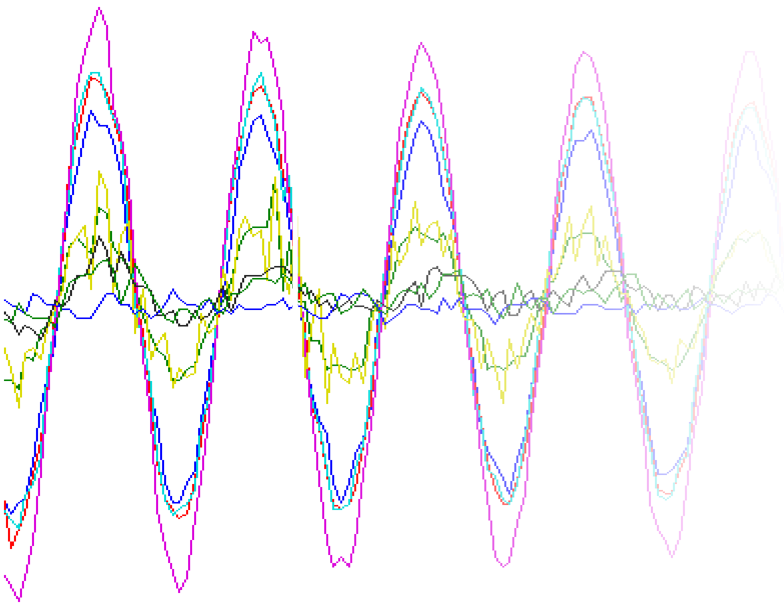
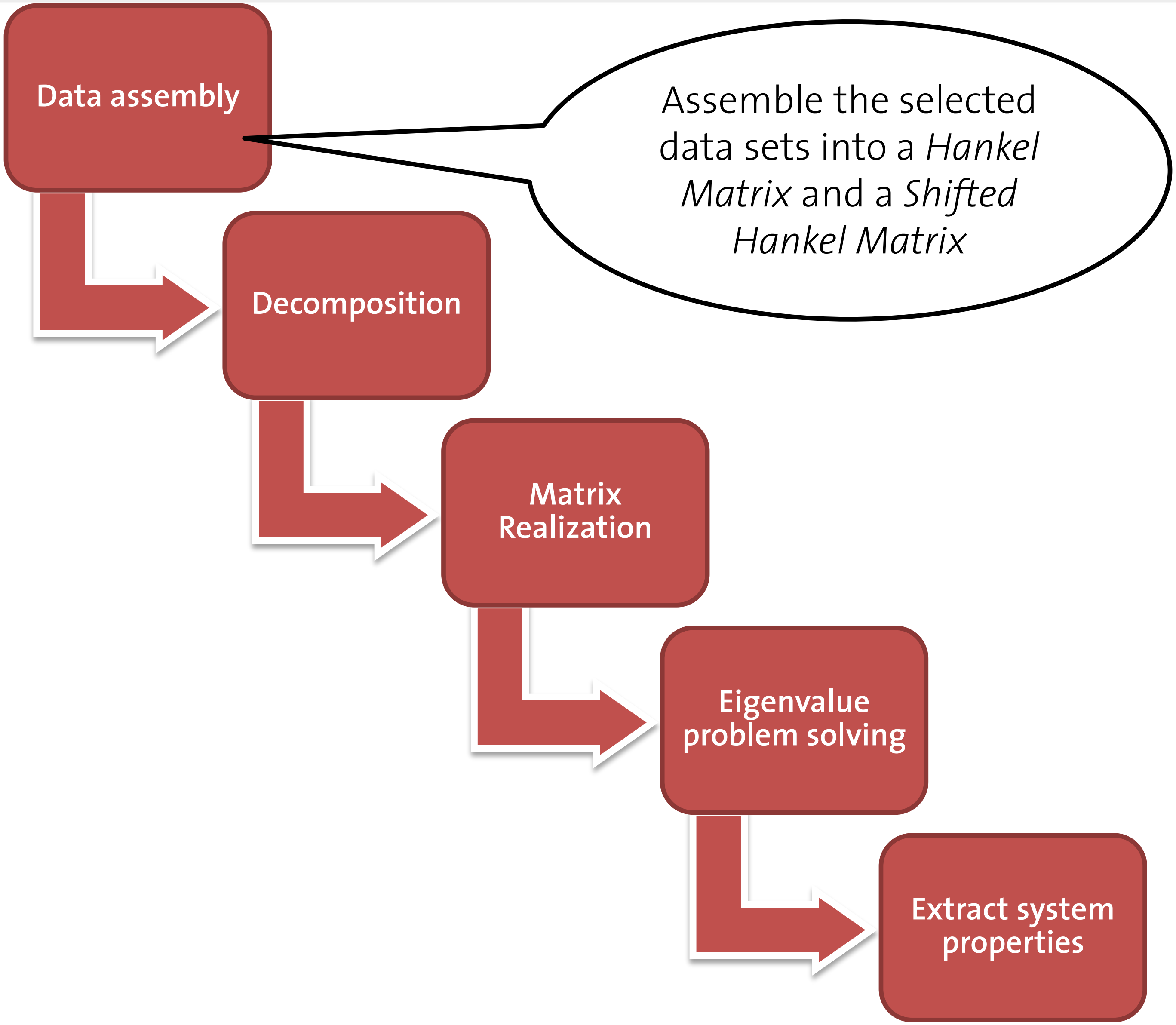
The Eigensystem Realization Algorithm (ERA)

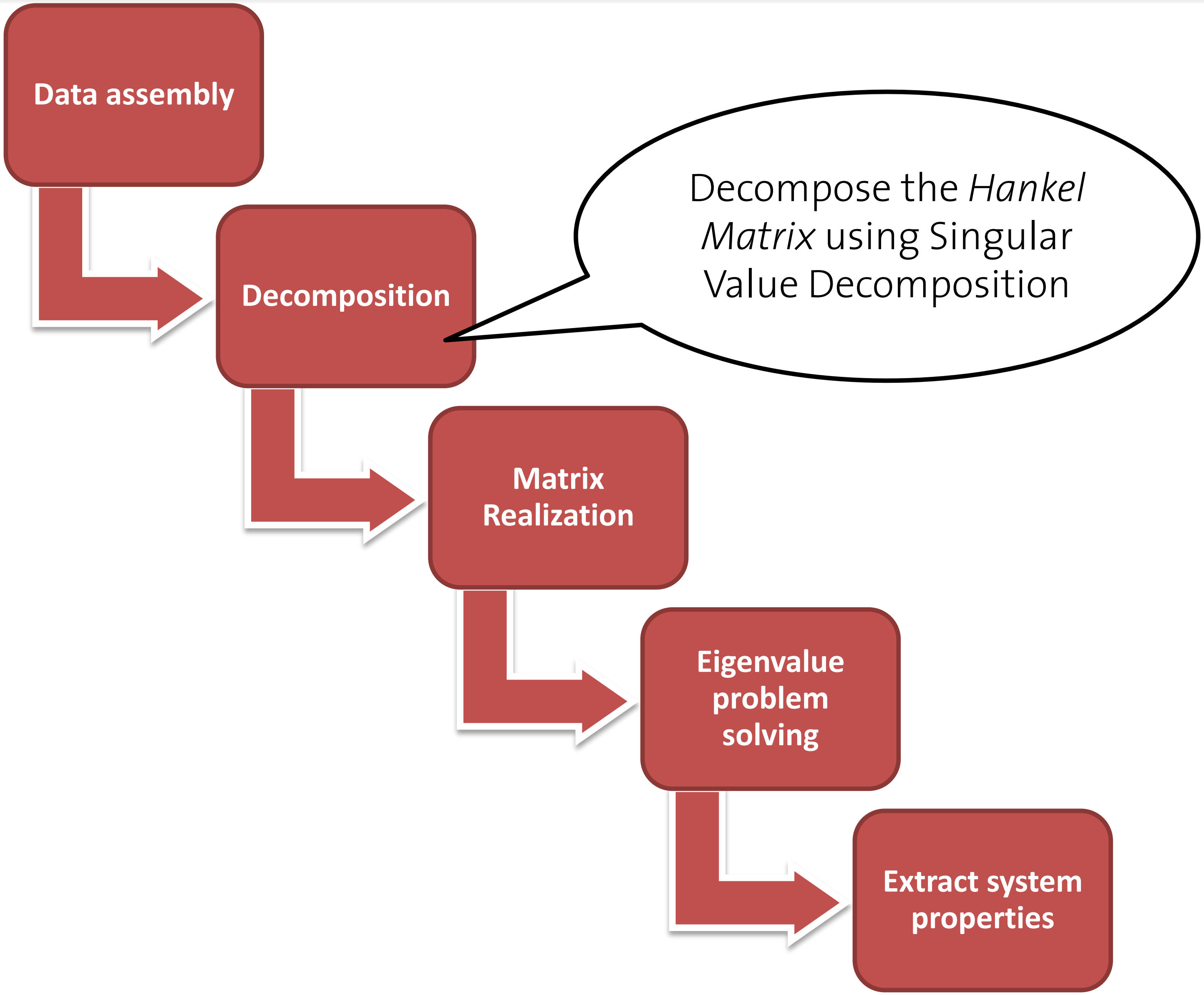


$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i + Du_i$$

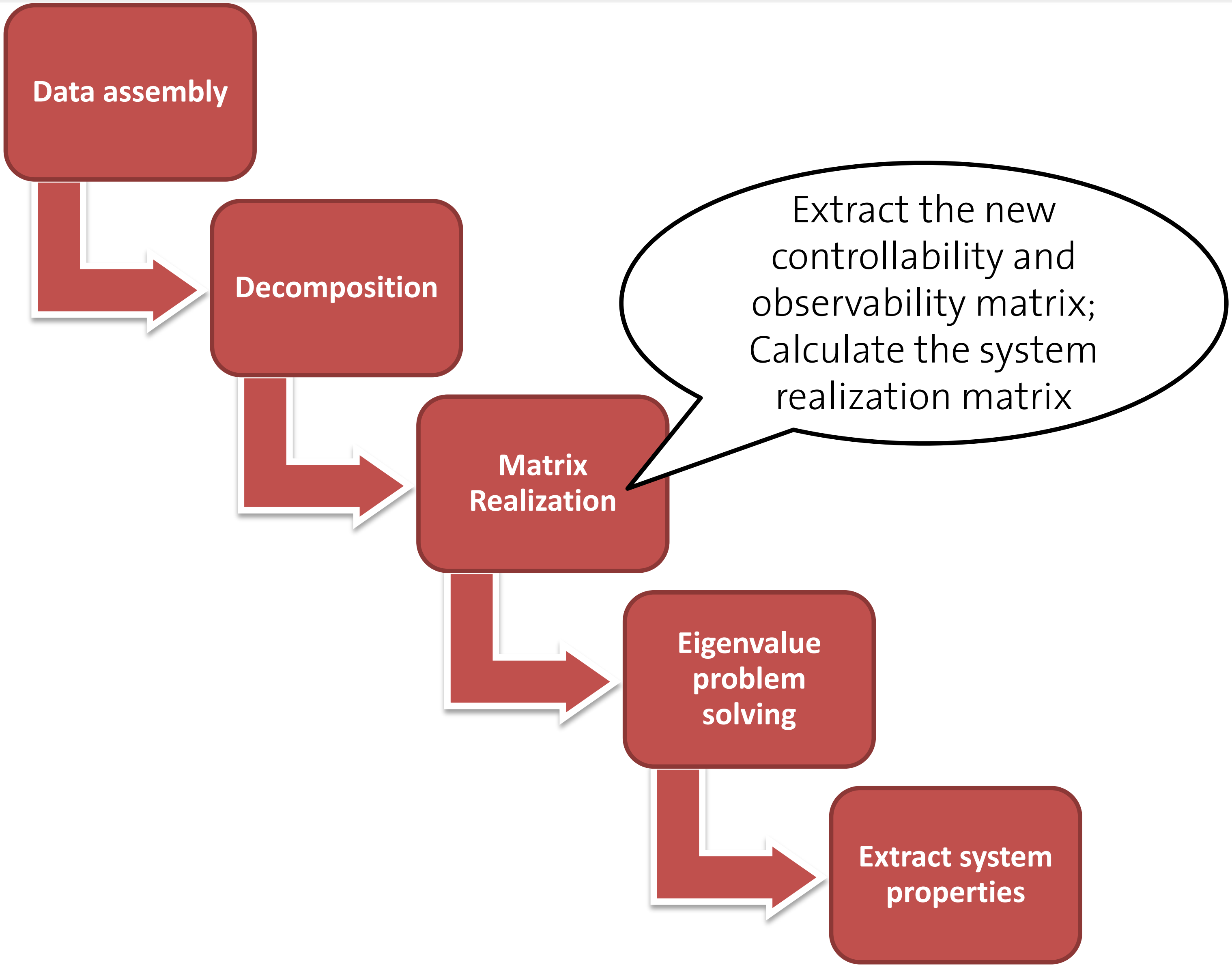
Workflow overview



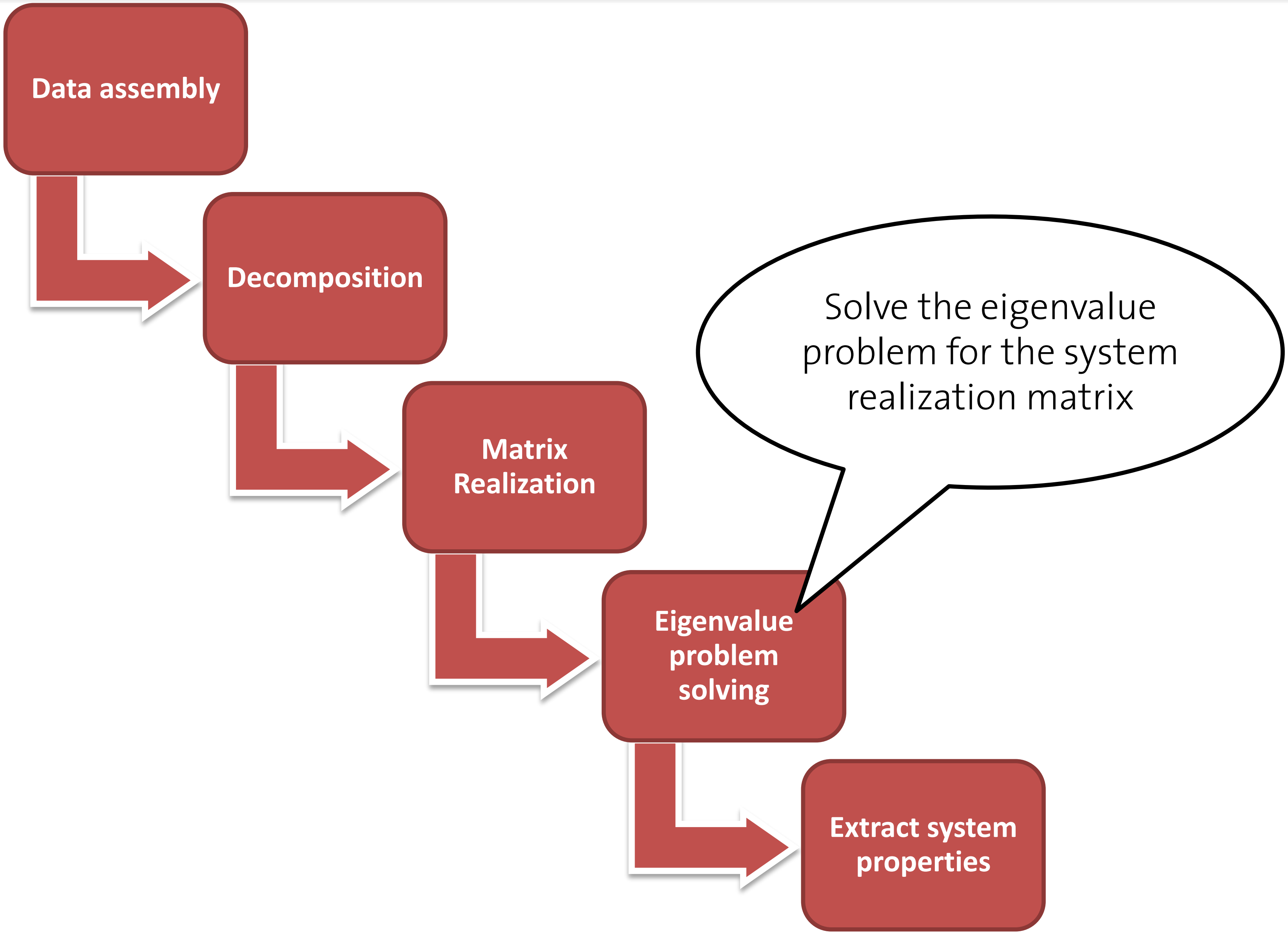
Workflow overview



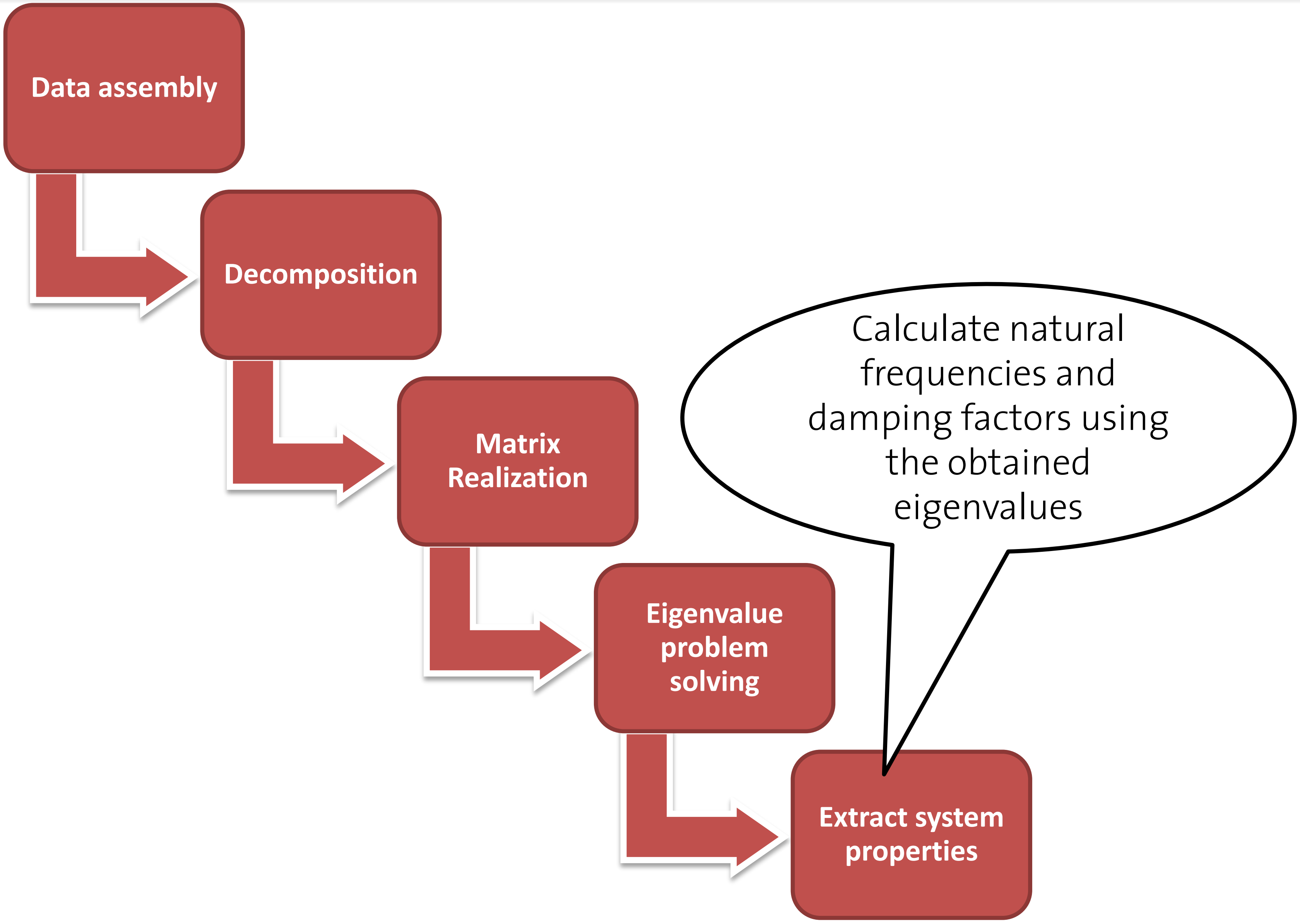
Workflow overview



Workflow overview

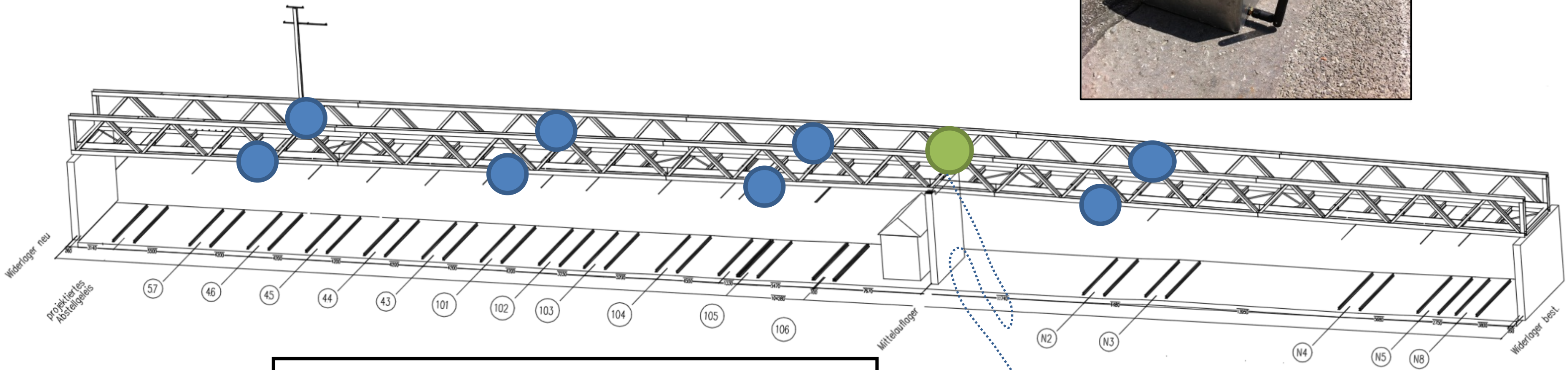


Workflow overview

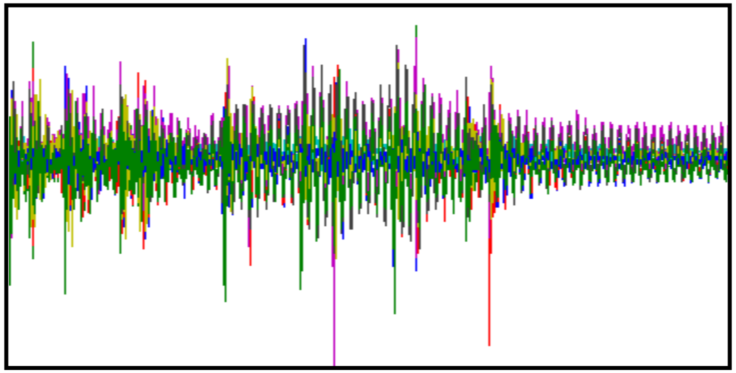


Data acquisition

Note: The ERA is implemented for the case of free response data. Therefore Impact (Hammer, drop-weight) tests would be generally suitable.



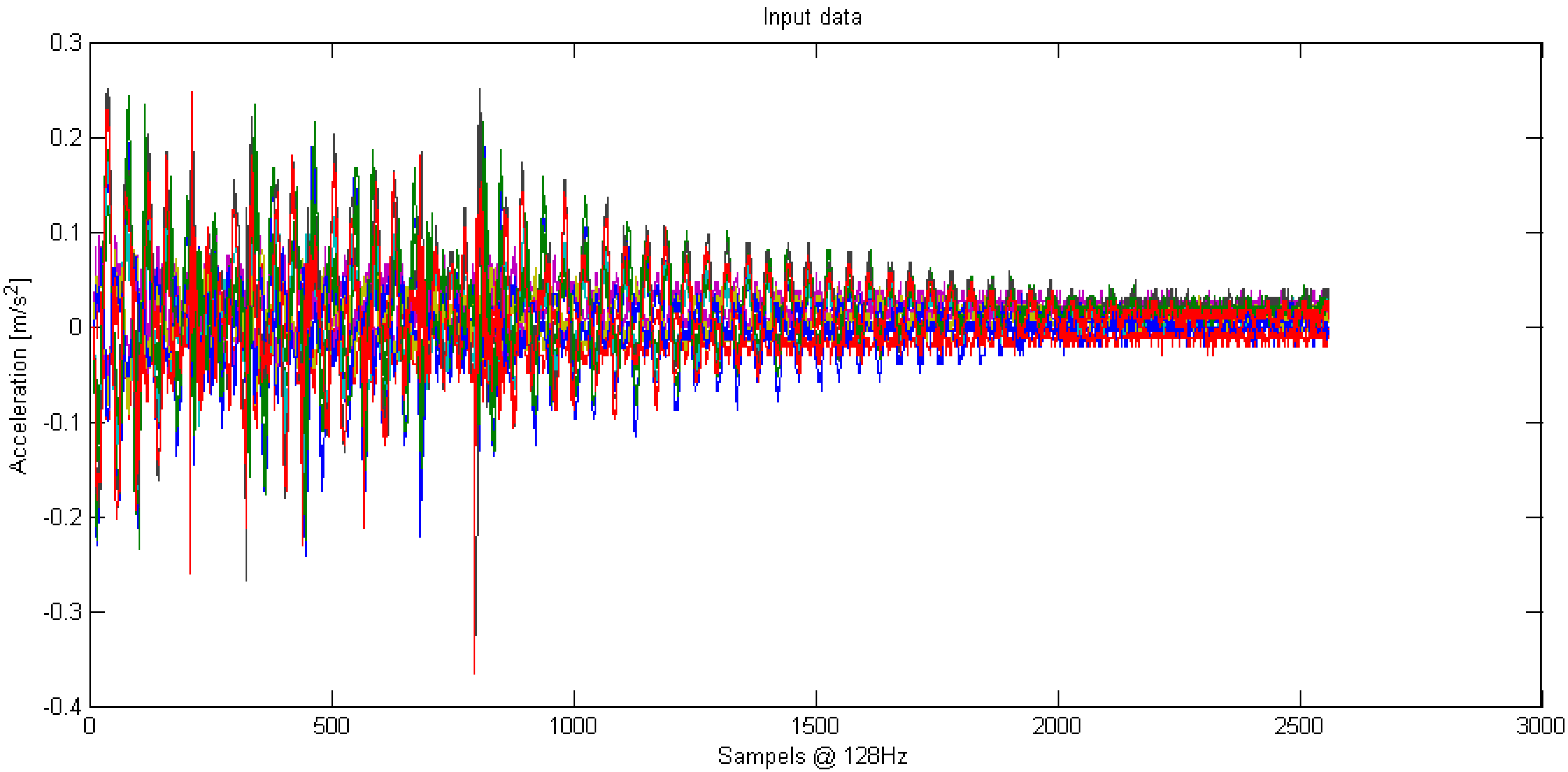
Acceleration signals



PC

Preprocessing

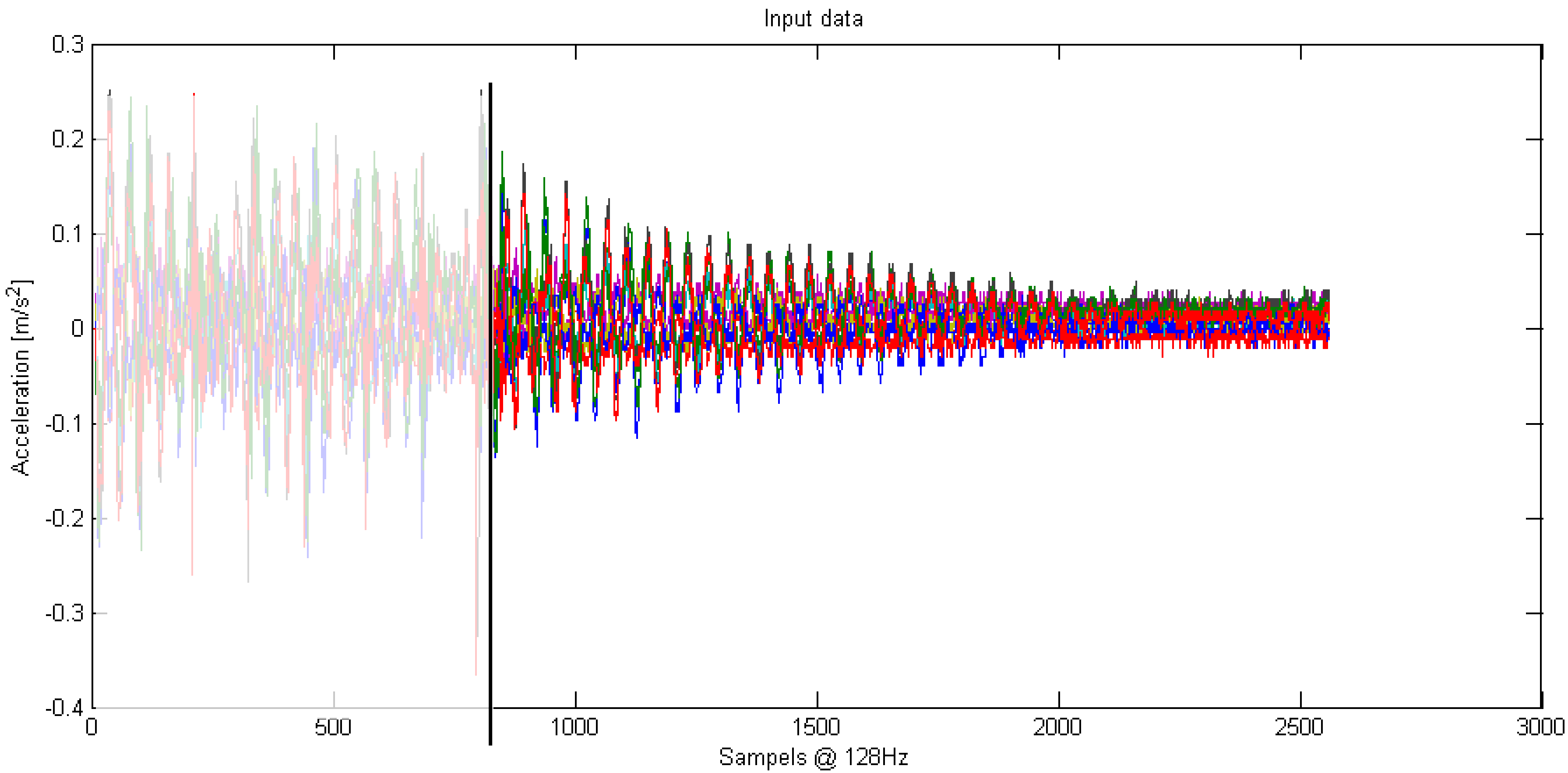
Selection



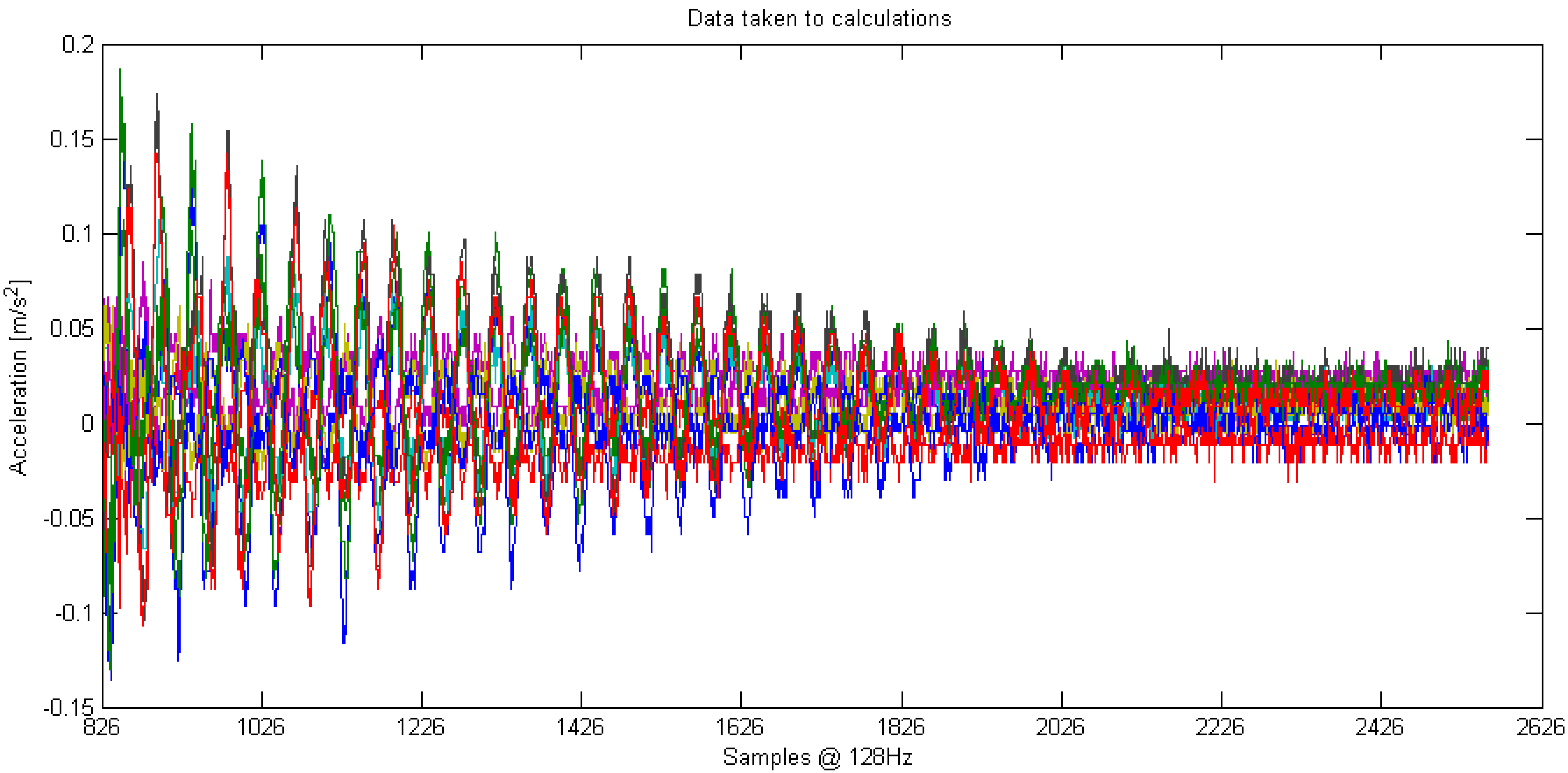
Preprocessing

Selection

(keep the part that corresponds to free response)

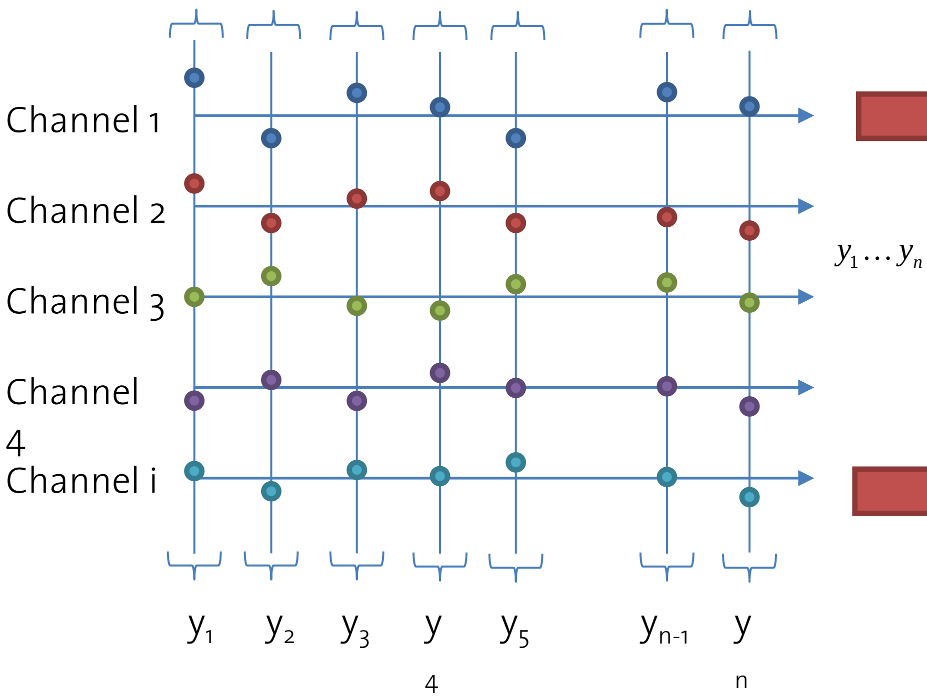


Selection


Data taken to calculations
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
Acceleration [m/s2]
826
1026
1226
1426
1626
1826
2026
2226
2426
2626
Samples @ 128Hz

Data assembly

The ERA works by exploiting the relationship of the series of outputs from different points **(channels)** of the structure to fundamental system properties **(Markov Parameters)**



Hankel Matrix:

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ y_3 & y_4 & \cdots & y_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n+1} & \cdots & y_{n+k} \end{bmatrix} = H_1$$

Shifted Hankel Matrix:

$$\begin{bmatrix} y_2 & y_3 & \cdots & y_{n+1} \\ y_3 & y_4 & \cdots & y_{n+2} \\ y_4 & y_5 & \cdots & y_{n+3} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n+1} & y_{n+2} & \cdots & y_{n+k+1} \end{bmatrix} = H_2$$

Decomposition

Assume the state – space representation of a dynamic system

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i + Du_i$$

Decomposition

Assume the state – space representation of a dynamic system

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$$\left.\begin{aligned} u_0 &= 1 \\ u_k &= 0 \quad \text{if } k > 0 \\ x_0 &= 0 \\ D &= 0 \end{aligned}\right\}$$

Assume an impulse force, at t = 0, and 0 Initial Conditions

Decomposition

Assume the state – space representation of a dynamic system

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i + Du_i$$

→

$$x_{i+1} = Ax_i + Bu_i$$
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→

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i$$

$$u_0 = 1$$
$$u_k = 0 \quad \text{if } k > 0$$
$$x_0 = 0$$
$$D = 0$$

By iterating system in time

$$x_0 = 0$$
$$x_1 = Ax_0 + B = B \qquad y_0 = 0$$
$$x_2 = Ax_1 = AB \qquad y_1 = CB$$
$$x_3 = Ax_2 = A^2B \qquad y_2 = CAB$$
$$\dots$$

Decomposition

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i + Du_i$$

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i$$

Iterate the system in time starting from I.C.

$$u_0 = 1$$
$$u_k = 0 \quad \text{if } k > 0$$
$$x_0 = 0$$
$$D = 0$$

$$x_0 = 0$$
$$x_1 = Ax_0 + B = B$$
$$x_2 = Ax_1 = AB$$
$$x_3 = Ax_2 = A^2B$$
$$\dots$$

$$y_0 = 0$$
$$y_1 = CB$$
$$y_2 = CAB$$

These constant parameters are termed & are system characteristics: Markov Parameters

Decomposition

By constructing the Hankel matrix of the Markov Parameters y_i :

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ y_3 & y_4 & \cdots & y_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n+1} & \cdots & y_{n+k} \end{bmatrix} = H_1 \Rightarrow \begin{bmatrix} CB & CAB & \ldots & CA^nB \\ CAB & CA^2B & \ldots & CA^{n+1}B \\ CA^2B & CA^3B & \ldots & CA^{n+2}B \\ \ldots & \ldots & \ldots & \ldots \\ CA^nB & CA^{n+1}B & \ldots & CA^{n+k}B \end{bmatrix} = H_1$$

Decomposition

By constructing the Hankel matrix of the Markov Parameters y_i :

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_2 & y_3 & \dots & y_{n+1} \\ y_3 & y_4 & \dots & y_{n+2} \\ \dots & \dots & \dots & \dots \\ y_n & y_{n+1} & \dots & y_{n+k} \end{bmatrix} = H_1 \Rightarrow \begin{bmatrix} CB & CAB & \dots & CA^nB \\ CAB & CA^2B & \dots & CA^{n+1}B \\ CA^2B & CA^3B & \dots & CA^{n+2}B \\ \dots & \dots & \dots & \dots \\ CA^nB & CA^{n+1}B & \dots & CA^{n+k}B \end{bmatrix} = H_1$$

Which is equivalent to the matrix product:

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^n \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^nB \end{bmatrix} = H_1$$

Decomposition

By constructing the **Hankel matrix** of the **Markov Parameters** y_i :

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ y_3 & y_4 & \cdots & y_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_n & y_{n+1} & \cdots & y_{n+k} \end{bmatrix} = H_1 \Rightarrow \begin{bmatrix} CB & CAB & \ldots & CA^nB \\ CAB & CA^2B & \ldots & CA^{n+1}B \\ CA^2B & CA^3B & \ldots & CA^{n+2}B \\ \ldots & \ldots & \ldots & \ldots \\ CA^nB & CA^{n+1}B & \ldots & CA^{n+k}B \end{bmatrix} = H_1$$

$$\Downarrow$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \ldots \\ CA^n \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \ldots & A^nB \end{bmatrix} = H_1$$

Controllability matrix

Observability matrix

Decomposition

By constructing the **Hankel matrix** of the **Markov Parameters** y_i :

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_2 & y_3 & \dots & y_{n+1} \\ y_3 & y_4 & \dots & y_{n+2} \\ \dots & \dots & \dots & \dots \\ y_n & y_{n+1} & \dots & y_{n+k} \end{bmatrix} = H_1 \Rightarrow \begin{bmatrix} CB & CAB & \dots & CA^nB \\ CAB & CA^2B & \dots & CA^{n+1}B \\ CA^2B & CA^3B & \dots & CA^{n+2}B \\ \dots & \dots & \dots & \dots \\ CA^nB & CA^{n+1}B & \dots & CA^{n+k}B \end{bmatrix} = H_1$$

Controllability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^n \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^nB \end{bmatrix} = H_1$$

Observability matrix

$$H_1 = O_p C_q$$

Decomposition

Controllability matrix

Observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^n \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & A^nB \end{bmatrix} = H_1$$

$$H_1 = O_p C_q$$

In order to obtain these matrices we perform Singular Value Decomposition for H1:

$$H_1 = U\Gamma^2 V^T$$

Matrix Realization

Product of
Singular Value Decomposition :

$$H_1 = U\Gamma^2 V^T$$

$$P = U\Gamma$$
$$Q = \Gamma V^T$$

New observability matrix

New controllability matrix

$$H_1 = PQ$$

TIP:

$$H_1 = O_p C_q$$

Matrix Realization

Note: The Decomposition $H_1 = PQ$ **is not unique!**

In fact by using a different number of shifts k, and total measurements n, different alternatives can occur. And this is due to the fact that if matrices (A, B, C) are a realization of the system:

$$x_{i+1} = Ax_i + Bu_i$$
$$y_i = Cx_i + Du_i$$

Then matrices, TAT^{-1}, TB, CT^{-1} are also a realization through the system:

$$\overline{x}_{i+1} = TAT^{-1}\overline{x}_i + TBu_i$$
$$y_i = CT^{-1}\overline{x}_i + Du_i$$

Under the transformation: $\overline{x} = Tx$

Therefore the state $\overline{x}$ that occurs from the ERA is not necessarily the x that corresponds to the structural dofs but some transformation of it.

Matrix Realization

Then, using the *Shifted Hankel Matrix* :

$$H_2 = O_p A C_q \quad \Longrightarrow \quad A = O_p^{-1} H_2 C_q^{-1}$$

By using the new observability and
the new controllability matrices:

O_p replaced by P

C_q replaced by Q

$$\hat{A} = P^{-1} H_2 Q^{-1}$$

Realization of A

Matrix Realization

Realization of C
“Output matrix”

Realization of B
“Control matrix”

$$\hat{C} \uparrow \qquad \hat{B} \uparrow$$

$$P = \begin{bmatrix} C \\ CA \\ CA^2 \\ \ldots \\ CA^n \end{bmatrix} \qquad Q = \begin{bmatrix} B & AB & A^2B & \ldots & A^nB \end{bmatrix}$$

Eigenvalue problem solving

$$x_{i+1} = \hat{A}x_i + \hat{B}u_i$$
$$y_i = \hat{C}x_i$$

$$\hat{A}v = \lambda v$$

By solving the eigenvalue problem

Eigenvalues

$$\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$$

$$V = \{v_1, v_2, ..., v_n\}$$

Eigenvectors

Extract system properties

Conversion for discrete time to continuous time representation

$$\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\} \Rightarrow \lambda_{c(i)} = \frac{ln(\lambda_i)}{dt}$$

$$\begin{cases} \omega_i = |\lambda_{c(i)}| \\ \zeta_i = \dfrac{Re(\lambda_{c(i)})}{|\lambda_{c(i)}|} \end{cases}$$

Natural frequencies;

Damping factors;

For obtaining the mode shapes:

$$V = \{v_1, v_2, ..., v_n\}$$

$$y_k = \hat{C} v_k$$

Extension for Random Input

The ERA as an input-output Id method

It has already been mentioned that the ERA operates using output measurements of **impulse response** data. However, it possible to appropriately extend the method so as to account for response to a **measured** input loading.

Assuming measurements of the input *f(t)* and output of the system *x(t)* are available from *m* measurement locations. The **Frequency Response Function (FRF)** may be extracted as:

$$H_i(j\omega) = \frac{S_{xf}(j\omega)}{S_{ff}(j\omega)}, \quad i = 1 \ldots m$$

Then by applying the **Inverse Fourier Transfom**, the **Impulse Response Functions (IRF)** per measurement channel (usually this implied per dof) are obtained. The ERA method, as described previously can then be implemented on the **IRFs** which essentially simulate the system's response to impulse.

Extension for Random Input

Proof of the FRF extraction formula:

As mentioned in Lecture 1, the system's response to a random input can be obtained via discrete convolution with the IRF:

$$x[t]=\sum_{\tau=0}^{\infty} h[t-\tau]\,f[\tau] \quad (1)$$

On the other hand, the ***cross-correlation*** of two discrete time signals is defined as:

$$R_{xf}[\tau]=\sum_{t=-\infty}^{\infty} x[t]\,f[t-\tau] \quad (2)$$

$$\sum_{t=-\infty}^{\infty} x[t]\,f[t-\tau]=\sum_{t=-\infty}^{\infty}\left\{\sum_{\tau=0}^{\infty} h[t-\tau]\,f[\tau]\right\} f[t-\tau] \Rightarrow R_{xf}[\tau]=R_{ff}[\tau]*h[\tau]$$

However, convolution in the time domain is multiplication in the frequency domain. Thus, by taking the Fourier Transform we obtain:

$$\boxed{S_{xf}(j\omega)=S_{xx}(j\omega)H(j\omega)}$$

Extension for White Noise (Ambient Data)

The Natural Excitation Technique (NExT)

For the case of ambient (operational) loads, it may be assumed that the excitation and responses are each stationary random processes. Assuming that the structural parameter matrices are deterministic, postmultiplying the Eq. of motion by a reference scalar response process $X_1(t_2)$ and taking the expected value of each side yields:

$$ME\left[\ddot{X}(t_1)X_i(t_2)\right]+CE\left[\dot{X}(t_1)X_i(t_2)\right]+KE\left[X(t_1)X_i(t_2)\right]=E\left[F(t_1)X_i(t_2)\right]$$

$$\Rightarrow MR_{\ddot{X}X_i}(t_1,t_2)+CR_{\dot{X}X_i}(t_1,t_2)+KR_{XX_i}(t_1,t_2)=R_{FX_i}(t_1,t_2)$$

where $X(t)$, $F(t)$ denote the displacement and excitation stochastic vector process respectively. Additionally, for weakly (or strongly) stationary processes, we know that:

$$R_{A^{(m)}B}(\tau)=R^{(m)}{}_{AB}(\tau),\quad \tau=t_2-t_1,\ \text{where } m \text{ denotes the m}^{\text{th}}\text{ derivative.}$$

Recognizing that the responses of the system are uncorrelated to the disturbance for *t>0*, and assuming that the random vector processes $X, \dot{X}, \ddot{X}$ are **weakly stationary,** we can write:

$$M\ddot{R}_{XX_i}(t_1\tau,t_2)+C\dot{R}_{XX_i}(\tau)+KR_{XX_i}(\tau)=0$$

Thus, the vector of displacement process correlation functions, satisfies the homogeneous differential equation of motion. Using a similar approach it can be shown that the acceleration process correlation functions also satisfy this equation (Beck et al. 1994). **We can therefore employ the ERA for the correlation signals!**