The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

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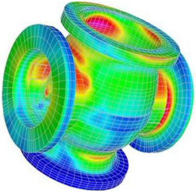
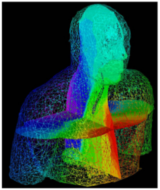




Newton's Method for FE

Let's start from the linear problem:

- Physics of the problem $\rightarrow$ Strong Form
- Derive the Weak Form of the problem (integral equation)
- Use Galerkin's method (weight functions) to approximate the solution
- Implement the FE method:
  - Shape Functions N^i, $i = 1 \dots \#dofs$
  - Element Stiffness Matrices $\mathbf{K}^e$, $i = 1 \dots \#elements$
  - Element Load Vectors f^e, $i = 1 \dots \#elements$
  - Assemble the global stiffness and force vector $\mathbf{K}$, F
- Finally, solve the problem equation:

$$\mathbf{K}U = F$$

to obtain the unknown displacement vector U.

Newton's Method for FE

Linear Analysis assumptions:

For a linear elastic material, the material properties E, A, I are treated as constants and can be moved outside of the relevant integrals:

- **Bar Element**

$$K_{ij} = \int_0^L \frac{dN_j(x)}{dx} AE \frac{dN_i(x)}{dx} dx$$

- **Beam Element**

$$K_{ij} = \int_0^L \frac{d^2 H_j(x)}{dx^2} EI \frac{d^2 H_i(x)}{dx^2} dx$$

However, when the material is described by a **nonlinear material law**, A, E, I may not be moved outside the integral and $\mathbf{K}$ inevitably is a nonlinear function of the level of displacement U.

Newton's Method for FE

Material Nonlinearity - General Equation

$$\mathbf{K}(U)U = F$$

This requires the solution of a nonlinear equation. The **Newton-Raphson** method can be implemented. In order to bring the problem in the form $f(x) = 0$, we define the residual $\mathbf{r}(U)$:

$$\mathbf{r}(U) = \mathbf{K}(U)U - \mathbf{F} \Rightarrow \mathbf{r}(U) = 0$$

Then, the Newton iteration formula for a multiple-degree-of-freedom (mdof) becomes:

$$U_{k+1} = U_k - \mathbf{T}^{-1}(U_k)\mathbf{r}(U_k)$$

where, the **tangent stiffness**, $\mathbf{T}$, is defined as:

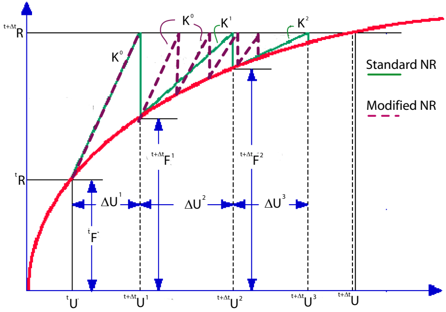
$$\mathbf{T}(U_k) = \frac{d\mathbf{r}(U_k)}{dU} = \frac{d\mathbf{K}(U)U - F}{dU}\Big|_{U=U_k}$$

$$\mathbf{T}(U_k) = \mathbf{K}(U_k) + \frac{d\mathbf{K}(U_k)}{dU}U_k$$

Modified Newton (Raphson) Method

Solution Process

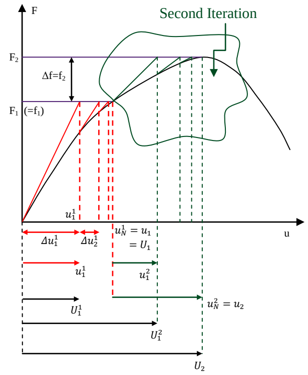
The N-R method provides the solution of the problem equation at a given load level F. Since, it may be computationally expensive to calculate the tangent stiffness matrix, an alternative is to apply a **Modified Newton-Raphson** iteration scheme where **T** is only calculated in the beginning of the analysis.



A larger number of iterations might be required but there is not cost for recalculating the stiffness - thus the overall time is significantly lower.

Solution Process

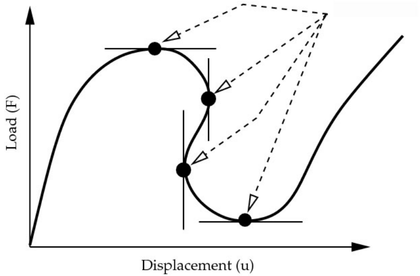
Implementation of the N-R method provides the displacement that would correspond to a specific load level. In order to track the evolution of the structure's response as the load level evolves, one needs to perform an incremental analysis, i.e., implement the load in sufficiently **small increments** ΔF (how small? - this is to be decided based on engineering judgment).

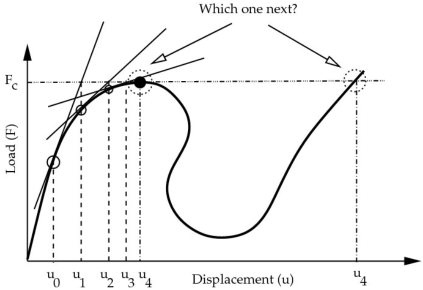


The N-R is implemented for every load increment:
The stiffness matrix $\mathbf{K}$, as well as the tangent matrix $\mathbf{T}$, should be calculated using the "global" displacement and force values (U, F), since their value depends on the actual (global) state of the element.
However, the Newton-Raphson equation is solved for each load increment i:
$\Delta F = f_i = \mathbf{K} u_i$, where u_i, f_i denote "incremental" quantities within each load step.

Special Considerations

The Standard Newton-Raphon methods perform poorly for bucking problems, where the slope at limit points is exactly equal to 0





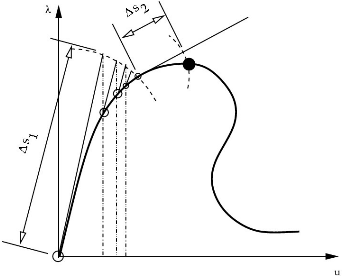
Standard solution techniques lead to instability near the limit points and also present problems in case of snap-through and snap-back points, failing to predict the complete load-displacement response.

How to solve this problem?

The Arc-Length Method for Nonlinear Post-Buckling

Also called the Modified Riks Method, originally developed by Riks (1972; 1979) and Wempner (1971).

- Control the size of the load step using a parameter λ.
- Solve for both λ and ΔU in each Newton iteration.



Assume F = independent of geometry. Then, λ can be thought of as a normalized load parameter and the residual is given by

$$\mathbf{r}(U,\lambda) = \mathbf{K}(U)U - \lambda\mathbf{F}$$

The load increment is computed using

$$\lambda = \pm\sqrt{\Delta s^2 - \Delta U_n^2}$$

where the reference arc length is

$$\Delta s_0^2 = \frac{\mathbf{F}}{n_{loadsteps}}$$