The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

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Overview so far
- Material Nonlinearity
- Large Displacements
- Dynamic Analysis

Special Topics
- Material Laws
- The Contact problem
- Fracture & Special Formulations (XFEM, SBFEM)
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Special Topics
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Constitutive Relations

Previously we examined the kinematic equations formulation (displacement, strain displacement relations)

The next step is to determine appropriate constitutive relationships of the form:

\[ \sigma = f(\varepsilon) \]

ex. linear analysis in 1D \( \Rightarrow \sigma = E\varepsilon \)

When dealing with higher dimensions & incremental analysis, this is written in tensor form for time \( t \):

\[ ^t\sigma = ^tC_{ijrs}^t\varepsilon_{rs} \]
It is necessary that kinematic and constitutive relations are appropriate. Previously we saw that in the large displacement formulation appropriate tensors need to be defined. e.g. TL Formulation $\Rightarrow$ Second Piola-Kirchhoff stress tensor, Green Lagrange strain tensor).

Therefore a problem involving large strains should also be combined with a material law that admits large strains.

However, we might be examining a problem of large displacements with small strains. In that case we can still use the material laws defined for classic engineering stress and strain measures (for small displacements) but this time combined with the SP-K stress and G-L strain tensors.
Main Stress - Strain pairs:

- **Material Nonlinearity** (small deformations)
  Engineering (or Nominal) Stress \( \sigma \)
  Engineering Strain \( \varepsilon \)

- **TL formulation** (large deformations)
  2nd Piola-Kirchhoff Stress \( S \)
  Green-Lagrange Strain \( \epsilon \)

- **UL formulation** (large deformations)
  Cauchy (or True) Stress \( \tau \)
  Almansi Strain \( \epsilon^A \)

Note that: \( \tau = \frac{L}{L_0} \sigma, \varepsilon = \frac{L - L_0}{L_0} \)
General Solution process in incremental nonlinear FE

**Known Solution at t:**
- Stresses $\sigma$, strains $\varepsilon$,
- Internal material parameters $\kappa$

**Known Quantities at iterations $i-1$:**
- Nodal Displacements at first Iteration: $\Delta U_{i-1}$
  - and hence
  - Element strains $\varepsilon_{i-1}$

**Calculate at $t+\Delta t$:**
- Stresses $\sigma_{i-1}$
- Tangent stress strain matrix $C_{i-1}$
- Internal material parameters $\kappa_{i-1}$
  - Elastic Analysis: directly obtain $\sigma_{i-1}$, $C_{i-1}$ from $\varepsilon_{i-1}$
  - Inelastic Analysis: Integrate to get $\sigma_{i-1} = \sigma + \int_{t}^{t+\Delta t} d\sigma$

**Calculate:**
- Incremental Displacement Vector $\Delta U_i$
  - $\Delta U_i = \Delta F_{i-1} - \Delta R$
  - Then,
  - $\Delta U_i = \Delta U_{i-1} + \Delta U_i$

Report till Convergence
Overview of Material Descriptions

We can discriminate amongst the following major classes of material behavior:

- Elastic, linear or nonlinear
- Hyperelastic
- Hypoelastic
- Elastoplastic
- Creep
- Viscoplastic
For an elastic material the stress is a function of strain only. The stress path is the same both in loading and unloading.

**Linear Elastic**
The elasticity (constitutive) tensor components, $C_{ijrs}$ are constant.

**Nonlinear Elastic**
The elasticity (constitutive) tensor components, $C_{ijrs}$ are a function of strain.

**Example:** Almost all materials under small stress.
1.2.1 Cauchy Elastic Models

Some nonlinear elastic models are based on the general premise that for an elastic material the current state of Cauchy stress is a function of the current state of strain, and not of the history of strain. That is, \( \sigma_{ij} = F_{ij}(\varepsilon_{kl}) \), where \( F_{ij} \) is some nonlinear function. Once this condition is satisfied, however, the inverse relation does not necessarily exist. Materials that satisfy this fundamental requirement of an elastic body are called Cauchy elastic materials.

More specifically, the general functional form relationship for Cauchy elastic models is written in indicial form as

\[
\sigma_{ij} = \alpha_0 \delta_{ij} + \alpha_1 \varepsilon_{ij} + \alpha_2 \varepsilon_{im} \varepsilon_{mj} + \alpha_3 \varepsilon_{im} \varepsilon_{mn} \varepsilon_{nj} + \cdots
\]  

(1.1)

or, in direct form

\[
\sigma = \alpha_0 I + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \alpha_3 \varepsilon^3 + \cdots
\]  

(1.2)

where \( I \) is the second-order identity tensor. The \( \alpha_i \) (\( i = 1, 2, 3, \cdots \)) represent model parameters whose values are determined from the results of laboratory experiments. Further details pertaining to such parameter determination are given in the following example.

**Example 1.1: First-Order Cauchy Elastic Model**

A first order Cauchy model corresponds to the case of isotropic linear elasticity. As such, only the first-order terms in equation (1.1) are retained, giving

\[
\sigma_{ij} = \alpha_0 \delta_{ij} + \alpha_1 \varepsilon_{ij}
\]  

(1.3)

In order to determine the values of the parameters \( \alpha_0 \) and \( \alpha_1 \), consider two simple laboratory experiments. First consider simple shear deformation. Since there is no volumetric strain, and since only one nonzero component of strain exists (say \( \varepsilon_{12} = \varepsilon_{21} \)), it follows that

\[
\sigma_{12} = \alpha_1 \varepsilon_{12} = \alpha_1 \gamma_{12}
\]  

(1.4)

Solving for \( \alpha_1 \) gives

\[
\alpha_1 = \frac{\sigma_{12}}{\gamma_{12}}
\]
For the case of an elastic material we already saw that the **TL Formulation** (used for large deformation analysis) yields:

\[ t_0 S_{ij} = t_0 C_{ijrs} t_0 \epsilon_{rs} \]

The elasticity tensor for 3D stress conditions is defined as:

\[ tC_{ijrs} = \lambda \delta_{ij} \delta_{rs} + \mu (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}) \]

where \( \lambda \) and \( \mu \) are the Lamé constants and \( \delta_{ij} \) is the Kronecker delta,

\[
\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \quad \text{(homogeneous isotropic material)}
\]

\[
\delta_{ij} = \begin{cases} 
0 & i \neq j \\
1 & i = j 
\end{cases}
\]
Important Note

“The 2nd Piola-Kirchhoff (PK2) stress and Green-Lagrange strain tensor components are invariant to rigid body motions.”

For problems with small strains we can take advantage of this observation and use any constitutive relationship that has been developed for engineering stress and strain measures by just substituting with the PK2 stress and Green-Lagrange strain

This observation can be extended to all problems with large deformations but small strain conditions such as the elastic or elastoplastic buckling problem and the collapse analysis of slender structures.
Hyperelastic (rubberlike) materials exhibit an incompressible response (volume preserving), path independence and no energy dissipation.

The stress is now calculated through the strain energy functional $W$

$$
^t S_{ij} = \frac{\partial W}{\partial^t \epsilon_{ij}}
$$

Figure: Stress-strain curves for various hyperelastic material models.
Hyperelastic Material

Hyperelastic Material Models

- **Saint Venant-Kirchhoff model**

  \[
  W(\varepsilon) = \frac{\lambda}{2} [tr(\varepsilon)]^2 + \mu tr(\varepsilon^2)
  \]

  and the second Piola-Kirchhoff stress can be derived as

  \[
  S = \lambda [tr(\varepsilon)] I + 2\mu \varepsilon
  \]

  \(\lambda, \mu\) are the Lamé constants

- **Mooney-Rivlin model**

  \[
  W(\varepsilon) = C_1 (I_1 - 3) + C_2 (I_2 - 3)
  \]

  where \(C_1\) and \(C_2\) are empirically determined material constants and

  \[
  I_1 = tr(C) = C_{11} + C_{22} + C_{33}
  \]

  where \(C\) is the Cauchy-Green deformation tensor (see Lecture 4) and

  \[
  I_2 = \frac{1}{2} [(I_1)^2 - tr(C')^2]
  \]
**Inelasticity**

**Elastoplasticity, Creep** and **Viscoplasticity** are types of **Inelastic** behavior

- Elastic behavior $\Rightarrow$ stresses can be directly calculated from the strain
- Inelastic behavior $\Rightarrow$ the stress at time $t$ depends on the stress strain history

In the incremental analysis of inelastic response we had three main scenarios

- Small displacements-rotations / small strains $\Rightarrow$ use linear elastic solution, engineering stress and strain measures
- Large displacements-rotations / small strains $\Rightarrow$ use TL formulation by substituting the appropriate stress - strain measures (PK2, Green-Lagrange) in the place of the engineering stress and strain measures
- Large displacements-rotations / large strains $\Rightarrow$ use either TL or UL formulation, more complex constitutive laws
Elastoplasticity

In this formulation we encounter a linearly elastic behavior until yield and usually a hardening post yield behavior.

Examples: Metals, soil, and Rocks when subjected to high stresses.
Elastoplasticity

The strain and stress increments are given by:

\[ d\epsilon_{rs} = d\epsilon^E_{rs} + d\epsilon^P_{rs} \]
\[ d\sigma_{ij} = C^E_{ijrs} (d\epsilon_{rs} - d\epsilon^P_{rs}) \]

where \( C^E_{ijrs} \) are the components of the elastic constitutive tensor and \( d\epsilon_{rs}, d\epsilon^E_{rs}, d\epsilon^P_{rs} \) are the components of the total strain increment.

To calculate the plastic strains we use the following three properties:

- **Yield Function** \( f_y(\sigma, \epsilon^P) \)
  \[ f_y < 0 \Rightarrow \text{Elastic behavior} \]
  \[ f_y \geq 0 \Rightarrow \text{Plastic or elastic behavior depending on the loading condition} \]

- **Flow rule**
  The yield function is used in the flow rule in order to obtain the plastic strain increments
  \[ d\epsilon^P_{ij} = \lambda \frac{\partial f_y}{\partial \sigma_{ij}} \]
  \( \lambda \) is a scalar to be determined

- **Hardening rule**
  This specifies how the yield function is modified during the progression of loading.
Elastoplasticity

Example: Von Mises yield criterion (in 3D):

\[ f_y = 0 \Rightarrow (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) - 2\sigma_y^2 = 0 \]
Elastoplasticity

Isotropic & Kinematic hardening Rules

- In the case of isotropic hardening, the yield surface expands uniformly.
- In the case of kinematic hardening, the size of the yield surface remains unchanged and the center location of the yield surface is shifted. (Bauschinger effect)
Obviously, for a reversed loading process like the one in the cyclic loading diagram of Fig. 1, the isotropic hardening will lead to a cyclic test behaviour according to the solid line $OABCD$ of Fig. 2 (in which the length of line segment $BC$ is the same as that of line segment $AB$). It is, however, a well-established fact that in most materials there is a Bauschinger effect, by which a reversed loading will lead to a behaviour that is different from the one in isotropic hardening. This is illustrated in Fig. 2, which shows the $\sigma\varepsilon$ diagram of a uniaxial cyclic test.

**Fig. 2** $\sigma\varepsilon$ diagram of uniaxial cyclic test.

- Isotropic hardening
- Kinematic hardening
Thermoelastoplasticity and Creep

This behavior exhibits time effect of increasing strains under constant loads or decreasing stress under constant deformations (relaxation). Typical examples of such behavior are metals at high temperatures.

The thermal strain ($\epsilon = \alpha \Delta T$) and the creep strain now enter the formulation of the stress strain relationships.
Viscoplasticity describes the rate-dependent inelastic behavior of solids. Rate-dependence in this context means that the deformation of the material depends on the rate at which loads are applied. The inelastic behavior that is the subject of viscoplasticity is plastic deformation which means that the material undergoes unrecoverable deformations when a load level is reached. Rate-dependent plasticity is important for transient plasticity calculations.

The main difference between rate-independent plastic and viscoplastic material models is that the latter exhibit not only permanent deformations after the application of loads but continue to undergo a creep flow as a function of time under the influence of the applied load.

Typical examples of such behavior are Polymers and Metals.
Viscoplasticity

(a) Dashpot Element (λ, N)

(b) Spring Element (E)

(c) Sliding Frictional Element (σy)

\[ \frac{d\varepsilon}{dt} = 100 \text{ /s} \]

\[ \frac{d\varepsilon}{dt} = 0.1 \text{ /s} \]

Hardening

Stress (σ) vs. Strain (ε)