Identification Methods for Structural Systems

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System Stability - 26 March, 2014
Overview

- System Stability
Assume given a dynamic system with input $u(t)$ and output $x(t)$. The stability property of a dynamic system can be defined from the system’s impulse response $x_\delta(t)$ as follows:

**Asymptotically stable system:** The steady state impulse response tends to zero

$$\lim_{t \to \infty} x_\delta(t) = 0$$

**Marginally stable system:** The steady state impulse response is non-zero, but bounded (does not grow infinite):

$$\lim_{t \to \infty} x_\delta(t) < \infty$$

**Unstable system:** The steady state impulse response is unbounded:

$$\lim_{t \to \infty} x_\delta(t) = \infty$$
**Stable System Example** - Assume the following SDOF system:

\[
\ddot{x}(t) + 3\dot{x}(t) + 8x(t) = u(t) \xrightarrow{L} s^2X(s) + 3sX(S) + 8X(s) = U(s)
\]

Therefore, the associated Transfer Function is: \( H(s) = \frac{1}{s^2 + 3s + 8} \)

Observe that the roots of \( s^2 + 3s + 8 \) are: \( s_1 = -1.5 + 2.4i \), \( s_2 = -1.5 - 2.4i \)
**Marginally System Example** - Assume the following SDOF system:

\[\ddot{x}(t) + 9x(t) = u(t) \xrightarrow{\mathcal{L}} s^2 X(s) + 9X(s) = U(s)\]

Therefore, the associated Transfer Function is:

\[H(s) = \frac{1}{s^2 + 9}\]

Observe that the roots of \(s^2 + 9\) are: \(s_1 = 3i, \ s_2 = -3i\)
**Unstable System Example** - Assume the following SDOF system:

\[
\ddot{x}(t) + 3\dot{x}(t) - \frac{3}{4}x(t) = u(t) \xrightarrow{\mathcal{L}} s^2X(s) + 3sX(S) - \frac{3}{4}X(s) = U(s)
\]

Therefore, the associated Transfer Function is:

\[
H(s) = \frac{1}{s^2 + 3s - \frac{3}{4}}
\]

Observe that the roots of \(s^2 + 9\) are: \(s_1 = -3.23, s_2 = 0.23\)
Transfer Function - Poles & Zeros

For the general case of linear systems examined herein. The TF will generally have the following form:

$$H(s) = \frac{K(s + z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} \quad m < n$$

*Note: If $m > n$, then the division can be carried out and the system can eventually be rewritten using a form analogous to the above.

The constants $z_i$ are called the zeros of the transfer function or signal, and $p_i$ are the poles.

Viewed in the complex plane, it is clear that the magnitude of $H(s)$ will be equal to zero at the zeros ($s = z_i$), and to infinity at the poles ($s = p_i$).
Reminder - By using partial fraction expansion we can rewrite the TF in the following form:

\[ H(s) = \frac{b_1 s + b_2}{(s - p_1)(s - p_2)} + \frac{b_3}{s - p_3} + \ldots \]

Where \( p_1 = p_2^* \) (complex conjugate roots) and \( p_3 \) is a real root.

In order to derive \( b_1, b_2 \) multiply both sides by \((s - p_1)(s - p_2)\), and then evaluate at \( s = p_1 \).

In order to derive \( b_3 \) multiply both sides by \((s - p_3)\), and then evaluate at \( s = p_3 \).

It is then easy to obtain the system’s response \( x(t) \) to an impulse input \( u(t) = \delta(t) \) by simply applying the Inverse Laplace transform on \( H(s) \):

\[ x(t) = \mathcal{L}^{-1} \{ H(s) \ast U(s) \} \]

\[ \quad \text{with} \quad U(s) = \mathcal{L}\{\delta(t)\} = 1 \rightarrow x(t) = \mathcal{L}^{-1} \{ H(s) \} \]
Linear System Stability

Stability Rules

For systems that are linear and time-invariant (i.e. their TF does not depend on time) their stability is defined by the roots of the characteristic polynomial, i.e., the poles of the TF. Specifically:

- **Asymptotically Stable**
  All the roots of the characteristic polynomial lie in the left half plane ($Re(s) < 0$)

- **Unstable**
  At least one root of the characteristic polynomial lies in the right half plane ($Re(s) > 0$)

- **Marginally Stable**
  No solution grows unbounded but some do not decay ($Re(s) \leq 0$)
Interpretation

The poles of the TF are essentially the roots of the characteristic polynomial that corresponds to the original ODE of the system. Therefore they govern the system’s homogeneous response.

A root $s = \sigma \pm \omega i$ signifies that the homogeneous response will be of the type:

$$x = e^{st} = e^{\sigma t}(C_1 \sin \omega t + C_2 \cos \omega t)$$

It is now apparent that a positive real part for $s$ would be linked to an exponentially increasing response which is the cause for instability.
Pole Zero Plots

Examining the Pole Zero plots of the previous examples, helps illustrate the previous point (MATLAB: pzmap)
Considerations

What happens in the case of a repeated root? Assume we have a repeated root at \( s = p_k \) with multiplicity \( l \). Then the Transfer Function is written as follows:

\[
H(s) = \frac{K(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_k)^l \ldots (s - p_n)}
\]

Then the partial fraction expansion, focusing on that term, will be of the form:

\[
H(s) = \cdots + \frac{b_1}{s - p_k} + \frac{b_2}{(s - p_k)^2} + \cdots + \frac{b_l}{(s - p_k)^l} + \cdots
\]

Hence, the impulse response will be of the form

\[
x(t) = h(t) = \cdots + b_1 e^{p_k t} + b_2 t e^{p_k t} + \cdots + \frac{b_l}{(l - 1)!} t^{l-1} e^{p_k t} + \cdots
\]
Considerations - Repeated Root

However for $p_k = \sigma + i\omega$ with $\Re(p_k) = \sigma < 0$ we have that:

$$\int_{0}^{\infty} t^{l-1} e^{p_k t} < \infty$$

This signifies that these terms are bounded, i.e., they do not grow to infinity and therefore our previously derived criteria for stability hold.

**Example:**
Impulse Response of $H(s) = \frac{1}{(s + 1)^2}$

Repeated root $s = -1$