Chapter 2

The Direct Stiffness Method
Direct Stiffness Method (DSM)

• **Computational method** for *structural analysis*
• **Matrix method** for computing the member forces and displacements in structures
• DSM implementation is the basis of most commercial and open-source finite element software
• Based on the displacement method (classical hand method for structural analysis)
• Formulated in the 1950s by Turner at Boeing and started a *revolution in structural engineering*
Goals of this Chapter

• DSM formulation

• DSM software workflow for …
  • linear static analysis (1\textsuperscript{st} order)
  • 2\textsuperscript{nd} order linear static analysis
  • linear stability analysis
Computational Structural Analysis

Physical problem → Continuous mathematical model → Discrete computational model

Modelling is the most important step in the process of a structural analysis!
System Identification (Modelling)

Global Coordinate System
- Nodes
- Elements
- Boundary conditions
- Loads

Node numbers

Element numbers and orientation

Method of Finite Elements I
Deformations

System Deformations

Nodal Displacements

nodes, elements, loads and supports
deformed shape

(deformational, nodal)
degrees of freedom = dofs
Degrees of Freedom

Truss Structure

\[ u_i = (u_{dx}, u_{dy}) \]

\[ 7 \times 2 = 14 \text{ dof} \]

Frame Structure

\[ u_i = (u_{dx}, u_{dy}, u_{rz}) \]

\[ 8 \times 3 = 24 \text{ dof} \]
Elements: Truss

\[ X/Y = \text{local coordinate system} \]
\[ u_x = \text{displacement in direction of local axis } X \]
\[ DX = \text{displacement of truss end} \]

\[ \varepsilon = \frac{DX}{L} \]

\[ \sigma = E \varepsilon \]

\[ P_2 = -P_1 = N \]

\[ N = \int E \sigma = EF \sigma = \frac{EF}{L} DX \]

\[ p_1 \]
\[ p_2 \]

\[ p = k u \]

\[ p : (\text{element}) \text{ stiffness matrix} \]
\[ k : (\text{element}) \text{ nodal forces} \]
\[ u : (\text{element}) \text{ displacement vector} \]
**Elements: Beam**

3 dof per node

\[ u_x = \text{displacement in direction of local axis } X \]

\[ u_y = \text{displacement in direction of local axis } Y \]
Elements: Global Orientation

\[ R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 & 0 & 0 \\
\sin(\theta) & \cos(\theta) & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\
0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 0 & 0 & 1 
\end{bmatrix} \]

\[ u_{\text{glob}} = u = R \ u_{\text{loc}} \]

\[ k_{\text{glob}} = k = R^T \ k_{\text{loc}} \ R \]
### Beam Stiffness Matrix

**Element stiffness matrix** in global orientation

\[
\begin{bmatrix}
U_{X_S} & U_{Y_S} & U_{Z_S} \\
U_{X_E} & U_{Y_E} & U_{Z_E}
\end{bmatrix} = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{33} & & k_{35} & k_{36}
\end{bmatrix}
\]

\[\begin{align*}
\{p\}_{iS} &= \begin{bmatrix} k_{iSS} \\ k_{iES} \end{bmatrix} \begin{bmatrix} k_{iSE} \\ k_{iEE} \end{bmatrix} \cdot \begin{bmatrix} u_{iS} \\ u_{iE} \end{bmatrix} \\
p &= k \cdot u
\end{align*}\]

- **FX_S** = \(k_{11}\) \(k_{12}\) \(k_{13}\) \(k_{14}\) \(k_{15}\) \(k_{16}\)
- **FY_S** = \(k_{22}\) \(k_{23}\) \(k_{24}\) \(k_{25}\) \(k_{26}\)
- **MZ_S** = \(k_{33}\) \(k_{35}\) \(k_{36}\)
- **FX_E** = symm.
- **FY_E** = \(k_{44}\) \(k_{45}\) \(k_{46}\)
- **MZ_E** = \(k_{55}\) \(k_{56}\) \(k_{66}\)

**Example:** \(k_{24}\) =

reaction in global direction Y at start node S

due to a

unit displacement in global direction X at end node E
Nodal Equilibrium

\( \mathbf{r}_4: \) Vector of all forces acting at node 4

\[
\mathbf{r}_4 = -k_{6E} \mathbf{u}_3 + \text{contribution of element 6 due to start node displacement } \mathbf{u}_3 \\
- k_{6E} \mathbf{u}_4 + \text{contribution of element 6 due to end node displacement } \mathbf{u}_4 \\
- k_{5E} \mathbf{u}_4 + \text{contribution of element 5 due to start node displacement } \mathbf{u}_4 \\
- k_{5E} \mathbf{u}_2 + \text{contribution of element 5 due to start node displacement } \mathbf{u}_2 \\
\mathbf{f}_4 \text{ external load}
\]

Equilibrium at node 4:

\[
\mathbf{r}_4 = -k_{5E} \mathbf{u}_2 - k_{6E} \mathbf{u}_3 - k_{5E} \mathbf{u}_4 - k_{6E} \mathbf{u}_4 + \mathbf{f}_4 = 0
\]
### Global System of Equations

<table>
<thead>
<tr>
<th></th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 ) = -</td>
<td>( k_{1EE}^+ )</td>
<td>( k_{3SE} )</td>
<td>( k_{4SE} )</td>
<td>+ ( f_1 = 0 )</td>
</tr>
<tr>
<td>( r_2 ) = -</td>
<td>( k_{3ES} )</td>
<td>( k_{2EE}^+ )</td>
<td>( k_{5SE} )</td>
<td>+ ( f_2 = 0 )</td>
</tr>
<tr>
<td>( r_3 ) = -</td>
<td>( k_{4ES} )</td>
<td>( k_{4EE}^+ )</td>
<td>( k_{6SE} )</td>
<td>+ ( f_3 = 0 )</td>
</tr>
<tr>
<td>( r_4 ) = -</td>
<td>( k_{5ES} )</td>
<td>( k_{6ES} )</td>
<td>( k_{5EE}^+ )</td>
<td>+ ( f_4 = 0 )</td>
</tr>
</tbody>
</table>

\(- K \quad U \quad + \quad F \quad = \quad 0\)  

\( F = K \quad U \)
Global System of Equations

\[ \mathbf{F} = \text{global load vector} = \text{Assembly of all } \mathbf{f}_e \]

\[ \mathbf{K} = \text{global stiffness matrix} = \text{Assembly of all } \mathbf{k}_e \]

\[ \mathbf{U} = \text{global displacement vector} = \text{unknown} \]

\[ \mathbf{F} = \mathbf{K} \mathbf{U} \] = equilibrium at every node of the structure
Solving the Equation System

What are the nodal displacements for a given structure (= stiffness matrix $\mathbf{K}$) due to a given load (= load vector $\mathbf{F}$)?

$$
\mathbf{K} \mathbf{U} = \mathbf{F} \quad \text{left multiply } \mathbf{K}^{-1}
$$

$$
\Rightarrow \quad \mathbf{K}^{-1} \mathbf{K} \mathbf{U} = \mathbf{K}^{-1} \mathbf{F}
$$

$$
\Rightarrow \quad \mathbf{U} = \mathbf{K}^{-1} \mathbf{F}
$$

Inversion possible only if $\mathbf{K}$ is non-singular (i.e. the structure is sufficiently supported = stable)
Beam Element Results

1. Element nodal displacements
   Disassemble $\mathbf{u}$ from resulting global displacements $\mathbf{U}$

2. Element end forces
   Calculate element end forces $\mathbf{p} = \mathbf{k} \mathbf{u}$

3. Element stress and strain along axis
   Calculate moment/shear from end forces (equilibrium equation)
   Calculate curvature/axial strain from moments/axial force

4. Element deformations along axis
   Calculate displacements from strain (direct integration)
Lateral Load

1. Adjust global load vector

\[ f = \begin{pmatrix} \frac{L}{2} \\ 0 \\ 0 \\ \frac{L}{2} \\ 0 \\ 0 \\ -\frac{L^2}{12} \end{pmatrix} = \begin{pmatrix} f_x+ \\ 0 \\ 0 \\ \frac{L^2}{12} \\ 0 \\ \frac{L}{2} \\ \frac{L^2}{12} \end{pmatrix} \]

- \[ f_x \to \frac{f_y L^2}{12} \]
- \[ f_y \to \frac{-f_y L^2}{24} \]

\( f = \) local load vector \( \Rightarrow \) add to global load vector \( F \)

2. Adjust element stresses

e.g. bending moment \( M \):

\[ M \text{ due to } f + \text{M due to } u = \text{M diagram} \]
Linear Static Analysis (1\textsuperscript{st} order)

Workflow of computer program

1. System identification: Elements, nodes, support and loads
2. Build element stiffness matrices and load vectors
3. Assemble global stiffness matrix and load vector
4. Solve global system of equations (\Rightarrow displacements)
5. Calculate element results

Exact solution for displacements and stresses