The very basics - Newton’s 2nd law of motion:

\[ m\ddot{x} = w - k(\Delta + x) \]

\[ m\ddot{x} - w + k(\Delta + x) = 0 \]

\[ m\ddot{x} - k\Delta + k(\Delta + x) = m\ddot{x} + kx = 0 \]

\[ \ddot{x} + \frac{k}{m} x = \ddot{x} + \omega_n^2 x = 0 \]

\[ x(t) = A\sin\omega_n t + B\cos\omega_n t \Rightarrow x(t) = \frac{\dot{x}_0}{\omega_n} \sin\omega_n t + x_0 \cos\omega_n t \]

where \( \omega_n = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \)
Introduction to Dynamic Analysis

We have previously considered the equilibrium equations governing the linear dynamic response of a system of finite elements:

\[ \mathbf{M} \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{R} \]

\[ \mathbf{F}_I(t) + \mathbf{F}_D(t) + \mathbf{F}_E(t) = \mathbf{R}(t) \]

where:

\[ \mathbf{M} : \text{Mass Matrix} \]
\[ \mathbf{K} : \text{Stiffness Matrix} \]
\[ \mathbf{U} : \text{Displacements} \]
\[ \dot{\mathbf{U}} : \text{Velocities} \]
\[ \ddot{\mathbf{U}} : \text{Accelerations} \]

\[ \mathbf{F}_I(t) : \text{Inertial Force} \]
\[ \mathbf{F}_D(t) : \text{Damping Force} \]
\[ \mathbf{F}_E(t) : \text{Internal Force} \]
When is Dynamic Analysis required?

Whether a dynamic analysis is needed or not is generally up to engineering judgment.

requires understanding of the interaction between loading and structural response!

In general, if the loading varies over time with frequencies higher than the Eigen-frequencies of the structure $\rightarrow$ dynamic analysis will be required.
Objective

Solve the dynamic Equation of motion (numerically)

\[ M\ddot{U} + C\dot{U} + KU = R \]

In principle the equilibrium equations may be solved by any standard numerical integration scheme BUT!

Efficiency numerical efforts must be considered and it is worthwhile to look at special techniques of integration which are especially suited for the analysis of finite element assemblies.

*The section on Direct Integration Methods is based on Prof. M. Faber's notes of the FEM II course - Fall 2009
Direct Integration Methods

These methods rely in discretizing the continuous problem. The original problem formulation is essentially the dynamic equation of motion which is for structural problems is a second order Ordinary Differential Equation (ODE).

This involves, after the solution is defined at time zero, the attempt to satisfy dynamic equilibrium at discrete points in time. Most methods use equal time intervals at $\Delta t, 2\Delta t, 3\Delta t........N\Delta t$.

However this is not mandatory; in some cases a variable time step might be employed. This is most commonly the case for special classes of problems such as Impact Problems.
**Direct** means: The equations are solved in their original form.

Two ideas are utilized:

1. The equilibrium equations are satisfied only at time steps, i.e., at discrete times with intervals $\Delta t$.

2. A particular variation of displacements, velocities, and accelerations within each time interval is assumed.

The accuracy depends on these assumptions as well as the choice of time intervals!
Assumptions

The displacements, velocities and accelerations

\[ \mathbf{U}_0 : \text{Displacement vector at time } t \]
\[ \dot{\mathbf{U}}_0 : \text{Velocity vector at time } t \]
\[ \ddot{\mathbf{U}}_0 : \text{Acceleration vector at time } t \]

are assumed to be known and we aim to establish the solution of the equilibrium equations for the period 0 - \( T \).

For this purpose we sub-divide \( T \) into \( n \) intervals of length \( \Delta t = T/n \) and establish solutions for the times \( \Delta t, 2\Delta t, 3\Delta t, \ldots, T \).
Direct Integration Methods

We distinguish principally between **Implicit and Explicit** methods

- **Explicit methods:**

  Solution is based on the equilibrium equations at time $t$

  1st order Example: The Forward Euler Method

  Assuming we want to approximate the solution of the initial value problem

  $$\dot{y} = f(t, y(t)), \quad y(t_0) = y_0$$

  by using the first two terms of the Taylor expansion of $y$, which represents the linear approximation around the point $(t_0, y(t_0))$ we obtain the forward Euler integration rule:

  $$y_{n+1} = y_n + hf(t_n, y_n)$$
Implicit methods:

Solution is based on the equilibrium equations at time $t + \Delta t$

1st order Example: The Backward Euler Method

Assuming we want to approximate the solution of the same initial value problem the backward Euler integration rule is obtained as:

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

Since $f(t, y)$ is, in general, a non-linear function of $y$, iteration is required to solve this last equation for $y_{n+1}$.
Direct Integration Methods

Explicit methods calculate the state of a system at a later time from the state of the system at the current time, while implicit methods find a solution by solving an equation involving both the current state of the system and the later one.

Note that:

Implicit integration is not necessarily more accurate than explicit. It can be less accurate! The major benefit of implicit integration is stability. Many of these methods are able to run with any arbitrarily large time step for any input unless we are lying at the limits of floating point math (unconditionally stable). Obviously a large time step implies throwing away accuracy.

For most real structures, which contain stiff elements, a very small time step is required in order to obtain a stable solution. Therefore, all explicit methods are conditionally stable. The condition is the size of the selected time step.
Direct Integration Methods

Most Commonly used Direct Integration Methods

(for the case of the Dynamic Equation of Motion)

1. The Central Difference Method (CDF)
2. The Houbolt method
3. The Newmark method
4. The Wilson $\theta$ method
5. Coupling of integration operators

The difference in items 1-4 lies in the way we choose a discretized equivalent of the derivatives. The overall setup of the solution is very much similar for all methods. Additionally depending on the resulting equations some schemes are explicit (CDF) and others implicit (Houbolt, Newmark, Wilson $\theta$)
## Direct Integration Methods

### Most Commonly used Direct Integration Methods

<table>
<thead>
<tr>
<th>Central Difference Method</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Velocity</strong></td>
<td><strong>Acceleration</strong></td>
</tr>
<tr>
<td>[ U(t) = \frac{U(t+\Delta t) - U(t-\Delta t)}{2\Delta t} ]</td>
<td>[ \ddot{U}(t) = \frac{\dot{U}(t+\Delta t) - \dot{U}(t)}{\Delta t} ]</td>
</tr>
</tbody>
</table>

**Reminder:** The difference lies in the way we choose a discretized equivalent of the derivatives.
**Direct Integration Methods**

**Most Commonly used Direct Integration Methods**

<table>
<thead>
<tr>
<th>The Houbolt Method</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Displacement</strong></td>
<td><strong>Velocity</strong></td>
</tr>
<tr>
<td>$U_i = U_{i+\Delta t} - \Delta t \dot{U}<em>{i+\Delta t} + \frac{\Delta t^2}{2} \ddot{U}</em>{i+\Delta t} - \frac{\Delta t^3}{6} \dddot{U}_{i+\Delta t}$</td>
<td>$\ddot{U}<em>{t+\Delta t} = \frac{1}{6\Delta t} \left(11U</em>{i+\Delta t} - 18U_i + 9U_{i-\Delta t} - 2U_{i-2\Delta t}\right)$</td>
</tr>
<tr>
<td>$U_{i-\Delta t} = U_{i+\Delta t} - 2\Delta t \dot{U}<em>{i+\Delta t} + \frac{(2\Delta t)^2}{2} \ddot{U}</em>{i+\Delta t} - \frac{(2\Delta t)^3}{6} \dddot{U}_{i+\Delta t}$</td>
<td></td>
</tr>
</tbody>
</table>

Houbolt’s method uses a third-order interpolation of displacements extending two steps back in time.

**Reminder**: The difference lies in the way we choose a discretized equivalent of the derivatives.
Direct Integration Methods

Most Commonly used Direct Integration Methods

<table>
<thead>
<tr>
<th>The Newmark Method</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Displacement</strong></td>
</tr>
<tr>
<td>The Newmark method uses a second order Taylor expansion for approximating Velocities and Accelerations:</td>
</tr>
<tr>
<td>$\dot{U}_{t+\Delta t} = \dot{U}_t + \Delta t \ddot{U}_t$</td>
</tr>
</tbody>
</table>

$\ddot{U}_t = (1-\delta)\dot{U}_t + \delta \dot{U}_{t+\Delta t}$

if $\delta = 0.5$

$\ddot{U}_t = (1-2\alpha)\dot{U}_t + 2\alpha \dot{U}_{t+\Delta t}$

if $\alpha = 1/4$

Reminder: The difference lies in the way we choose a discretized equivalent of the derivatives.
The Central Difference Method

Approximate the velocity (first derivative) as:

\[ t\dot{U} = \frac{1}{2\Delta t} (t^{+\Delta t} U - t^{-\Delta t} U) \]  

(1)

Approximate the acceleration (second derivative) as:

\[ t\ddot{U} = \frac{1}{\Delta t^2} (t^{+\Delta t} U - 2t^t U + t^{-\Delta t} U) \]  

(2)

Substitute Eqns (1),(2) into \( \mathbf{M}\ddot{U} + \mathbf{C}\dot{U} + \mathbf{K}U = \mathbf{R} \):

\[
\left( \frac{1}{\Delta t^2} \mathbf{M} + \frac{1}{\Delta t^2} \mathbf{C} \right)^{t+\Delta t} U = ^t \mathbf{R} - \left( \mathbf{K} - \frac{2}{\Delta t^2} \mathbf{M} \right)^t U - \left( \frac{1}{\Delta t^2} \mathbf{M} - \frac{1}{2\Delta t} \mathbf{C} \right)^{t-\Delta t} U
\]

\[
\Rightarrow (a_0 \mathbf{M} + a_1 \mathbf{C})^{t+\Delta t} U = ^t \mathbf{R} - (\mathbf{K} - a_2 \mathbf{M})^t U - (a_0 \mathbf{M} - a_1 \mathbf{C})^{t-\Delta t} U
\]
The Central Difference Method

- We see that we do not need to factorize the stiffness matrix, i.e., we only rely on info from the current step $t$ to go to the next one $t + \Delta t$ (explicit method)

- We also see that in order to calculate the displacements at time $\Delta t$ we need to know the displacements at time 0 and $\Delta t$

- In general, $^0U$, $^0\dot{U}$, $^0\ddot{U}$ are known and we can use Eqns (1),(2) to obtain $-\Delta t \dot{U}$:

$$-\Delta t \dot{U} = ^0U - \Delta t^0\dot{U} + \frac{\Delta t^2}{2}^0\ddot{U}$$
The Central Difference Method

Solution Procedure

A. Initial Calculations

1) Form stiffness matrix, mass matrix and damping matrix

2) Initialize $^0U$, $^0\dot{U}$ and $^0\ddot{U}$

3) Select time step $\Delta t$, $\Delta t \leq \Delta t_{cr}$ and calculate integration constants

$$a_0 = \frac{1}{\Delta t^2}, \quad a_1 = \frac{1}{2\Delta t}, \quad a_2 = 2a_0, \quad a_3 = \frac{1}{a_2}$$

4) Calculate $^{-\Delta t}U = ^0U - \Delta t\, ^0\dot{U} + a_3\, ^0\ddot{U}$

5) Form effective mass matrix $\hat{M} = (a_0M + a_1C)$

6) Triangularize $\hat{M} = LDL^T$
Solution Procedure

B. For each time step

1) Calculate effective loads at time $t$:

$$
\hat{t} \mathbf{R} = \hat{t} \mathbf{R} - (\mathbf{K} - a_2 \mathbf{M}) \hat{t} \mathbf{U} - (a_0 \mathbf{M} - a_1 \mathbf{C}) t^{-\Delta t} \mathbf{U}
$$

2) Solve for the displacements $\mathbf{U}$ at time $t + \Delta t$

$$
\mathbf{LDL}^T (t+\Delta t) \mathbf{U} = \hat{t} \mathbf{R}
$$

3) If required, solve for the corresponding velocities and accelerations

$$
\hat{t} \mathbf{\ddot{U}} = a_0 (t^{-\Delta t} \mathbf{U} - 2 \hat{t} \mathbf{U} + t^{+\Delta t} \mathbf{U})
$$

$$
\hat{t} \mathbf{U} = a_1 (t^{-\Delta t} \mathbf{U} + t^{+\Delta t} \mathbf{U})
$$
The effectiveness of the central difference method depends on the efficiency of the time step solution since generally a small discretization is required.

For this reason the method is usually only applied when a **lumped (diagonal) mass matrix** can be assumed and when the velocity dependent damping \((C)\) can be neglected, i.e.:

\[
\frac{1}{\Delta t^2} M^{t+\Delta t} U = \hat{t} \hat{R} \\
\hat{t} \hat{R} = t R - \left( K - \frac{2}{\Delta t^2} M \right) t U - \left( \frac{1}{\Delta t^2} M \right) t-\Delta t U \\
t-\Delta t U = \sum_i \left( \frac{\Delta t^2}{m_i} \right) \hat{R}_i, m_i > 0 \\
K^t U = \sum_i K^{(i)} t U = \sum_i t F^{(i)}
\]
The Central Difference Method

Example - 2 DOF system

For this system the natural periods are $T_1 = 4.45, \ T_2 = 2.8$
The Central Difference Method

Example - 2 DOF system

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{U}_1 \\
\ddot{U}_2 \\
\end{bmatrix}
+ \begin{bmatrix}
6 & -2 \\
-2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
10 \\
\end{bmatrix}
\]

We will calculate the response of the system for \(\Delta t = T_2/10\) and for \(\Delta t = 10 T_2\) over 12 time steps

First we calculate \(0\ddot{U}\)

\[
\begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0\ddot{U}_1 \\
0\ddot{U}_2 \\
\end{bmatrix}
+ \begin{bmatrix}
6 & -2 \\
-2 & 4 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
10 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0\ddot{U}_1 \\
0\ddot{U}_2 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
10 \\
\end{bmatrix}
\]

Then we continue with the steps
The Central Difference Method

Example - 2 DOF system

\[ a_0 = \frac{1}{\Delta t^2}, \quad a_1 = \frac{1}{2\Delta t}, \quad a_2 = 2a_0, \quad a_3 = \frac{1}{a_2} \]

\[ a_0 = \frac{1}{(0.28)^2} = 12.8, \quad a_1 = \frac{1}{2 \cdot 0.28} = 1.79, \]

\[ a_2 = 2 \cdot \frac{1}{(0.28)^2} = 25.5, \quad a_3 = \frac{1}{25.5} = 0.0392 \]

\[ \Delta t = 0.28 \]

\[ -\Delta t \mathbf{U}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.28 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.0392 \begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.0392 \end{bmatrix} \]

\[ \mathbf{M} = 12.8 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 1.79 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix} \]

\[ \hat{\mathbf{R}}^t = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 45.0 & 2 \\ 2 & 21.5 \end{bmatrix}^t \mathbf{U} - \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}^t - \Delta t \mathbf{U} \]
Example - 2 DOF system

The equation which must be solved for each time step is:

\[
\begin{bmatrix}
25.5 & 0 \\
0 & 12.8
\end{bmatrix}
\begin{bmatrix}
U_1^{t+\Delta t} \\
U_2^{t+\Delta t}
\end{bmatrix}
= \hat{t} \mathbf{R}
\]

\[
\hat{t} \mathbf{R} = \begin{bmatrix} 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 45.0 & 2 \\ 2 & 21.5 \end{bmatrix} \begin{bmatrix} U_1^t \\ U_2^t \end{bmatrix} - \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix} \begin{bmatrix} U_1^{t-\Delta t} \\ U_2^{t-\Delta t} \end{bmatrix}
\]
The Central Difference Method

Example - 2 DOF system

The results are:

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>0.28</th>
<th>0.56</th>
<th>0.84</th>
<th>1.12</th>
<th>1.4</th>
<th>1.68</th>
<th>1.96</th>
<th>2.24</th>
<th>2.52</th>
<th>2.8</th>
<th>3.08</th>
<th>3.36</th>
</tr>
</thead>
<tbody>
<tr>
<td>U_1</td>
<td>0</td>
<td>0.030529</td>
<td>0.16643</td>
<td>0.483346</td>
<td>1.007119</td>
<td>1.679429</td>
<td>2.35714</td>
<td>2.849397</td>
<td>2.980892</td>
<td>2.658201</td>
<td>1.913392</td>
<td>0.90657</td>
<td></td>
</tr>
<tr>
<td>U_2</td>
<td>0</td>
<td>0.38925</td>
<td>1.435068</td>
<td>2.807237</td>
<td>4.087467</td>
<td>4.915203</td>
<td>5.10715</td>
<td>4.706874</td>
<td>3.948481</td>
<td>3.151808</td>
<td>2.592586</td>
<td>2.39952</td>
<td>2.518075</td>
</tr>
<tr>
<td>R_1</td>
<td>0</td>
<td>0.7785</td>
<td>4.24396</td>
<td>12.32531</td>
<td>25.68153</td>
<td>42.82543</td>
<td>60.10707</td>
<td>72.65961</td>
<td>76.01274</td>
<td>67.78413</td>
<td>48.79149</td>
<td>23.11753</td>
<td></td>
</tr>
<tr>
<td>R_2</td>
<td>0</td>
<td>4.9824</td>
<td>18.36888</td>
<td>35.93263</td>
<td>52.31957</td>
<td>62.91459</td>
<td>65.37152</td>
<td>60.24799</td>
<td>50.54056</td>
<td>40.34314</td>
<td>33.1851</td>
<td>30.71386</td>
<td>32.23136</td>
</tr>
</tbody>
</table>

![Graph showing CentralDifference_U1 and CentralDifference_U2]
Houbolt Method Derivative Approximation

\[
M\dddot{U} + C\dot{U} + KU = R \quad (c)
\]

\[
t^{+\Delta t}\dddot{U} = \frac{1}{\Delta t^2} (2^{t+\Delta t}U - 5^{t\Delta t}U + 4^{t-\Delta t}U - 2^{t-2\Delta t}U) \quad (a)
\]

\[
t^{+\Delta t}\dot{U} = \frac{1}{6\Delta t} (11^{t+\Delta t}U - 18^{t\Delta t}U + 9^{t-\Delta t}U - 2^{t-2\Delta t}U) \quad (b)
\]

a and b inserted in c

\[
\left(\frac{2}{\Delta t^2} M + \frac{12}{6\Delta t} C + K\right)^{t+\Delta t} U =
\]

\[
t^{+\Delta t}R + \left(\frac{5}{\Delta t^2} M + \frac{3}{\Delta t} C\right)^{t\Delta t} U - \left(\frac{4}{\Delta t^2} M + \frac{3}{2\Delta t} C\right)^{t-\Delta t} U + \left(\frac{1}{\Delta t^2} M + \frac{1}{3\Delta t} C\right)^{t-2\Delta t} U
\]
The Houbolt Method

We will not consider the Houbolt in more detail however it is noted that it is necessary to factorize the stiffness matrix (implicit method)

Furthermore, if the mass and damping terms are neglected, the Houbolt method results in the static analysis equations:

\[ K^{t+\Delta t}U = r^{t+\Delta t} \]

Notice how this is not true for the CDM!
The Newmark Method

In 1959 Newmark presented a family of single-step integration methods for the solution of structural dynamic problems for both blast and seismic loading.

\[ t^{+\Delta t} \dot{U} = t \dot{U} + \left[ (1 - \delta) t \ddot{U} + \delta t^{+\Delta t} \dddot{U} \right] \Delta t \]

\[ t^{+\Delta t} U = t U + t \dot{U} \Delta t + \left[ \left( \frac{1}{2} - \alpha \right) t \ddot{U} + \alpha t^{+\Delta t} \dddot{U} \right] \Delta t^2 \]

The terms essentially result from the use of a Taylor series expansion:

\[ tU = t^{-\Delta t} U + \Delta t t^{-\Delta t} \dot{U} + \frac{\Delta t^2}{2!} t^{-\Delta t} \ddot{U} + \frac{\Delta t^3}{3!} t^{-\Delta t} \dddot{U} + ... \]

\[ t \dot{U} = t^{-\Delta t} \dot{U} + \Delta t t^{-\Delta t} \ddot{U} + \frac{\Delta t^2}{2!} t^{-\Delta t} \dddot{U} + ... \]
The Newmark Method

Notes: $\delta$ and $a$ are parameters, effectively acting as weights for calculating the approximation of the acceleration, and may be adjusted to achieve accuracy and stability

- $\delta = 0.5, \ a = 1/6$ is known as the linear acceleration method, which also correspond to the Wilson $\theta$ method with $\theta = 1$
- Newmark originally proposed $\delta=0.5, \ a = 1/4$, which results in an unconditionally stable scheme (the trapezoidal rule)
The Newmark Method

We now solve for the displacements, velocities and accelerations by inserting the above into the dynamic equilibrium equation:

\[ M^{t+\Delta t} \ddot{U} + C^{t+\Delta t} \dot{U} + K^{t+\Delta t} U = R^{t+\Delta t} \]

where

\[ ^{t+\Delta t} \dot{U} = \dot{U}^t + \left[ (1 - \delta)^t \ddot{U} + \delta^{t+\Delta t} \ddot{U} \right] \Delta t \]

\[ ^{t+\Delta t} U = U^t + \dot{U}^t \Delta t + \left[ \left( \frac{1}{2} - \alpha \right)^t \ddot{U} + \alpha^{t+\Delta t} \ddot{U} \right] \Delta t^2 \]
The Newmark Method

Solution Procedure

A. For each time step

1) Form stiffness matrix, mass matrix and damping matrix
2) Initialize $0U$, $0\dot{U}$ and $0\ddot{U}$
3) Select time step $\Delta t$ and parameters $\alpha$ and $\delta$

$$a_0 = \frac{1}{\alpha \Delta t^2}, \quad a_1 = \frac{\delta}{\alpha \Delta t}, \quad a_2 = \frac{1}{\alpha \Delta t}, \quad a_3 = \frac{1}{2\alpha} - 1,$$

$$a_4 = \frac{\delta}{\alpha} - 1, \quad a_5 = \frac{\Delta t}{2} \left( \frac{\delta}{\alpha} - 2 \right), \quad a_6 = \Delta t (1 - \delta), \quad a_7 = \delta \Delta t$$

4) Form effective stiffness matrix $\hat{K} = K + a_0 M + a_1 C$
5) Triangularize $\hat{K} = LDL^T$

Implicit procedure!
Solution Procedure

B. For each time step

1) Calculate effective loads at time $t$: 
\[ t + \Delta t \begin{bmatrix} \hat{R} = t + \Delta t \begin{bmatrix} R + M (a_0 \dot{U} + a_2 \ddot{U} + a_3 \dddot{U}) \\
+ C (a_1 \dot{U} + a_4 \ddot{U} + a_5 \dddot{U}) \end{bmatrix} \end{bmatrix} \]

2) Solve for the displacements $U$ at time $t + \Delta t$ 
\[ LDL^T \begin{bmatrix} t + \Delta t \dot{U} = t + \Delta t \begin{bmatrix} \hat{R} \end{bmatrix} \end{bmatrix} \]

3) Solve for the corresponding velocities and accelerations 
\[ t + \Delta t \dddot{U} = a_0 (t + \Delta t \begin{bmatrix} U - \dot{U} \end{bmatrix}) - a_2 \dot{U} - a_3 \dddot{U} \]
\[ t + \Delta t \dddot{U} = \dot{U} + a_7 (t + \Delta t \dddot{U}) + a_6 \dddot{U} \]
Stability of the Newmark Method

For zero damping the Newmark method is conditionally stable if

\[ \delta \geq \frac{1}{2}, \quad \alpha \leq \frac{1}{2} \quad \text{and} \quad \Delta t \leq \frac{1}{\omega_{max} \sqrt{\frac{\delta}{2} - \alpha}} \]

where \( \omega_{max} \) is the maximum natural frequency.

The Newmark method is unconditionally stable if

\[ 2\alpha \geq \delta \geq \frac{1}{2} \]
The Newmark Method

Stability of the Newmark Method

However, if $\delta \geq \frac{1}{2}$, errors are introduced. These errors are associated with “numerical damping” and “period elongation”, i.e. a seemingly larger damping and period of oscillation than in reality.

Because of the unconditional stability of the average acceleration method, it is the most robust method to be used for the step-by-step dynamic analysis of large complex structural systems in which a large number of high frequencies, short periods, are present.

The only problem with the method is that the short periods, which are smaller than the time step, oscillate indefinitely after they are excited. The higher mode oscillation can however be reduced by the addition of stiffness proportional (artificial) damping.

source: csiberkeley.com
The Newmark Method

2 dof system example $\Delta t = 0.28s$
The Newmark Method

2 dof system example $\Delta t = 1 \text{s}$

Discrete Time Step, $\Delta t = 1 \text{ sec}$

- 1st Floor CMD
- 2nd Floor CMD
- 1st Floor NM
- 2nd Floor NM
- 1st Floor true
- 2nd Floor true
APPENDIX The Wilson $\theta$ Method In 1973, the general Newmark method was made unconditionally stable by the introduction of a $\theta$ factor.

The introduction of this factor is motivated by the observation that an unstable solution tends to oscillate about the true solution.

Therefore, if the numerical solution is evaluated within the time increment the spurious oscillations are minimized. This can be accomplished by a simple modification to the Newmark method by using a time step defined by:

$$\Delta t' = \theta \Delta t, \quad \theta \geq 1.0$$

Reminder: For $\theta = 1$ the Wilson $\theta$ method is equivalent to the Newmark linear acceleration method with $\delta = 0.5, \ a = 1/6$. 

source: csiberkeley.com
The Wilson $\theta$ Method

In this method the acceleration is assumed to vary linearly from time $t$ to $t + \Delta t$

$$t^+ = t \ddot{U} + \frac{\tau}{\theta \Delta t} \left( (t + \theta \Delta t) \ddot{U} - t \ddot{U} \right)$$

By integration we obtain

$$t^+ = t \ddot{U} + t \dddot{U} + \frac{\tau^2}{2 \theta \Delta t} \left( (t + \theta \Delta t) \dddot{U} - t \dddot{U} \right)$$

$$t^+ = t \dddot{U} + \frac{1}{2} t \dddot{U} + \frac{1}{6 \theta \Delta t} \tau^3 \left( (t + \theta \Delta t) \dddot{U} - t \dddot{U} \right)$$
The Wilson $\theta$ Method

Setting $\tau = \theta \Delta t$ we get

$$t^{+\theta\Delta t} \ddot{U} = \dot{U} + \frac{\theta\Delta t}{2} \left( t^{+\theta\Delta t} \ddot{U} + \dot{U} \right)$$

$$t^{+\theta\Delta t} U = t^{+\theta\Delta t} \dot{U} \theta\Delta t + \frac{1}{6} (\theta\Delta t)^2 \left( t^{+\theta\Delta t} \ddot{U} + 2 \dot{U} \right)$$

from which we can solve

$$t^{+\theta\Delta t} \dddot{U} = \frac{6}{(\theta\Delta t)^2} \left( t^{+\theta\Delta t} U - t \dot{U} \right) - \frac{6}{\theta\Delta t} \dot{U} - 2 \dddot{U}$$

$$t^{+\theta\Delta t} \ddot{U} = \frac{3}{\theta\Delta t} \left( t^{+\theta\Delta t} U - t \dot{U} \right) - 2 \dddot{U} - \frac{\theta\Delta t}{2} \dot{U}$$

We now solve for the displacements, velocities and accelerations by inserting into the dynamic equilibrium equation

$$M^{t^{+\theta\Delta t}} \dddot{U} + C^{t^{+\theta\Delta t}} \ddot{U} + K^{t^{+\theta\Delta t}} U = t^{+\theta\Delta t} \overline{R}$$

$$t^{+\theta\Delta t} \overline{R} = t \overline{R} + \theta (t^{+\Delta t} \overline{R} - t \overline{R})$$
The Wilson $\theta$ Method

Solution Procedure

A. Initial Calculations

1) Form stiffness matrix, mass matrix and damping matrix

2) Initialize $^0U$, $^0\dot{U}$ and $^0\ddot{U}$

3) Select time step $\Delta t$ and calculate integration constants $\theta = 1.4$

   \[
   a_0 = \frac{6}{(\theta \Delta t)^2}, \quad a_1 = \frac{3}{\theta \Delta t}, \quad a_2 = 2a_1, \quad a_3 = \frac{\theta \Delta t}{2}, \quad a_4 = \frac{a_0}{\theta},
   \]

   \[
   a_5 = -\frac{a_2}{\theta}, \quad a_6 = 1 - \frac{3}{\theta}, \quad a_7 = \frac{\Delta t}{2}, \quad a_8 = \frac{\Delta t^2}{6}
   \]

4) Form effective stiffness matrix $\hat{K} = K + a_0M + a_1C$

5) Triangularize $\hat{K} = LDL^T$

Implicit Procedure!
The Wilson $\theta$ Method

Solution Procedure

**B. For each time step**

1) **Calculate effective loads at time $t+\Delta t$:**

$$
^{t+\Delta t}{\hat{R}} = ^{t}{R} + \theta(^{t+\Delta t}{R} - ^{t}{R}) + M(a_0^{t}{U} + a_2^{t}{\dot{U}} + 2^{t}{\ddot{U}}) \\
+ C(a_1^{t}{U} + 2^{t}{\dot{U}} + a_3^{t}{\ddot{U}})
$$

2) **Solve for the displacements $U$ at time $t + \Delta t$**

$$
LDLT^{t+\Delta t}U = ^{t+\Delta t}{\hat{R}}
$$

3) **Solve for the corresponding velocities and accelerations**

$$
^{t+\Delta t}{\ddot{U}} = a_4(^{t+\Delta t}{U} - ^{t}{U}) + a_5^{t}{\dot{U}} + a_6^{t}{\ddot{U}} \\
^{t+\Delta t}{\dot{U}} = ^{t}{\ddot{U}} + a_7(^{t+\Delta t}{\ddot{U}} + ^{t}{\ddot{U}}) \\
^{t+\Delta t}{U} = ^{t}{U} + \Delta t^{t}{\dot{U}} + a_8(^{t+\Delta t}{\ddot{U}} + 2^{t}{\ddot{U}})
$$
Coupling of integration operators

For some problems it may be an advantage to combine the different types of integration schemes e.g. if a structure is subjected to dynamic load effect from hydrodynamic loading then the analysis of the hydrodynamic forces may be assessed using an explicit scheme and the structural response by using an implicit scheme.

The best choice of strategy will depend on the problem at hand in regard with stability and accuracy!