observe the similarity of these formulas with those of (3.87). With the definitions

$$\mathfrak{D}(t) = \frac{1}{2\pi i}\int \overline{D(s)}\,e^{st}ds \quad ;$$

$$\mathfrak{D}(t)\,\mathfrak{N}(t) = \frac{1}{2\pi i}\int \overline{D(s)}\;\overline{N(s)}e^{st}\,ds \quad , \tag{4.62}$$

where the integration is along the Bromwich contour, we may write

$$M_x = \int_0^t \mathfrak{D}(t-\tau)\{\frac{d}{d\tau}(\tilde{E}_{11}^{(1)}(\tau)) + \mathfrak{N}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{22}^{(1)}(\tau))\}d\tau \quad ;$$

$$M_y = \int_0^t \mathfrak{D}(t-\tau)\{\frac{d}{d\tau}(\tilde{E}_{22}^{(1)}(\tau)) + \mathfrak{N}(t-\tau)\frac{d}{d\tau}(\tilde{E}_{22}^{(1)}(\tau))\}d\tau \quad , \tag{4.63}$$

which are the formulas analogous to (3.86)

This brings us into the position to list the governing equations for this plate theory. With the definitions (3.88) - (3.90) and (4.58) and (4.62) we may write

$$Q_x = \int_0^t \mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\frac{\partial\eta}{\partial x}(\tau) - \varphi(\tau))d\tau \quad ;$$

$$Q_y = \int_0^t \mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}(\frac{\partial\eta}{\partial y}(\tau) - \psi(\tau))d\tau \quad , \tag{4.64}$$

and

$$M_x = -\int_0^t \mathfrak{D}(t-\tau)\{\frac{d}{d\tau}(\frac{\partial\varphi}{\partial x}(\tau)) + \mathfrak{N}(t-\tau)\frac{d}{d\tau}(\frac{\partial\psi}{\partial y}(\tau))\}d\tau \quad ;$$

$$M_y = -\int_0^t \mathfrak{D}(t-\tau)\{\frac{d}{d\tau}(\frac{\partial\varphi}{\partial x}(\tau)) + \mathfrak{N}(t-\tau)\frac{d}{d\tau}(\frac{\partial\psi}{\partial y}(\tau))\}d\tau \quad ; \tag{4.65}$$

$$M_{xy} = -\int_0^t \mathfrak{M}^{(2)}(t-\tau)\{\frac{d}{d\tau}(\frac{\partial\psi}{\partial x}(\tau) + \frac{\partial\varphi}{\partial y}(\tau))\}d\tau \quad .$$

Furthermore, the dynamic equations (3.93) must hold; we refrain from repeating them. It is worth noting that the same correspondence principle between the Laplace transformed viscoelastic and the elastic solution applies also in this theory. To see this, let

$$\begin{aligned} &\eta_o(x_1,x_2) \;;\; \dot{\eta}_o(x_1,x_2) \;; \\ &\varphi_o(x_1,x_2) \;;\; \dot{\varphi}_o(x_1,x_2) \;; \\ &\psi_o(x_1,x_2) \;;\; \dot{\psi}_o(x_1,x_2) \;, \end{aligned} \tag{4.66}$$

be the initial values of η, $\dot{\eta}$, φ, $\dot{\varphi}$, ψ and $\dot{\psi}$. The Laplace transformed equations (3.93) then become

$$\begin{aligned} -\rho_o A s^2 \bar{\eta} &= \overline{\frac{\partial Q_x}{\partial x}} + \overline{\frac{\partial Q_y}{\partial y}} + \bar{q} + (\dot{\eta}_o + s\eta_o)\rho_o A \;; \\ -\rho_o I^{(2)} s^2 \bar{\varphi} &= \overline{\frac{\partial M_x}{\partial x}} - \overline{\frac{\partial M_{xy}}{\partial y}} - \overline{Q_x} + \overline{T_x} + \rho_o I^{(2)}(\dot{\varphi}_o + s\varphi_o) \;; \\ -\rho_o I^{(2)} s^2 \bar{\psi} &= \overline{\frac{\partial M_y}{\partial y}} - \overline{\frac{\partial M_{xy}}{\partial x}} - \overline{Q_y} + \overline{T_y} + \rho_o I^{(2)}(\dot{\psi}_o + s\psi_o) \;, \end{aligned} \tag{4.67}$$

where

$$\begin{aligned} \bar{M}_x &= -s\overline{D(s)}\left\{\overline{\frac{\partial\varphi}{\partial x}(s)} + \overline{N(s)}\,\overline{\frac{\partial\psi}{\partial y}(s)}\right\} \;; \\ \bar{M}_y &= -s\overline{D(s)}\left\{\overline{\frac{\partial\psi}{\partial y}(s)} + \overline{N(s)}\,\overline{\frac{\partial\varphi}{\partial x}(s)}\right\} \;; \\ \bar{M}_{xy} &= s\,\overline{\mathfrak{M}^{(2)}}\left\{\overline{\frac{\partial\psi}{\partial x}(s)} + \overline{\frac{\partial\varphi}{\partial y}(s)}\right\} \;, \end{aligned} \tag{4.68}$$

and

$$\begin{aligned} \bar{Q}_x &= \overline{\mathfrak{M}^{(0)}}\left\{\overline{\frac{\partial\eta}{\partial x}(s)} - \overline{\varphi(s)}\right\} \;; \\ \bar{Q}_y &= \overline{\mathfrak{M}^{(0)}}\left\{\overline{\frac{\partial\eta}{\partial y}(s)} - \overline{\psi(s)}\right\} \;. \end{aligned} \tag{4.69}$$

By comparison with the Laplace transform of the dynamic equations for the elastic plate or the static equations the following correspondences are obtained:

Correspondence I The Laplace transforms of the solution of the elastic plate equations in the generalized Reissner theory and the Laplace transforms of those in the corresponding viscoelastic plate theory can be obtained from each other by interchanging the following quantities:

$$\begin{aligned} &\overline{\mathfrak{M}^{(0)}(s)} \leftrightarrow M^{(0)} \ ; \\ &\overline{sD(s)} \leftrightarrow D \ ; \\ &s\,\overline{\mathfrak{M}^{(2)}} \leftrightarrow M^{(2)} \end{aligned} \tag{4.70}$$

Correspondence II The solutions of the static elastic plate equations in the generalized Reissner theory and the Laplace transforms of those in the corresponding viscoelastic theory can be obtained from each other by interchanging the quantities (4.70).

c) Generalized von Kàrmàn Theory for Viscoelastic Plates

The preceding viscoelastic plate theories are, strictly speaking, only correct if the temperature distribution within the plate is uniform or satisfies certain restrictions which will be investigated below. We now allow the temperature to be distributed arbitrarily; only later shall we assume that the temperature distribution does not vary with the coördinates x_1 and x_2, which define the reference plane in the plate. In the purely elastic theory (with nonuniform temperature distribution) we have fixed this reference plane by requiring that (3.95) holds. Here, in the viscoelastic isotropic theory we cannot simply assume that $\mathfrak{L}^{(1)}(\tau) = 0$, since the value of this macroscopic relaxation function generally will vary with time, viz:

$$\mathcal{L}^{(1)}(\tau) \;=\; \int_h x_3 \lambda^{(1)}(\tau\chi(\vartheta_X(x_3)))dx_3 \quad . \tag{4.71}$$

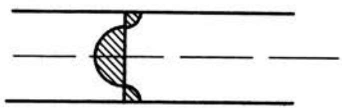
Here, χ satisfies the side conditions (4.29) and $\lambda^{(1)}$ is given by

$$G_{ijk\ell}(t) \;=\; \lambda^{(1)}(t)\delta_{ij}\delta_{k\ell} + \mu^{(1)}(t)(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}) \quad .$$

It is obvious that the integral on the right hand side of (4.71) is only time independent for special situations. If, for instance, the temperature distribution is symmetric with respect to the middle plane then irrespective of the explicit form of its functional form $\chi(\cdot)$ is symmetric with respect to this plane; and so is $\lambda^{(1)}(\cdot)$. Thus $x_3\lambda^{(1)}(\tau\chi(\vartheta_X(x_3)))$ is odd and $\mathcal{L}^{(1)} = 0$. This situation, certainly does not apply for floating ice. On the other hand, let $\chi(x_3) = \chi(\vartheta_X(x_3))$. Then, using the transformation

$$\tau\hat{\chi}(x_3) \;=\; \xi \quad , \tag{4.72}$$

Fig 4.2



symmetric temperature distribution

we can show that (4.71) may be written as

$$\mathcal{L}^{(1)}(\tau) \;=\; \int_{\xi_0}^{\xi_1} \frac{\hat{\chi}^{-1}(\xi/\tau)}{\tau}\lambda^{(1)}(\xi)\frac{d\chi^{-1}(\xi/\tau)}{d(\xi/\tau)}d\xi \quad , \tag{4.73}$$

which immediately proves that $\mathcal{L}^{(1)}$ is independent of τ provided

$$\frac{d}{d(\xi/\tau)}[\hat{\chi}^{-1}(\xi/\tau)]^2 \;=\; \alpha\tau \quad , \tag{4.74}$$

or, setting $\xi/\tau = x$

$$[\hat{\chi}^{-1}(x)]^2 = \alpha\, x \ln(cx/\tau) \quad , \tag{4.75}$$

where c is an arbitrary constant. Since this involves the time τ, $\hat{\chi}(\cdot)$ must also be time dependent, which contradicts (4.72), where $\hat{\chi}(\cdot)$ has been assumed to be time independent.

We thus conclude that generally there does not exist a reference plane for which (4.71) vanishes for all τ; we thus choose the middle plane as reference plane. The viscoelastic analogue of the generalized von Kàrmàn plate theory must therefore be considerably more complex than the corresponding elastic theory when the temperature distribution is nonuniform.

Guided by the experience of the calculations in the previous subsections we perform most subsequent calculations in the Laplace transformed domain. We further make the "von Kàrmàn assumption" regarding the strain displacement relations (see Chapter 3, formulas 3.10c). We assume isotropy so that the stress strain relations can be assumed as given in (4.46a) with the relaxation function (4.46b). The nonvanishing zeroth order stress resultants are:

$$\overline{T_{13}^{(0)}} = \overline{Q_x} = s\left\{2\,\overline{\mathfrak{M}^{(0)}}\,\overline{E_{13}^{(0)}} + 2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E_{13}^{(1)}}\right\} ;$$

$$\overline{T_{23}^{(0)}} = \overline{Q_y} = s\left\{2\,\overline{\mathfrak{M}^{(0)}}\,\overline{E_{23}^{(0)}} + 2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E_{23}^{(1)}}\right\} ;$$

$$T_{11}^{(1)} = \overline{N_x} = s\Big\{\overline{\mathfrak{L}^{(0)}}\,\overline{E_{kk}^{(0)}} + 2\,\overline{\mathfrak{M}^{(0)}}\,\overline{E_{11}^{(0)}} - \omega(3\,\overline{\mathfrak{L}^{(0)}} + 2\,\overline{\mathfrak{M}^{(0)}})\overline{\Theta^{(0)}} + \overline{\mathfrak{L}^{(1)}}\,\overline{E_{kk}^{(1)}}$$

$$+ 2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E_{11}^{(1)}} - \omega(3\,\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}})\overline{\Theta^{(1)}}\Big\} ;$$

$$\overline{T^{(0)}_{22}} = \overline{N_y} = s\Big\{\overline{\mathcal{L}^{(0)}E^{(0)}_{kk}}+2\,\overline{\mathcal{M}^{(0)}E^{(0)}_{11}} - \omega(3\,\overline{\mathcal{L}^{(0)}}+2\,\overline{\mathcal{M}^{(0)}})\overline{\Theta^{(0)}}$$

$$+\,\overline{\mathcal{L}^{(1)}E^{(1)}_{kk}}+2\,\overline{\mathcal{M}^{(1)}E^{(1)}_{22}}-\omega(3\,\overline{\mathcal{L}^{(1)}}+2\,\overline{\mathcal{M}^{(1)}})\overline{\Theta^{(1)}}\Big\}$$

$$\overline{T^{(0)}_{12}} = \overline{N_{xy}} = s\Big\{2\,\overline{\mathcal{M}^{(0)}E^{(0)}_{12}}+2\,\overline{\mathcal{M}^{(1)}E^{(1)}_{12}}\Big\} \quad . \tag{4.76}$$

But we assume that

$$\overline{T^{(0)}_{33}} = s\Big\{\overline{\mathcal{L}^{(0)}E^{(0)}_{kk}}+2\,\overline{\mathcal{M}^{(0)}E^{(0)}_{33}}-\omega(3\,\overline{\mathcal{L}^{(0)}}+2\,\overline{\mathcal{M}^{(0)}})\overline{\Theta^{(0)}}$$

$$+\,\overline{\mathcal{L}^{(1)}E^{(1)}_{kk}}+2\,\overline{\mathcal{M}^{(1)}E^{(1)}_{33}}-\omega(3\,\overline{\mathcal{L}^{(1)}}+2\,\overline{\mathcal{M}^{(1)}})\overline{\Theta^{(1)}}\Big\} = 0 \quad . \tag{4.77}$$

Following the corresponding calculations in Section 3.8d (see formulas (3.99) and (3.100)), we set

$$T^{(1)}_{11} = \overline{M_x} = s\Big\{\overline{\mathcal{L}^{(1)}}(\overline{E^{(0)}_{kk}}-3\omega\overline{\Theta^{(0)}})+2\,\overline{\mathcal{M}^{(1)}}(\overline{E^{(0)}_{11}}-\omega\overline{\Theta^{(0)}})$$

$$+\,\overline{\mathcal{L}^{(2)}}(\overline{E^{(1)}_{kk}}-3\omega\overline{\Theta^{(1)}})+2\,\overline{\mathcal{M}^{(2)}}(\overline{E^{(1)}_{11}}-\omega\overline{\Theta^{(1)}})\Big\}$$

$$\overline{T^{(1)}_{22}} = \overline{M_y} = s\Big\{\overline{\mathcal{L}^{(1)}}(\overline{E^{(0)}_{kk}}-3\omega\overline{\Theta^{(0)}})+2\,\overline{\mathcal{M}^{(1)}}(\overline{E^{(0)}_{11}}-\omega\overline{\Theta^{(0)}})$$

$$+\,\overline{\mathcal{L}^{(2)}}(\overline{E^{(1)}_{kk}}-3\omega\overline{\Theta^{(1)}})+2\,\overline{\mathcal{M}^{(2)}}(\overline{E^{(1)}_{11}}-\omega\overline{\Theta^{(1)}})\Big\}$$

$$\overline{T^{(1)}_{12}} = -\overline{M_{xy}} = s\Big\{2\,\overline{\mathcal{M}^{(1)}E^{(0)}_{12}}+2\,\overline{\mathcal{M}^{(2)}E^{(0)}_{12}}\Big\} \tag{4.78}$$

and

$$\overline{T^{(1)}_{13}} = s\left\{2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E^{(0)}_{13}} + 2\,\overline{\mathfrak{M}^{(2)}}\,\overline{E^{(1)}_{13}}\right\} = 0 \; ;$$

$$\overline{T^{(1)}_{23}} = s\left\{2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E^{(0)}_{23}} + 2\,\overline{\mathfrak{M}^{(2)}}\,\overline{E^{(1)}_{23}}\right\} = 0 \; ; \tag{4.79}$$

$$\overline{T^{(1)}_{33}} = s\left\{\overline{\mathfrak{L}^{(1)}}\,\overline{E^{(0)}_{kk}} + 2\,\overline{\mathfrak{M}^{(1)}}\,\overline{E^{(0)}_{33}} - \omega(3\,\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}})\overline{\Theta^{(0)}} + \overline{\mathfrak{L}^{(2)}}\,\overline{E^{(1)}_{kk}} \right.$$
$$\left. + 2\,\overline{\mathfrak{L}^{(2)}}\,\overline{E^{(1)}_{33}} - \omega(3\,\overline{\mathfrak{L}^{(2)}} + 2\,\overline{\mathfrak{M}^{(2)}})\overline{\Theta^{(1)}}\right\} = 0 \; .$$

From $(4.79)_{1,2}$ the first order shear strains may be expressed in terms of the second order ones:

$$E^{(1)}_{13} = \frac{\overline{\mathfrak{M}^{(1)}(s)}}{\overline{\mathfrak{M}^{(2)}(s)}}E^{(0)}_{13} \quad ; \quad \overline{E^{(1)}_{23}} = \frac{\overline{\mathfrak{M}^{(1)}(s)}}{\overline{\mathfrak{M}^{(2)}(s)}}\overline{E^{(0)}_{23}} \; . \tag{4.80}$$

Furthermore, (4.77) and $(4.79)_3$ imply

$$\overline{E^{(0)}_{33}} = \overline{A^{(0)}}\,\overline{E^{(0)}_{\alpha\alpha}} + \overline{B^{(0)}}\,\overline{E^{(1)}_{\alpha\alpha}} + \overline{C^{(0)}}$$
$$\overline{E^{(1)}_{33}} = \overline{A^{(1)}}\,\overline{E^{(0)}_{\alpha\alpha}} + \overline{B^{(1)}}\,\overline{E^{(1)}_{\alpha\alpha}} + C^{(1)} \tag{4.81}$$

with

$$\overline{A^{(0)}} = \left\{\overline{\mathfrak{L}^{(1)}}(\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}}) - \overline{\mathfrak{L}^{(0)}}(\overline{\mathfrak{L}^{(2)}} + 2\,\overline{\mathfrak{M}^{(2)}})\right\}/\Delta \; ;$$
$$\overline{B^{(0)}} = \left\{\overline{\mathfrak{L}^{(0)}}(\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}}) - \overline{\mathfrak{L}^{(1)}}(\overline{\mathfrak{L}^{(2)}} + 2\,\overline{\mathfrak{M}^{(2)}})\right\}/\Delta \; ;$$
$$\overline{A^{(1)}} = \left\{\overline{\mathfrak{L}^{(0)}}(\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}}) - \overline{\mathfrak{L}^{(1)}}(\overline{\mathfrak{L}^{(0)}} + 2\,\overline{\mathfrak{M}^{(0)}})\right\}/\Delta \; ; \tag{4.82}$$
$$\overline{B^{(1)}} = \left\{\overline{\mathfrak{L}^{(1)}}(\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}}) - \overline{\mathfrak{L}^{(2)}}(\overline{\mathfrak{L}^{(0)}} + 2\,\overline{\mathfrak{M}^{(0)}})\right\}/\Delta \; ,$$

and

$$\overline{C^{(0)}} = \Big\{\Big[(\overline{\mathfrak{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}})(3\overline{\mathfrak{L}^{(0)}}+2\overline{\mathfrak{M}^{(0)}})-(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})(3\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})\Big]\overline{\Theta^{(0)}}$$

$$+\Big[(\overline{\mathfrak{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}})(3\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})-(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})(3\overline{\mathfrak{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}})\Big]\overline{\Theta^{(1)}}\Big\}$$

$$\overline{C^{(1)}} = \Big\{\Big[(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})(3\overline{\mathfrak{L}^{(0)}}+2\overline{\mathfrak{M}^{(0)}})-(\overline{\mathfrak{L}^{(0)}}+2\overline{\mathfrak{M}^{(0)}})(3\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})\Big]\overline{\Theta^{(0)}}$$

$$\Big[(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})(3\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})-(\overline{\mathfrak{L}^{(0)}}+2\overline{\mathfrak{M}^{(0)}})(3\overline{\mathfrak{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}})\Big]\overline{\Theta^{(1)}}\Big\}/\Delta$$

$$\Delta = (\overline{\mathfrak{L}^{(0)}}+2\overline{\mathfrak{M}^{(0)}})(\overline{\mathfrak{L}^{(2)}}+2\overline{\mathfrak{M}^{(2)}})-(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}})(\overline{\mathfrak{L}^{(1)}}+2\overline{\mathfrak{M}^{(1)}}) \quad . \tag{4.83}$$

We further define

$$\overline{\Gamma_x} = \overline{\Gamma_y} = 2\overline{\mathfrak{M}^{(0)}} + \frac{2\overline{\mathfrak{M}^{(1)}}^2}{\overline{\mathfrak{M}^{(2)}}} \quad . \tag{4.84}$$

With the abbreviations as defined in (4.82) - (4.85) the Laplace transformed macroscopic stress strain relations are almost identical to those in the elastic case (see 3.104 and subsequent formulas). In fact, substituting (4.80) and (4.81) into (4.76) and (4.78) we obtain

$$\begin{aligned} \overline{Q_x} &= s\overline{\Gamma_x}\,\overline{E_{13}^{(0)}} \quad ; \\ \overline{Q_y} &= s\overline{\Gamma_y}\,\overline{E_{23}^{(0)}} \quad ; \end{aligned} \tag{4.85}$$

$$\overline{N_x} = s\left\{\overline{\mathfrak{D}^{(0)}}\left(\overline{E_{11}^{(0)}} + \overline{\mathfrak{N}^{(0)}}\overline{E_{22}^{(0)}}\right) + \overline{\mathfrak{D}^{(1)}}\left(\overline{E_{11}^{(1)}} + \overline{\mathfrak{N}^{(1)}}\overline{E_{22}^{(1)}}\right) + \overline{\mathfrak{T}_x}\right\} ;$$

$$\overline{N_y} = s\left\{\overline{\mathfrak{D}^{(0)}}\left(\overline{E_{22}^{(0)}} + \overline{\mathfrak{N}^{(0)}}\overline{E_{11}^{(0)}}\right) + \overline{\mathfrak{D}^{(1)}}\left(\overline{E_{22}^{(1)}} + \overline{\mathfrak{N}^{(1)}}\overline{E_{11}^{(2)}}\right) + \overline{\mathfrak{T}_y}\right\} ; \qquad (4.86)$$

$$\overline{N_{xy}} = s\left\{2\,\overline{\mathfrak{M}^{(0)}}\overline{E_{12}^{(0)}} + 2\,\overline{\mathfrak{M}^{(1)}}\overline{E_{12}^{(1)}}\right\} ,$$

and

$$\overline{M_x} = s\left\{\overline{\boldsymbol{\mathfrak{D}}^{(0)}}\left(\overline{E_{11}^{(0)}} + \overline{\boldsymbol{\mathfrak{N}}^{(0)}}\overline{E_{22}^{(0)}}\right) + \overline{\boldsymbol{\mathfrak{D}}^{(1)}}\left(\overline{E_{11}^{(1)}} + \overline{\boldsymbol{\mathfrak{N}}^{(1)}}\overline{E_{22}^{(1)}}\right) + \overline{\boldsymbol{\mathfrak{T}}_x}\right\};$$

$$\overline{M_y} = s\left\{\overline{\boldsymbol{\mathfrak{D}}^{(0)}}\left(\overline{E_{22}^{(0)}} + \overline{\boldsymbol{\mathfrak{N}}^{(0)}}\overline{E_{22}^{(0)}}\right) + \overline{\boldsymbol{\mathfrak{D}}^{(1)}}\left(\overline{E_{22}^{(1)}} + \overline{\boldsymbol{\mathfrak{N}}^{(1)}}\overline{E_{11}^{(1)}}\right) + \overline{\boldsymbol{\mathfrak{T}}_y}\right\}; \qquad (4.87)$$

$$\overline{M_{xy}} = -s\left\{2\,\overline{\mathfrak{M}^{(1)}}\overline{E_{12}^{(0)}} + 2\,\overline{\mathfrak{M}^{(2)}}\overline{E_{12}^{(1)}}\right\}$$

where

$$\overline{\mathfrak{D}^{(0)}} = \left(\overline{\mathfrak{L}^{(0)}} + 2\,\overline{\mathfrak{M}^{(0)}}\right) + \overline{\mathfrak{L}^{(0)}A^{(0)}} + \overline{\mathfrak{L}^{(1)}A^{(1)}} ;$$

$$\overline{\mathfrak{D}^{(1)}} = \left(\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}}\right) + \overline{\mathfrak{L}^{(0)}B^{(0)}} + \overline{\mathfrak{L}^{(1)}B^{(1)}} ;$$

$$\overline{\mathfrak{N}^{(0)}} = \frac{1}{1 + 2\,\overline{\mathfrak{M}^{(0)}} / \left[\overline{\mathfrak{L}^{(0)}} + \overline{\mathfrak{L}^{(0)}A^{(0)}} + \overline{\mathfrak{L}^{(1)}A^{(1)}}\right]} ;$$

$$\overline{\mathfrak{N}^{(1)}} = \frac{1}{1 + 2\,\overline{\mathfrak{M}^{(1)}} / \left[\overline{\mathfrak{L}^{(1)}} + \overline{\mathfrak{L}^{(0)}B^{(0)}} + \overline{\mathfrak{L}^{(1)}B^{(1)}}\right]} ; \qquad (4.88)$$

$$\overline{\boldsymbol{\mathfrak{D}}^{(0)}} = \left[\overline{\mathfrak{L}^{(1)}} + 2\,\overline{\mathfrak{M}^{(1)}} + \overline{\mathfrak{L}^{(1)}A^{(0)}} + \overline{\mathfrak{L}^{(2)}A^{(1)}}\right] ;$$

$$\overline{\boldsymbol{\mathfrak{D}}^{(1)}} = \left[\overline{\mathfrak{L}^{(2)}} + 2\,\overline{\mathfrak{M}^{(2)}} + \overline{\mathfrak{L}^{(1)}B^{(0)}} + \overline{\mathfrak{L}^{(2)}B^{(1)}}\right] ;$$

$$\overline{\boldsymbol{\mathfrak{N}}^{(0)}} = \frac{1}{1 + 2\,\overline{\mathfrak{M}^{(1)}} / \left[\overline{\mathfrak{L}^{(1)}} + \overline{\mathfrak{L}^{(1)}A^{(0)}} + \overline{\mathfrak{L}^{(2)}A^{(1)}}\right]} ;$$

$$\overline{\mathcal{R}^{(1)}} = \frac{1}{1+2\,\overline{\mathfrak{M}^{(2)}}/[\overline{\mathcal{L}^{(2)}}+\overline{\mathcal{L}^{(1)}}\,\overline{B^{(0)}}+\overline{\mathcal{L}^{(2)}}\,\overline{B^{(1)}}]} \; ; \qquad (4.88) \text{ cont.}$$

and

$$\overline{\mathcal{T}_x} = \overline{\mathcal{T}_y} = \overline{\mathcal{L}^{(0)}}\,\overline{C^{(0)}}+\overline{\mathcal{L}^{(1)}}\,\overline{C^{(1)}}-\omega(3\,\overline{\mathcal{L}^{(0)}}+2\,\overline{\mathfrak{M}^{(0)}})\overline{\Theta^{(0)}}-\omega(3\,\overline{\mathcal{L}^{(1)}}+2\,\overline{\mathfrak{M}^{(1)}})\overline{\Theta^{(1)}}$$

(4.8

$$\overline{\mathcal{I}_x} = \overline{\mathcal{I}_y} = \overline{\mathcal{L}^{(1)}}\,\overline{C^{(0)}}+\overline{\mathcal{L}^{(2)}}\,\overline{C^{(1)}}-\omega[(3\,\overline{\mathcal{L}^{(1)}}+2\,\overline{\mathfrak{M}^{(1)}})\overline{\Theta^{(0)}}+(3\,\overline{\mathcal{L}^{(2)}}+2\,\overline{\mathfrak{M}^{(2)}})\overline{\Theta^{(1)}}] \,.$$

Taking inverse transforms the linear constitutive equations are obtained.

$$N_x = \int_0^t \mathcal{D}^{(0)}(t-\tau)\{\frac{d}{d\tau}E_{11}^{(0)}(\tau)+ \mathcal{N}^{(0)}(t-\tau)\frac{d}{d\tau}E_{22}^{(0)}(\tau)\}d\tau$$

$$+\int_0^t \mathcal{D}^{(1)}(t-\tau)\{\frac{d}{d\tau}E_{11}^{(1)}(\tau)+ \mathcal{N}^{(1)}(t-\tau)\frac{d}{d\tau}E_{22}^{(1)}(\tau)\}d\tau + \mathcal{T}_x \; ;$$

$$N_y = \int_0^t \mathcal{D}^{(0)}(t-\tau)\{\frac{d}{d\tau}E_{22}^{(0)}(\tau)+ \mathcal{N}^{(0)}(t-\tau)\frac{d}{d\tau}E_{11}^{(0)}(\tau)\}d\tau \qquad (4.90)$$

$$+\int_0^t \mathcal{D}^{(1)}(t-\tau)\{\frac{d}{d\tau}E_{22}^{(1)}(\tau)+ \mathcal{N}^{(1)}(t-\tau)\frac{d}{d\tau}E_{11}^{(1)}(\tau)\}d\tau + \mathcal{T}_y \; ;$$

$$N_{xy} = 2\int_0^t \mathfrak{M}^{(0)}(t-\tau)\frac{d}{d\tau}E_{12}^{(0)}(\tau)d\tau + 2\int_0^t \mathfrak{M}^{(1)}\frac{d}{d\tau}E_{12}^{(1)}(\tau)d\tau$$

and

$$M_x = \int_0^t \mathfrak{D}^{(0)}(t-\tau)\{\frac{d}{d\tau}E_{11}^{(0)}(\tau) + \mathfrak{N}^{(0)}(t-\tau)\frac{d}{d\tau}E_{22}^{(0)}(\tau)\}d\tau$$

$$+\int_0^t \mathfrak{D}^{(1)}(t-\tau)\{\frac{d}{dt}E_{11}^{(1)}(\tau) + \mathfrak{N}^{(1)}(t-\tau)\frac{d}{d\tau}E_{22}^{(1)}(\tau)\}d\tau + \mathfrak{I}_x \quad ;$$

$$M_y = \int_0^t \mathfrak{D}^{(0)}(t-\tau)\{\frac{d}{d\tau}E_{22}^{(0)}(\tau) + \mathfrak{N}^{(0)}(t-\tau)\frac{d}{d\tau}E_{11}^{(0)}(\tau)\}d\tau$$

$$+\int_0^t \mathfrak{D}^{(1)}(t-\tau)\{\frac{d}{d\tau}E_{22}^{(1)}(\tau) + \mathfrak{N}^{(1)}(t-\tau)\frac{d}{d\tau}E_{11}^{(1)}(\tau)\}d\tau + \mathfrak{I}_y \quad ; \qquad (4.91)$$

$$M_{xy} = -2\int_0^t \mathfrak{M}^{(1)}(t-\tau)\frac{d}{d\tau}E_{12}^{(0)}(\tau)d\tau - 2\int_0^t \mathfrak{M}^{(2)}(t-\tau)\frac{d}{d\tau}E_{12}^{(0)}(\tau)d\tau \quad ;$$

$$Q_x = \int_0^t \Gamma_x(t-\tau)\frac{d}{d\tau}E_{13}^{(0)}(\tau)d\tau \quad ;$$

$$Q_y = \int_0^t \Gamma_y(t-\tau)\frac{d}{d\tau}E_{23}^{(0)}(\tau)d\tau \quad .$$

where $\mathfrak{I}_x$, $\mathfrak{I}_y$ are obtained from (4.89) by an inverse Laplace transform. The strains may be expressed in terms of the displacements with the aid of the formulas (3.117). For chiefly flexural deformations, horizontal accelerations may be neglected. The equations for the membrane forces are then satisfied, if

$$N_x = \frac{\partial^2 F}{\partial y^2} \quad ; \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \quad ; \quad N_y = \frac{\partial^2 F}{\partial x^2} \quad . \qquad (4.92)$$

Corresponding expressions also hold for the Laplace transformed variables.

Using these expressions, the equations (4.86) can be solved for the zeroth order strain. The result is

$$\overline{E_{11}^{(0)}} = \frac{1}{\overline{\mathcal{D}}^{(0)}(1-\overline{\mathcal{N}}^{(0)2})}\left\{\frac{1}{s}\frac{\partial^2\bar{F}}{\partial y^2} - \overline{\mathcal{N}}^{(0)}\frac{1}{s}\frac{\partial^2\bar{F}}{\partial x^2} - \left[\overline{\mathcal{D}}^{(1)}(\overline{E_{11}^{(1)}} + \overline{\mathcal{N}}^{(1)}\overline{E_{22}^{(1)}}) - \overline{\mathcal{N}}^{(0)}\overline{\mathcal{D}}^{(1)}(\overline{E_{22}^{(1)}} + \overline{\mathcal{N}}^{(1)}\overline{E_{11}^{(1)}}) + \overline{\mathcal{T}}_x - \overline{\mathcal{N}}^{(0)}\overline{\mathcal{T}}_y\right]\right\}$$

$$\overline{E_{22}^{(0)}} = \frac{1}{\overline{\mathcal{D}}^{(0)}(1-\overline{\mathcal{N}}^{(0)2})}\left\{\frac{1}{s}\frac{\partial^2\bar{F}}{\partial x^2} - \overline{\mathcal{N}}^{(0)}\frac{1}{s}\frac{\partial^2\bar{F}}{\partial y^2} - \left[\overline{\mathcal{D}}^{(1)}(\overline{E_{22}^{(1)}} + \overline{\mathcal{N}}^{(1)}\overline{E_{11}^{(1)}}) - \overline{\mathcal{N}}^{(0)}\overline{\mathcal{D}}^{(1)}(\overline{E_{11}^{(1)}} + \overline{\mathcal{N}}^{(1)}\overline{E_{22}^{(1)}}) + \overline{\mathcal{T}}_y - \overline{\mathcal{N}}^{(0)}\overline{\mathcal{T}}_x\right]\right\}$$

$$\overline{E_{12}^{(0)}} = -\frac{1}{2\overline{\mathfrak{M}}^{(0)}}\frac{1}{s}\frac{\partial^2\bar{F}}{\partial x\partial y} - \frac{\overline{\mathfrak{M}}^{(1)}}{\overline{\mathfrak{M}}^{(0)}}\bar{E}_{12}^{(1)} \quad ; \qquad (4.93)$$

Taking inverse Laplace transforms, substituting the results into (3.119), which is also valid for viscoelastic material, and using (3.117), we obtain

$$\int_0^t \mathfrak{H}^{(1)}(t-\tau)\left\{\frac{\partial^4 F(\tau)}{\partial x^4} + \frac{\partial^4 F(\tau)}{\partial y^4}\right\}d\tau + \int_0^t \mathfrak{H}^{(2)}(t-\tau)\frac{\partial^4 F(\tau)}{\partial x^2\partial y^2}d\tau$$

$$+\int_0^t \mathfrak{H}^{(3)}(t-\tau)\left\{\frac{\partial^3\varphi(\tau)}{\partial x\partial y^2} - \frac{\partial^3\psi(\tau)}{\partial x^2\partial y}\right\}d\tau + \int_0^t \mathfrak{H}^{(4)}(t-\tau)\left\{\frac{\partial^3\varphi(\tau)}{\partial x^3} + \frac{\partial^3\psi(\tau)}{\partial y^3}\right\}d\tau$$

$$+\int_0^t \mathfrak{H}^{(5)}(t-\tau)\left\{\frac{\partial^3\varphi(\tau)}{\partial x^2\partial y} + \frac{\partial^3\psi(\tau)}{\partial x\partial y^2}\right\}d\tau = \left(\frac{\partial^2\eta}{\partial x\partial y}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2} \qquad (4.94)$$

where

$$
\begin{aligned}
\overline{\mathfrak{F}}^{(1)} &= \frac{1/s}{\overline{\mathfrak{D}}^{(0)}(1-\overline{\mathfrak{N}}^{(0)2})} \; ; \\
\overline{\mathfrak{F}}^{(2)} &= 2\Gamma^{(1)}(t-\tau)-2\frac{1/s}{\overline{\mathfrak{D}}^{(0)}(1-\overline{\mathfrak{N}}^{(0)2})} \; ; \\
\overline{\mathfrak{F}}^{(3)} &= \frac{\overline{\mathfrak{D}}^{(1)}(1-\overline{\mathfrak{N}}^{(0)}\overline{\mathfrak{N}}^{(1)})}{\overline{\mathfrak{D}}^{(0)}(1-\overline{\mathfrak{N}}^{(0)2})} \; ; \\
\overline{\mathfrak{F}}^{(4)} &= \frac{\overline{\mathfrak{D}}^{(1)}(\overline{\mathfrak{N}}^{(1)}-\overline{\mathfrak{N}}^{(0)})}{\overline{\mathfrak{D}}^{(0)}(1-\overline{\mathfrak{N}}^{(0)2})} \; ; \\
\overline{\mathfrak{F}}^{(5)} &= \bar{\Gamma}^{(2)} \; .
\end{aligned}
\tag{4.95}
$$

Equation (4.94), interesting in its similarity to the corresponding equation (3.120) of the elastic case is not of much help, however, at least not on the level of this generality. This is easily seen when one is trying to write down the analogous equations of (3.122). The reason is the following: With the introduction of the stress function F three variables (namely N_x, N_y and N_{xy}) are reduced to one, but the success of this reduction is bound to a simultaneous disappearance of the displacement vector $u_i^{(0)}$. This was the case under the simplifying assumption $\mathfrak{D}^{(0)} = 0$ (which required that the Poisson ratio be not a function of the temperature). In the present situation of viscoelastic plate theory $\mathfrak{D}^{(0)}(t)$ cannot be assumed to vanish.

An approximate theory, however, can be developed by a perturbation expansion procedure. Although it cannot be assumed that $\mathfrak{L}^{(1)}(t)$ and $\mathfrak{M}^{(1)}(t)$ vanish they must be small as compared to $\mathfrak{L}^{(0)}(t)$, $\mathfrak{L}^{(2)}(t)$, $\mathfrak{M}^{(0)}(t)$ and $\mathfrak{M}^{(2)}(t)$. We thus write in all formulas

$$\mathfrak{L}^{(1)}(t) = \epsilon\tilde{\mathfrak{L}}^{(1)}(t) \quad ; \quad \mathfrak{M}^{(1)}(t) = \epsilon\tilde{\mathfrak{M}}^{(1)}(t) \tag{4.96}$$

where $\epsilon << 1$ and $\tilde{\mathfrak{L}}^{(1)} = O(1)$, $\tilde{\mathfrak{M}}^{(1)} = O(1)$. It is now reasonable to search for a solution in the form of a regular perturbation expansion by writing

$$\begin{aligned} N_x &= \mathring{N}_x + \epsilon N_x' + \ldots \\ \eta &= \mathring{\eta} + \epsilon\eta' + \ldots \\ \varphi &= \mathring{\varphi} + \epsilon\varphi' + \ldots \end{aligned} \tag{4.97}$$

Using the representations (4.96), expanding all variables in the form (4.97), substituting these expressions into (3.114), (3.117), (4.90) and (4.91) and collecting terms of like powers results in a hierarchy of equations the first set of which reduces to a viscoelastic generalization of the von Kàrmàn equations. This set of nonlinear integro-differential equations is

$$\int_0^t \mathfrak{G}^{(1)}(t-\tau)\left\{\frac{\partial^4\mathring{F}(\tau)}{\partial x^4} + \frac{\partial^4\mathring{F}(\tau)}{\partial y^4}\right\}d\tau + \int_0^t \mathfrak{G}^{(2)}(t-\tau)\frac{\partial^4\mathring{F}(\tau)^2}{\partial x^2\partial y^2}\,d\tau = \left(\frac{\partial^2\mathring{\eta}}{\partial x\partial y}\right)^2 - \frac{\partial^2\mathring{\eta}}{\partial x^2}\frac{\partial^2\mathring{\eta}}{\partial y^2}$$

$$\mathfrak{M}\ddot{\mathring{\eta}} = \frac{\partial\mathring{Q}_x}{\partial x} + \frac{\partial\mathring{Q}_y}{\partial y} + q + \frac{\partial^2\mathring{F}}{\partial y^2}\frac{\partial^2\mathring{\eta}}{\partial x^2} + 2\frac{\partial^2\mathring{F}}{\partial x\partial y}\frac{\partial^2\mathring{\eta}}{\partial x\partial y} + \frac{\partial^2\mathring{F}}{\partial x^2}\frac{\partial^2\eta}{\partial y^2} + \left(\frac{\partial\mathring{M}_x}{\partial x} - \frac{\partial\mathring{M}_{xy}}{\partial y}\right)\frac{\partial\mathring{\eta}}{\partial x} + \left(\frac{\partial\mathring{M}_y}{\partial y} - \frac{\partial\mathring{M}_{xy}}{\partial x}\right)\frac{\partial\mathring{\eta}}{\partial y}$$

$$-\mathfrak{J}\frac{\partial^2\mathring{\varphi}}{\partial t^2} = \frac{\partial\mathring{M}_x}{\partial x} - \frac{\partial\mathring{M}_{xy}}{\partial y} - \mathring{Q}_x \quad ; \quad -\mathfrak{J}\frac{\partial^2\mathring{\psi}}{\partial t^2} = \frac{\partial\mathring{M}_{xy}}{dx} - \frac{\partial\mathring{M}_y}{\partial y} - \mathring{Q}_y \quad ; \tag{4.98a}$$

$$\overset{\circ}{M}_x = -\int_0^t \overset{\circ}{\mathfrak{D}}^{(1)}(t-\tau)\left\{\frac{d}{d\tau}\left(\frac{\partial\overset{\circ}{\Phi}}{\partial x}(\tau)\right) + \mathfrak{n}^{(1)}(t-\tau)\frac{d}{d\tau}\left(\frac{\partial\overset{\circ}{\Psi}}{\partial y}(\tau)\right)\right\}d\tau + \overset{\circ}{\mathfrak{I}}_x \quad ;$$

$$\overset{\circ}{M}_y = -\int_0^t \overset{\circ}{\mathfrak{D}}^{(1)}(t-\tau)\left\{\frac{d}{d\tau}\left(\frac{\partial\overset{\circ}{\Psi}}{\partial y}(\tau)\right) + \mathfrak{n}^{(1)}(t-\tau)\frac{d}{d\tau}\left(\frac{\partial\overset{\circ}{\Phi}}{\partial x}(\tau)\right)\right\}d\tau + \overset{\circ}{\mathfrak{I}}_y \quad ;$$

$$\overset{\circ}{M}_{xy} = -\int_0^t \mathfrak{m}^{(2)}(t-\tau)\frac{d}{dt}\left\{\frac{\partial\overset{\circ}{\Phi}}{\partial x} + \frac{\partial\overset{\circ}{\Psi}}{\partial y}\right\}d\tau \quad ; \tag{4.98b}$$

$$\overset{\circ}{Q}_x = \frac{1}{2}\int_0^t \overset{\circ}{\Gamma}_x(t-\tau)\frac{d}{d\tau}\left\{\frac{\partial\overset{\circ}{\eta}}{\partial x} - \overset{\circ}{\Phi}\right\}d\tau \quad ;$$

$$\overset{\circ}{Q}_y = \frac{1}{2}\int_0^t \overset{\circ}{\Gamma}_y(t-\tau)\frac{d}{d\tau}\left\{\frac{\partial\overset{\circ}{\eta}}{\partial y} - \overset{\circ}{\Psi}\right\}d\tau \quad ,$$

where

$$\overset{\circ}{\Gamma}_x = \overset{\circ}{\Gamma}_y = 2\,\mathfrak{m}^{(0)}(\tau) \quad ;$$

$$\overset{\circ}{\mathfrak{D}}^{(1)} = \mathfrak{D}^{(1)}\Big|_{\mathfrak{L}^{(1)}\to 0,\ \mathfrak{m}^{(1)}\to 0} \quad ; \tag{4.99}$$

$$\overset{\circ}{\mathfrak{I}}_x = \overset{\circ}{\mathfrak{I}}_y = \mathfrak{I}_x\Big|_{\mathfrak{L}^{(1)}\to 0,\ \mathfrak{m}^{(1)}\to 0} \quad .$$

In the above equations F is the stress function for N_x, N_y and N_{xy}. Observe that no zeroth order displacement $u^{(o)}$, $v^{(o)}$ appears in this set of equations. Note further that this set of zeroth order equations is accurate provided the temperature field in the ice is symmetric to the plate middle surface.

The equations for the primed quantities are not listed here, because they are far too complicated to have any chance to be solved analytically or numerically.

In concluding this section it seems proper to summarize as follows: Plate theories for viscoelastic material become much more complex than their elastic counterparts in particular when the temperature varies with depth. Only in special situations does the viscoelastic theory reflect a connection to the corresponding elastic theory. In this case the class of physical situations where the elastic equations apply is, however, much broader than the class where the viscoelastic equations hold.

4.5) Determination of the Relaxation Functions

The preceding calculations have been performed on a rather general level at least as far as relaxation functions are concerned. We have refrained from interpreting the relaxation functions in terms of mechanical models, as is customary in the older literature. Accounts on this modelistic approach are by Zener [50], Gross [51] and others. A recent review of the modelistic approach is given by Caputo and Mainardi [52], [53].

It should have become apparent that the abstract treatment presented in this article is the only one that made a consistent rigorous derivation of the plate equations possible. In fact it is not clear to us how the equations could have been derived when based upon the modelistic approach.

The most commonly used reheological models can be summarized by the model for the standard linear solid which is represented by a spring arranged in parallel fashion with a dashpot in series with a spring (see Fig. 4.3). $k_2 \to 0$ corresponds

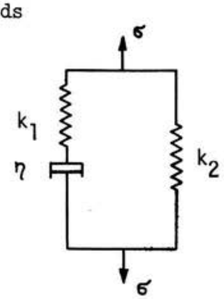


Fig 4.3

to the Maxwell body, consisting of a spring and a dashpot arranged in series. Similarly, $k_1 \to 0$ gives the so called Voigt model. It is easy to show by methods known in viscoelasticity, [46], that the standard linear solid gives rise to the following relaxation function

$$G = k_2 + k_1 \exp(-k_1 t/\eta) \quad . \tag{4.100}$$

Here, G stands for any one of $G_{ijk\ell}$. Its time dependence is sketched in Fig. 4.4.

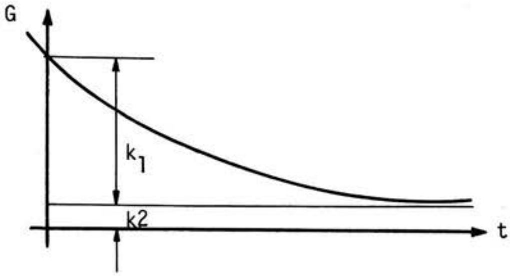


Fig 4.4
Relaxation function for the standard linear solid.

The relaxation function (4.100) for a standard linear solid is determined once the constants k_1, k_2 and η are fixed. This would mean that 3 experimental points would suffice to determine the free parameters in (4.100). It is thus natural to generalize (4.100) by the representation

$$G = G^{(0)} + \sum_{i=1}^{N} G^{(i)} \exp(-t/\tau^{(i)}) \quad , \tag{4.101}$$

where $G^{(0)}$, $G^{(i)}$ and $\tau^{(i)}$ are the free constants; $\tau^{(i)}$ is called <u>relaxation time</u>. To simplify notation we shall henceforth write

$$G_{ijk\ell}(t) = \sum_{m=o}^{\infty} G^{(m)}_{ijk\ell} \exp(-t/\tau^{(m)}_{ijk\ell}) \qquad \text{(no summation over } ijk\ell) \tag{4.102}$$

where

$$\tau^{(0)}_{ijk\ell} = \infty \quad .$$

Thermorheologically simple plates are characterized by equation (4.45). With the use of (4.102) we obtain

$$\mathfrak{G}^{(p)}_{ijk\ell}(t) = \int_h x_3^p \sum_{m=o}^{N} G^{(m)}_{ijk\ell} \exp(-t\chi(\vartheta_x)/\tau^{(m)}_{ijk\ell})dx_3$$

$$= \sum_{m=o}^{\infty} \int_h x_3^p G^{(m)}_{ijk\ell} \exp(-t/(\tau^{(m)}_{ijk\ell}/\chi(\vartheta_x)))dx_3 \tag{4.103}$$

$$= \sum_{m=o}^{\infty} \int_h x_3^p G^{(m)}_{ijk\ell} \exp(-t/\tilde{\tau}^{(m)}_{ijk\ell}(\vartheta_x))dx_3 \text{ (no summation over } ijk\ell)$$

where $\tilde{\tau}^{(m)}_{ijk\ell}(\vartheta_x)$ is now a temperature dependent relaxation time

$$\tilde{\tau}^{(m)}_{ijk\ell}(\vartheta_X) = \tilde{\tau}^{(m)}_{ijk\ell}(\vartheta_Y)f(\vartheta_X) \tag{4.104}$$

with

$$f(\vartheta_X) = 1/\chi(\vartheta_X) \quad ; \quad f(\vartheta_Y) = 1 \ . \tag{4.105}$$

The postulate of thermorheologically simple behavior is thus equivalent to the specification that the relaxation times $\tau^{(m)}_{ijk\ell}$ have a temperature dependence. The representation $(4.103)_3$, however, is not the most convenient one. Defining

$$\chi(\vartheta_X) = \ln(\Xi(\vartheta_X)) \tag{4.106}$$

the expression (4.103) can be written in the form

$$\mathfrak{G}^{(p)}_{ijk\ell}(t) = \sum_{m=o}^{N} G^{(m)}_{ijk\ell} \int_h x_3^p [\Xi(\vartheta(x_3))]^{-t/\tilde{\tau}_{ijk\ell}(\vartheta_Y)} dx_3 \ . \tag{4.107}$$

If the functions $\chi(\vartheta_X)$ or Ξ are known, it is a straightforward matter to calculate the resulting viscoelastic plate constants in a similar fashion as the corresponding constants in the elastic theory have been obtained. However, we have been unable to find experimental investigations which would provide sufficient information for the determination of the time temperature shifting function $\chi(\cdot)$. That this function must be of substantial influence follows from the fact that ice may be considered to be brittle at low temperature, while near the melting point it becomes highly viscous (see [54]). This is exactly in agreement with the postulate of thermorheologically simple materials.

It is thus suggested that research groups with the appropriate experimental equipments perform systematic experiments on the relaxation functions of polycrystalline ice with special emphasis to the variation of the temperature. Only after these experiments will have been performed, will we be in a position to accurately calculate the plate relaxation functions.

4.6) Further Applications

In the preceding sections on viscoelastic plates with nonuniform base temperature the postulate of thermorheologically simple solids has been used in its simplest form for a base temperature constant in time. The general theory, however was presented also for nonconstant temperature.

A situation somewhat between these two limiting cases is the one, where the time scale of the processes which determine the base temperature distribution and the one for the processes governed by the listed equations are different enough so that the time variation of the base temperature can be neglected except for the calculation of the relaxation functions. In other words, the relaxation functions not only depend on the present values of the temperature but also on the history of it. This seems to be important in particular in temperate regions where the bearing capacity of floating ice plates suddenly collapses when the air temperature rises above the freezing temperature for a considerable amount of time. On the other hand, short durations of such climatic variation are of little influence to the gross behavior of ice.

Although there is no experimental evidence that such processes are modeled by the history dependent form of the thermorheologically simple behavior, it is clear that formula (4.26) provides a means of modeling such behavior at least qualitatively. If for instance

$$1/\xi_s = \chi(\vartheta)s\int_{-\infty}^{t} H\Big(\vartheta(t-u) - \vartheta_o(t-u)\Big)h(u)du \qquad (4.108)$$

where $H(\cdot)$ is the Heaviside step function, ϑ_o a fixed reference temperature and h a monotonic decreasing influence function, then such behavior is modeled. Choosing namely $h(u) = \delta(u)$ (Dirac distribution) and ϑ_o below any occuring temperature then (4.108) reduces to (4.28), which was used throughout Chapter 4. If, however, we choose

$$h(u) = -A \exp(-u/\tau) \quad ,$$

where τ is some fixed relaxation time, then we account for an exponentially fading memory. Choosing further ϑ_o as the freezing temperature then the integral in (4.108) will give a contribution whenever a particle of the ice has reached the melting temperature. This contribution is larger the longer such periods of high air temperature persist and the more recent they occur in the past.

Obviously the form (4.108) is probably not the appropriate one, but it certainly demonstrates that the apparent temperature - time shifting function

$$\chi(\vartheta)\int_{-\infty}^{t} H(\vartheta(t-u)-\vartheta_o(t-u))\, h(u)du$$

increases with increasing duration of relatively high temperature. This corresponds to an effective decrease of the relaxation function. In the limit when this apparent temperature-time shifting function resides to infinity only $G^{(o)}$(see 4.101) will survive. However, ice is a viscoelastic fluid and not a solid, as istacitly assumed by all workers on viscoelastic behavior of ice (see [1]). Thus $G^{(o)} = 0$ and this means that all bearing capacity has disappeared.

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